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Research Article

On the identification numbers of lobster graphs

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Abstract: Given a nontrivial connected undirected graph G with diameter d, a vertex coloring c of G that uses only the colors red and white induces, for each $v \in V(G)$, the d-vector $\vec{d}(v) = [a_1 a_2 \cdots a_d]$, where each a_i is equal to the number of red vertices of distance i from v. Then c is called an ID-coloring of G if $\vec{d}(v) \neq \vec{d}(w)$ for all distinct $v, w \in V(G)$. If G has at least one ID-coloring, then it is called an ID-graph and its identification number $\mathrm{ID}(G)$ is defined to be the minimum number of red vertices among all ID-colorings of G. The notions of ID-colorings and identification number have been shown to be equivalent to the notions of multiset resolving sets and multiset dimension, respectively. Previous works on this topic have focused on characterizing ID-caterpillars and ID-lobsters and on the identification numbers of some ID-caterpillars. In this paper, we focus on the identification numbers of ID-lobsters. Specifically, we establish a sharp lower bound for the identification number of all ID-lobsters. Furthermore, we characterize and determine the identification numbers of all uniform ID-lobsters.

Keywords: identification colorings, multiset dimension, lobster graph.

AMS Subject classification: 05C12, 05C15, 05C78

1. Introduction

The problem of distinguishing the vertices of a graph has captured the interest of different mathematicians. In their paper *Distance Vertex Identification in Graphs* [2], Chartrand, Kono and Zhang have introduced an approach to this problem using

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red-white colorings (i.e., vertex colorings in which each vertex is colored red or white and at least one vertex is colored red). Given a nontrivial connected undirected graph G with diameter $d \geq 2$, a red-white coloring c of G induces a code $\vec{d}(v) = [a_1 a_2 \cdots a_d]$ for every vertex v in G, where for $1 \leq i \leq d$, the component a_i is the number of red vertices at distance i from v. If no two vertices have the same code, then c called is an *identification coloring* or *ID-coloring* of G. Any graph possessing an ID-coloring is called an *ID-graph*. Thus, if a graph G is an ID-graph, then its vertices can be distinguished from each other using an ID-coloring of G, even in the case that G has nontrivial automorphisms.

It turns out that the notion of ID-colorings is equivalent to the earlier notion of multiset resolvings sets independently introduced by Saenpholphat [11] and Simanjuntak, Siagian, & Vetrik [12]. This equivalence is established in [4] and implies that we can replace the vector code $\vec{d}(v)$ by the *multiset code* $M(v)$, which is defined to be the multiset of distances of v to each of the red vertices in G. More precisely, a red-white coloring of a graph G is an ID-coloring if and only if no two vertices have the same multiset codes.

Given an ID-graph G, the minimum number of red vertices among all ID-colorings of G is called the *identification number* $\mathrm{ID}(G)$ (or *ID-number* or *multiset dimension*) of G. The ID-numbers of different graph families have been studied in the literature. For instance in [2, 12], it has been shown that any nontrivial path has ID-number 1, and any cycle with at least 6 vertices has ID-number 3. In [9, 10, 12], it has also been shown that the grid $P_m \Box P_n$, for $m \geq 1$ and $n \geq 4$, has identification number 3.

ID-colorings of different tree families have also been studied in [3, 5–8], where different results on conditions for trees to be ID-graphs have been presented. Particular attention has also been devoted to a specific family of trees called *caterpillars*, which are trees of order 3 or more, for which the removal of its leaves yields a path graph. The ID-number of some ID-caterpillars has been determined in [8].

In [3], one of the graph families considered is that of *lobster graphs*, which are trees for which the removal of their leaves yields a caterpillar graph. One of the proposed main results in [3] is a characterization theorem for lobster graphs that are ID-graphs. However, the identification number (or multiset dimension) of such graphs has not been studied. Thus, in this paper, we extend these previous works by focusing on the identification numbers of lobster graphs that are ID-graphs.

After presenting some preliminary definitions and notations below, we investigate in the next section properties of ID-colorings for general lobster graphs. This leads to our first result that provides a sharp lower bound for the ID-number of general ID-lobsters. In the third section, we focus our attention on characterizing and determining the ID-number of a general family of lobster graphs called uniform ID-lobsters.

1.1. Preliminaries

All graphs to be considered in this paper are simple, connected, nontrivial, and undirected. As is customary, for a positive integer n, the path graph of order n is denoted by P_n while the star graph of order $n+1$ is denoted by $K_{1,n}$.

Given a red-white coloring c of a graph G, recall that, for each $v \in V(G)$, the multiset code of v is given by the multiset

$$M(v) = \{d(u,v) : u \text{ is a red vertex}\}.$$

Further, we denote by $\max M(v), \min M(v)$, and $\sigma(M(v))$ the maximum element, the minimum element, and the sum of all elements, respectively, of $M(v)$.

From [2], recall that a set S of t vertices, where $t \geq 2$, of a graph G is called a *t-tuplet* if either

(a) the vertices form an independent set and every two vertices in S have the same neighborhood; or

(b) S is a clique and every two vertices in S have the same closed neighborhood.

A 2-tuplet is also called a *twin* while a 3-tuplet is also called a *triplet.* The following results from [2] are used in the next sections.

Proposition 1 ([2]). *Let c be an ID-coloring of a connected graph G. If u and v are twins of G, then $c(u) \neq c(v)$. Consequently, if G is an ID-graph, then G is triplet-free.*

Proposition 2 ([2]). *There is no ID-coloring of a connected graph with exactly two red vertices.*

Theorem 1 ([2]). *A nontrivial connected graph G has $\mathrm{ID}(G) = 1$ if and only if G is a path.*

As previously mentioned, a caterpillar graph is a tree for which the removal of its pendant vertices yields a path or a zero-order graph. On the other hand, a lobster graph is a tree for which the removal of its pendant vertices yields a caterpillar graph. In the following, we present an equivalent formulation of lobster graphs as well as corresponding notations to be used throughout the paper.

Definition 1. A *lobster graph* is a tree $G = (V,E)$ such that $V = V_0 \cup V_1 \cup V_2$, where $V_0 \neq \emptyset$ and V_0, V_1, V_2 are pairwise disjoint, and

1. the subgraph of G induced by the vertices in V_0 is a path, called the *central path* of G;
2. for each $w \in V_1$, there is exactly one $v \in V_0$ such that $vw \in E$;
3. for each $y \in V_2$, there is exactly one $w \in V_1$ such that $wy \in E$.

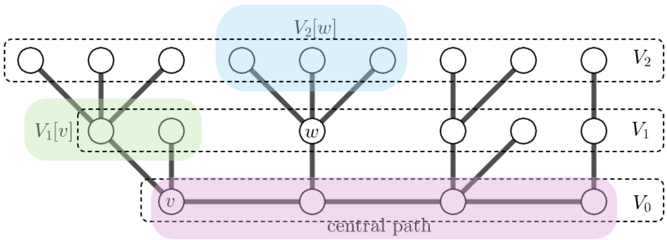


Figure 1. A lobster graph with vertex set $V_0 \cup V_1 \cup V_2$. For the identified central vertex v and distance-1 vertex w, the sets $V_1[v]$ and $V_2[w]$ are shown.

Following the notations in the above definition, the central path of a lobster graph $G = (V_0 \cup V_1 \cup V_2, E)$ will be denoted by $P(G)$ while the vertices in V_0 will be referred to as *central vertices* of G. It can be easily observed that any vertex in V_1 (resp. V_2) is of distance 1 (resp. distance 2) to the central path $P(G)$. Thus, we refer to the vertices in V_1 (resp. V_2) as *distance-1* (resp. *distance-2*) *vertices*. For any central vertex v of G, we define$V_1[v] = \{w \in V_1 : vw \in E\}$. Similarly, for any distance-1 vertex w of G, we define $V_2[w] = \{y \in V_2 : wy \in E\}$. These definitions and notations are illustrated using the lobster graph shown in Fig. 1.

Now, let v be a central vertex of the lobster graph G. By a *branch* at v, we mean a subgraph of G induced by v, a vertex $w \in V_1[v]$, and all the vertices, if any, in $V_2[w]$. For example, there are two branches at the central vertex v in the lobster graph in Fig. 1; one branch is isomorphic to $K_{1,4}$ while the other is isomorphic to P_2.

2. Lobster Graphs

A lobster graph that is also an ID-graph is called an *ID-lobster*. In this section, we investigate some properties of ID-lobsters towards establishing a sharp lower bound for the ID-number of these graphs.

Following a similar development as in [3], we begin by presenting restrictions on the branches that can be present in ID-lobsters. The first restriction is given by the following observation, which is also part of Theorem 2 in [3]. We include a proof for completeness.

Observation 2 ([3]). Let G be an ID-lobster. Then for any central vertex v of G, any branch at v must be isomorphic to P_2, P_3, or $K_{1,3}$.

Proof. Since G is an ID-graph, it must be triplet-free by Proposition 1. Thus, $|V_2[w]| \leq 2$ for all $w \in V_1$. Let v be a central vertex of G and let B be a branch at v. By definition, B is a subgraph induced by v, a vertex $w \in V_1[v]$, and all the vertices, if any, in $V_2[w]$. If $V_2[w] = \emptyset$, then B is isomorphic to P_2. If $|V_2[w]| = 1$, then B is isomorphic to P_3. If $|V_2[w]| = 2$, then B is isomorphic to $K_{1,3}$. $\square$

In light of this observation, for any central vertex v of an ID-lobster G, we refer to the possible branch types at v as *P_2-type, P_3-type, and $K_{1,3}$-type.* Moreover, we denote by $s_1(v), s_2(v)$, and $s_3(v)$ the number of branches at v that are of P_2-type, P_3-type, and $K_{1,3}$-type, respectively.

The following observation implies further restrictions on the branches that may be present in ID-lobsters. Once again, this observation is also part of Theorem 2 in [3] but we also include a proof for completeness. (See also Observation 2.1 in [7].)

Lemma 1 ([3]). *Let G be an ID-lobster. Then for any central vertex v of G, we have $s_1(v) \leq 2$, $s_3(v) \leq 2$, and $s_1(v) + s_2(v) + s_3(v) \leq 4$.*

Proof. By Proposition 1, G must be triplet-free; thus, $s_1(v) \leq 2$. The inequality $s_3(v) \leq 2$ follows immediately from Claim 2A below.

Claim 2A. Suppose G has two $K_{1,3}$-type branches B_1, B_2 at v. For $i \in \{1, 2\}$, suppose $V(B_i) = \{v, w_i, y_{i,1}, y_{i,2}\}$ and $E(B_i) = \{w_i v, w_i y_{i,1}, w_i y_{i,2}\}$. Then for any ID-coloring c of G, we must have **(a)** $c(y_{1,1}) \neq c(y_{1,2})$, **(b)** $c(y_{2,1}) \neq c(y_{2,2})$, and **(c)** $c(w_1) \neq c(w_2)$.

Proof of Claim 2A. Refer to Fig. 2(a). Let c be an ID-coloring of G. Since $y_{1,1}$ and $y_{1,2}$ (resp. $y_{2,1}$ and $y_{2,2}$ are twins, (a) and (b) follow immediately from Proposition 1. Now, note that for all $x \in V(G) \setminus \{w_1, w_2, y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}\}$, we have $d(w_1, x) = d(w_2, x)$. This implies that $\vec{d}(w_1)[k] = \vec{d}(w_2)[k]$ for $k \geq 4$. By Claim 2A(a) (resp. Claim 2A(b)), exactly one of $y_{1,1}, y_{1,2}$ (resp. $y_{2,1}, y_{2,2}$) is red. This implies that $\vec{d}(w_1)[1] = \vec{d}(w_2)[1]$ and $\vec{d}(w_1)[3] = \vec{d}(w_2)[3]$ as well. Thus, since c is an ID-coloring, we must have $\vec{d}(w_1)[2] \neq \vec{d}(w_2)[2]$, which only happens if $c(w_1) \neq c(w_2)$.

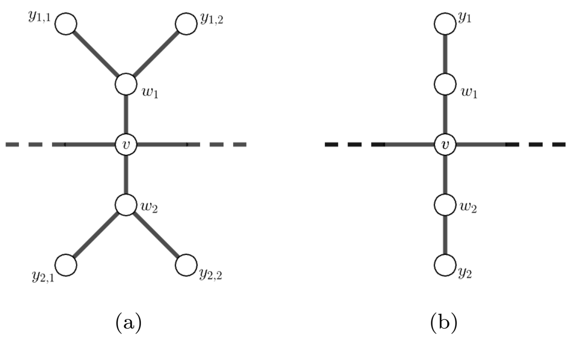


Figure 2. A central vertex v in a lobster graph having (a) two $K_{1,3}$-type branches at v; (b) two P_3-type branches at v.

We now focus on the P_3-type branches at v. We first prove the following claim.

Claim 2B. Suppose G has two P_3-type branches B_1, B_2 at v. For $i \in \{1, 2\}$, suppose $V(B_i) = \{v, w_i, y_i\}$ and $E(B_i) = \{vw_i, w_i y_i\}$. Then for any ID-coloring c of G, we must have $(c(w_1), c(y_1)) \neq (c(w_2), c(y_2))$.

Proof of Claim 2B. Refer to Fig. 2(b). Let c be an ID-coloring of G with $(c(w_1), c(y_1)) = (c(w_2), c(y_2))$. Since $d(w_1, x) = d(w_2, x)$ for all $x \in V(G) \smallsetminus \{w_1, w_2, y_1, y_2\}$, it follows that $\vec{d}(w_1) = \vec{d}(w_2)$, which contradicts the assumption that c is an ID-coloring.

Since any ID-coloring uses only two colors, Claim 2B implies that $s_2(v) \leq 4$. We are now ready to prove that $s_1(v) + s_2(v) + s_3(v) \leq 4$, which follows immediately from Claim 2B and the succeeding Claim 2C.

Claim 2C. Let c be an ID-coloring of G.

(1) Suppose G has a P_2-type branch B_1 and a P_3-type branch B_2 at v. Let $V(B_1) = \{v, w_1\}, E(B_1) = \{vw_1\}, V(B_2) = \{v, w_2, y_2\}$, and $E(B_2) = \{vw_2, w_2y_2\}$. Then:

   (a) $c(w_1) = \text{white}$ implies $(c(w_2), c(y_2)) \neq (\text{white}, \text{white})$;

   (b) $c(w_1) = \text{red}$ implies $(c(w_2), c(y_2)) \neq (\text{red}, \text{white})$.

(2) Suppose G has a $K_{1,3}$-type branch B_1 and a P_3-type branch B_2 at v. Let $V(B_1) = \{v, w_1, y_{1,1}, y_{1,2}\}, E(B_1) = \{vw_1, w_1y_{1,1}, w_1y_{1,2}\}$, $V(B_2) = \{v, w_2, y_2\}$, and $E(B_2) = \{vw_2, w_2y_2\}$. Then:

   (a) $c(w_1) = \text{white}$ implies $(c(w_2), c(y_2)) \neq (\text{white}, \text{red})$;

   (b) $c(w_1) = \text{red}$ implies $(c(w_2), c(y_2)) \neq (\text{red}, \text{red})$.

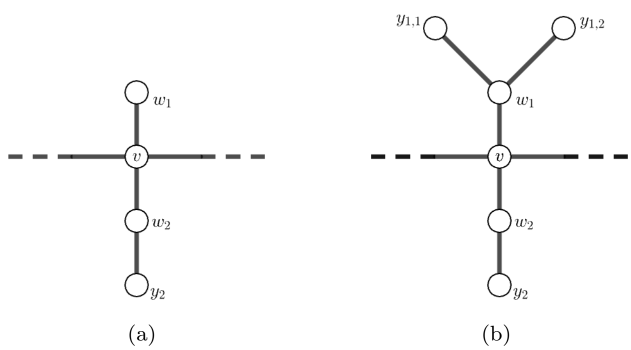


Figure 3. A central vertex v in a lobster graph having (a) one P_2-type branch and one P_3-type branch at v; (b) one $K_{1,3}$-type branch and one P_3-type branch at v.

Proof of Claim 2C. Refer to Fig. 3. For (1), note that $d(w_1, x) = d(w_2, x)$ for all $x \in V(G) \smallsetminus \{w_1, w_2, y_2\}$. On the other hand, for (2), we have $d(w_1, x) = d(w_2, x)$ for all $x \in V(G) \smallsetminus \{w_1, w_2, y_{1,1}, y_{1,2}, y_2\}$. The desired conclusions now follow using similar arguments as the ones used for Claims 2A and 2B. $\square$

Observation 2 and Lemma 1 provide necessary conditions for a lobster graph to be an ID-graph. We summarize these conditions in the following corollary.

Corollary 1 ([3]). *Let G be a lobster graph that has a central vertex v for which at least one of the following conditions holds:* (a) *there is a branch at v that is not isomorphic to P_2, P_3, or $K_{1,3}$; or* (b) $s_1(v) + s_2(v) + s_3(v) > 4$. *Then G is not an ID-graph.*

We now provide a lower bound for the identification number of ID-lobsters. The sharpness of this lower bound is established through Proposition 6 in the next section.

Lemma 2. *Let G be an ID-lobster and V_0 be the set of all central vertices of G. Then*

$$\mathrm{ID}(G) \geq \max\left\{1, \sum_{v \in V_0} r(v)\right\},$$

where for each $v \in V_0$,

$$r(v) = \begin{cases} 4, & \textit{if } s_1(v) + s_2(v) + s_3(v) = 4, \\ 3, & \textit{if } s_3(v) = 2 \textit{ and } s_1(v) + s_2(v) + s_3(v) \leq 3, \\ 2, & \textit{if } s_3(v) \leq 1 \textit{ and } s_1(v) + s_2(v) + s_3(v) = 3, \\ 1, & \textit{if } s_3(v) \leq 1 \textit{ and } s_1(v) + s_2(v) + s_3(v) = 2, \\ & \quad \textit{or } s_3(v) = 1 \textit{ and } s_1(v) + s_2(v) = 0, \\ 0, & \textit{if } s_3(v) = 0 \textit{ and } s_1(v) + s_2(v) \leq 1. \end{cases}$$

Proof. In general, the result is established by applying Proposition 1 and Claims 2A, 2B, and 2C from the proof of Lemma 1.

Let c be any ID-coloring of G and v be a central vertex of an ID-lobster G. For simplicity, we will denote $s_1(v), s_2(v), s_3(v)$ by s_1, s_2, s_3, respectively. It is sufficient to show that there are at least $r(v)$ red vertices, under the coloring c, in the branches at v.

Case 1. Suppose $s_1 + s_2 + s_3 = 4$. Then $(s_1, s_2, s_3) \in \{(0,2,2), (0,3,1), (0,4,0), (1,1,2), (1,2,1), (1,3,0), (2,0,2), (2,1,1), (2,2,0)\}$. If $(s_1, s_2, s_3) = (0,2,2)$, Proposition 1 and Claim 2A imply that, under the coloring c, the two $K_{1,3}$-type branches must have 3 red vertices. Claim 2C(2b) then implies that the two P_3-type branches must have only 1 red vertex. Thus, the branches at v must have at least 4 red vertices, as required. The proof for other values of (s_1, s_2, s_3) is similar.

Case 2. Suppose $s_3 = 2$ and $s_1 + s_2 + s_3 \leq 3$. Then $(s_1, s_2, s_3) \in \{(0,0,2), (0,1,2), (1,0,2)\}$. As in the previous case, the two $K_{1,3}$-type branches at v must have 3 red vertices under the coloring c. The third branch at v, if it exists, may have all of its vertices colored white. Thus, branches at v must have at least 3 red vertices, as required.

Case 3. Suppose $s_3 \leq 1$ and $s_1 + s_2 + s_3 = 3$. Then $(s_1, s_2, s_3) \in \{(0,2,1), (0,3,0), (1,1,1), (1,2,0), (2,0,1), (2,1,0)\}$. If $s_3 = 1$, then the one $K_{1,3}$-type branch at v must have at least 1 red vertex; moreover, the other two branches must have at least 1 red vertex as well. On the other hand, if $s_3 = 0$, then $(s_1, s_2) \in \{(0,3), (1,2), (2,1)\}$ for which it is easy to see that the branches at v must have at least 2 red vertices.

Case 4. Suppose $s_3 \leq 1$ and $s_1 + s_2 + s_3 = 2$; or $s_3 = 1$ and $s_1 + s_2 + s_3 = 1$. If $s_1 + s_2 + s_3 = 2$, then the two branches at v must have at least 1 red vertex under the coloring c. Similarly, if $s_3 = 1$, then the one $K_{1,3}$-type branch at v must have at least 1 red vertex.

Case 5. Suppose $s_3 \leq 0$ and $s_1 + s_2 + s_3 = 1$; or $s_1 + s_2 + s_3 = 0$. For this case, the desired result is trivial.

Thus, we have established the desired formula for $r(v)$ for any vertex $v \in V_0$. Moreover, we have shown that under an arbitrary ID-coloring c of G, each central vertex v must have in its branches at least $r(v)$ vertices that are colored red. Thus, the number of red vertices under c is at least $\sum_{v \in V_0} r(v)$. The desired lower bound for $\mathrm{ID}(G)$ follows immediately. $\square$

To illustrate an application of Lemma 2, we characterize and compute the identification number of all ID-lobsters with only one central vertex.

Proposition 3. *Let G be a lobster graph with exactly one central vertex v and whose order is at least 2. Then G is an ID-graph if and only if the following conditions hold:* (a) *any branch at v is isomorphic to P_2, P_3, or $K_{1,3}$,* (b) $1 \leq s_1(v) + s_2(v) + s_3(v) \leq 4$, *and* (c) $(s_1(v), s_2(v), s_3(v)) \neq (0, 0, 1)$. *Moreover, if* (a), (b), (c) *hold (i.e., G is an ID-graph), then*

$$\mathrm{ID}(G) = \begin{cases} 4, & \text{if } s_1(v) + s_2(v) + s_3(v) = 4, \\ 1, & \text{if } s_3(v) = 0 \text{ and } 1 \leq s_1(v) + s_2(v) \leq 2, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. For simplicity, we denote $s_1(v), s_2(v), s_3(v)$ by s_1, s_2, s_3, respectively. Let G be an ID-graph. Since G has order at least 2, we must have $s_1 + s_2 + s_3 \geq 2$. Then (a) and (b) hold by Corollary 1. Moreover, when $(s_1, s_2, s_3) = (0, 0, 1)$, then $G \equiv K_{1,3}$, which is not an ID-graph. Thus, (c) also holds.

We now prove the converse. Suppose that G satisfies (a), (b), (c).

First, we consider the case when $s_3 = 0$ and $1 \leq s_1 + s_2 \leq 2$. Equivalently, (s_1, s_2, s_3) must be $(0, 2, 0)$, $(2, 0, 0)$, $(1, 1, 0)$, $(1, 0, 0)$, or $(0, 1, 0)$. In any of these cases, G isomorphic to a path graph. By Theorem 1, G is an ID-graph and $\mathrm{ID}(G) = 1$.

Now, consider the case when $s_1 + s_2 + s_3 = 4$. Then it is sufficient to show that G has an ID-coloring. In Table 1, an ID-coloring of G (with 4 red vertices) is shown for each possible (s_1, s_2, s_3). Note that, in this case, we also have $r(v) = 4$; thus, by Lemma 2, it also follows that $\mathrm{ID}(G) = 4$.

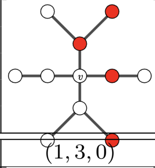
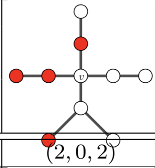
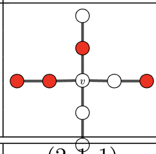
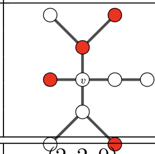
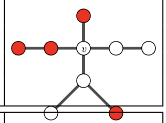
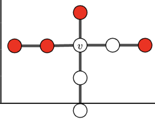
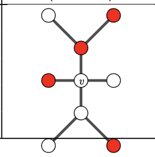
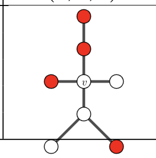
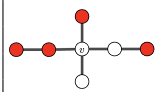
(s_1,s_2,s_3)	$(0,2,2)$	$(0,3,1)$	$(0,4,0)$	$(1,1,2)$	$(1,2,1)$
ID-coloring of G					
(s_1,s_2,s_3)	$(1,3,0)$	$(2,0,2)$	$(2,1,1)$	$(2,2,0)$	
ID-coloring of G					

Table 1. ID-colorings of a lobster graph G that has a single central vertex v, for different values of (s_1,s_2,s_3) satisfying $s_1+s_2+s_3=4$.

Finally, for the last case, we consider the remaining possible values of (s_1,s_2,s_3) subject to conditions (b) and (c). All these remaining values are shown in Table 2, from which it is evident that G is not isomorphic to a path graph. We also provide therein an ID-coloring of G for each possible value of (s_1,s_2,s_3); note that the ID-colorings shown have exactly 3 red vertices. Thus, G is an ID-graph. Since Theorem 1 and Proposition 2 imply that $\mathrm{ID}(G)\geq 3$, it also follows that $\mathrm{ID}(G)=3$ in this case.

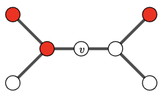
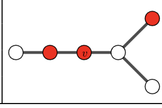
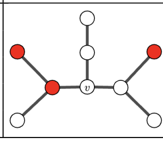
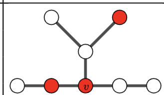
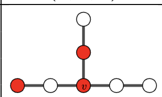
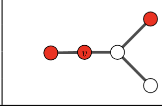
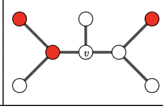
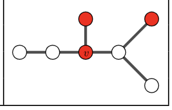
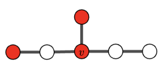
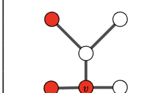
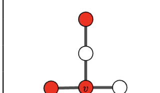
(s_1,s_2,s_3)	$(0,0,2)$	$(0,1,1)$	$(0,1,2)$	$(0,2,1)$
ID-coloring of G				
(s_1,s_2,s_3)	$(0,3,0)$	$(1,0,1)$	$(1,0,2)$	$(1,1,1)$
ID-coloring of G				
(s_1,s_2,s_3)	$(1,2,0)$	$(2,0,1)$	$(2,1,0)$	
ID-coloring of G				

Table 2. ID-colorings of a lobster graph G that has a single central vertex v, for different values of (s_1,s_2,s_3) satisfying $s_1+s_2+s_3\neq 4$ and ($s_3\neq 0$ or $s_1+s_2=3$).

$\square$

3. Uniform Lobster Graphs

Following the notations of Definition 1, recall that the vertex set of a lobster graph G is a disjoint union of three sets V_0, V_1, and V_2. The vertices in V_0 induce the central path of G while vertices in V_1 and V_2 are of distance 1 and distance 2, respectively, to the central path. Thus, vertices in V_1 and in V_2 are called distance-1 and distance-2 vertices, respectively. We now present the following definition.

Definition 2. Let n, a, b be positive integers such that $n \geq 2$. The *uniform lobster graph* $L(n, a, b)$ is the lobster graph having exactly n central vertices such that each central vertex is adjacent to exactly a distance-1 vertices while each distance-1 vertex is adjacent to exactly b distance-2 vertices.

In this section, we characterize and determine the identification number of all uniform lobster graphs that are ID-graphs. The following is an immediate consequence of Corollary 1.

Corollary 2. *Let n, a, b be positive integers such that $n \geq 2$. If $a > 4$, or $b > 2$, or $(a > 2$ and $b > 1)$, then the uniform lobster graph $L(n, a, b)$ is not an ID-graph.*

Thus, we are left to consider the following values of (a, b): $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 1)$, and $(4, 1)$, each of which is considered in the following propositions.

Proposition 4. *The uniform lobster graph $L(n, 1, 1)$ is an ID-graph for any integer $n \geq 2$. Moreover,*

$$\mathrm{ID}(L(n,1,1)) = \begin{cases} 1, & \text{if } n = 2, \\ 3, & \text{if } n \geq 3. \end{cases}$$

Proof. The case $n = 2$ is trivial. For $n \geq 3$, it is clear that $L(n, 1, 1)$ is not a path; thus, its identification number is at least 3. We now provide an identification coloring of $L(n, 1, 1)$ that uses 3 red vertices. For $n = 3$, refer to Fig. 4(a).

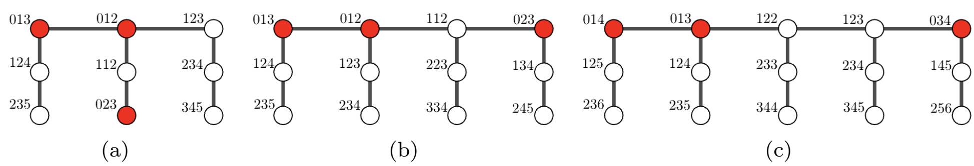


Figure 4. ID-colorings for (a) $L(3, 1, 1)$, (b) $L(4, 1, 1)$, (c) $L(5, 1, 1)$; shown beside each vertex is its multiset code.

We now assume that $n \geq 4$. Let $G = L(n, 1, 1)$; we will use the vertex notations as shown in Fig. 5.

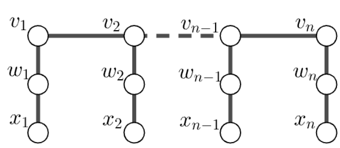


Figure 5. Notations for the vertices of the uniform lobster graph $L(n, 1, 1)$.

Let c be a red-white coloring of G in which the vertices v_1, v_2, v_n are colored red and the other vertices are colored white. We show that c is an ID-coloring of G. When $n = 4$ or 5, this can be verified from Fig. 4 (b) & (c), respectively.

We now consider the general case $n \geq 6$. Based on the definition of c, it can be easily verified that

$$\sigma(M(v_i)) = \begin{cases} n, & i = 1, \\ n + i - 3, & i \in \{2, 3, \ldots, n\}, \end{cases} \tag{3.1}$$

and that

$$\max(M(v_i)) = \begin{cases} n - i, & i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}, \\ i - 1, & i \in \{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \ldots, n\}. \end{cases} \tag{3.2}$$

Moreover, for each $i \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned} \sigma(M(w_i)) &= \sigma(M(v_i)) + 3, \quad \text{and} \quad \max M(w_i) = \max M(v_i) + 1, \\ \sigma(M(x_i)) &= \sigma(M(v_i)) + 6, \quad \text{and} \quad \max M(x_i) = \max M(v_i) + 2. \end{aligned}$$

Now, let u, y be two arbitrary distinct vertices of G. We show that $M(u) \neq M(y)$. For this, note that it is sufficient to show that $\sigma(M(u)) \neq \sigma(M(y))$ or $\max M(u) \neq \max M(y)$ or $\min M(u) \neq \min M(y)$.

Case 1. Suppose $u = v_i$ and $y = v_j$, where $1 \leq i < j \leq n$. If $i \neq 1$, (3.1) implies that $\sigma(M(v_i)) \neq \sigma(M(v_j))$. If $i = 1$ and $2 \leq j \leq n$, then $\sigma(M(v_1)) = n \neq n + j - 3 = \sigma(M(v_j))$ if and only if $j \neq 3$. We are left to consider the case where $(i, j) = (1, 3)$: since v_1 is red and v_3 is white, we have $M(v_1) \neq M(v_3)$.

Case 2. Suppose $u = w_i$ and $y = w_j$, where $1 \leq i < j \leq n$. Fix i, j arbitrarily. Note that adding 1 to each element of $M(v_i)$ (resp. $M(v_j)$) yields $M(w_i)$ (resp. $M(w_j)$). By Case 1, $M(v_i) \neq M(v_j)$, which implies that $M(w_i) \neq M(w_j)$ as well.

Case 3. Suppose $u = x_i$ and $y = x_j$, where $1 \leq i < j \leq n$. Note that adding 2 to each element of $M(v_i)$ (resp. $M(v_j)$) yields $M(x_i)$ (resp. $M(x_j)$). The proof proceeds similarly as in Case 2.

Case 4. Suppose $u = v_i$ and $y = w_j$, where $i, j \in \{1, 2, \ldots, n\}$. The cases where $i = 1$ or $j = 1$ can be verified easily; thus, we now assume that $i, j \in \{2, 3, \ldots, n\}$. Notice that $\sigma(M(v_i)) \neq \sigma(M(w_j))$ if and only if $n + i - 3 \neq n + j$ or $i - j \neq 3$. Hence, we are left to consider the case where $i - j = 3$. Given that both i, j must be in $\{2, 3, \ldots, n\}$, we must have $(i, j) \in \{(5, 2), (6, 3), \ldots, (n, n - 3)\}$.

(4.1) Suppose $5 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Then $2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 3$ and

$$\max M(v_i) = n - i \neq n - (i - 3) + 1 = \max M(w_j),$$

as desired.

(4.2) Suppose $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 3$. Then $\lfloor \frac{n}{2} \rfloor - 2 \leq j \leq \lfloor \frac{n}{2} \rfloor$. It follows that $\max M(v_i) = i - 1 \neq n - (i-3) + 1 = \max M(w_j)$ if and only if $i \neq \frac{n+5}{2}$. Thus, we are left to consider the case where n is odd and $i = \frac{n+5}{2}$; it follows that $j = i - 3 = \frac{n-1}{2}$. In this case, we have

$$\min M(v_i) = \min\left\{n - \frac{n+5}{2}, \frac{n+5}{2} - 2, \frac{n+5}{2} - 1\right\} = \frac{n-5}{2}$$

while

$$\min M(w_j) = \min\left\{n - \frac{n-1}{2} + 1, \frac{n-1}{2} - 2 + 1, \frac{n-1}{2} - 1 + 1\right\} = \frac{n-3}{2}.$$

Thus, $\min M(v_i) \neq \min M(w_j)$, as desired.

(4.3) Suppose $\lfloor \frac{n}{2} \rfloor + 4 \leq i \leq \lfloor n \rfloor$. Then $\lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n - 3$ and

$$\max M(v_i) = i - 1 \neq i - 3 = \max M(w_j),$$

as desired.

Case 5. Suppose $u = w_i$ and $y = x_j$, where $i, j \in \{1, 2, ..., n\}$. Fix i, j arbitrarily. Note that adding 1 to each element of $M(v_i)$ (resp. $M(w_j)$) yields $M(w_i)$ (resp. $M(x_j)$). Using Case 4, the proof proceeds similarly as in Case 2.

Case 6. Suppose $u = v_i$ and $y = x_j$, where $i, j \in \{1, 2, ..., n\}$. As the proof is simlar to that for Case 4, the details for this case are skipped. □

Proposition 5. *For any integer $n \geq 2$, the uniform lobster graph $L(n, 1, 2)$ is an ID-graph and* $\mathrm{ID}(L(n, 1, 2)) = n + 1$.

Proof. Let $n \geq 2$ and $G = L(n, 1, 2)$. Then at each central vertex v of G, there is exactly one branch and this branch is of $K_{1,3}$-type; thus, $s_3(v) = 1$ and $s_1(v) = s_2(v) = 0$. Using the notation in Lemma 2, we have $r(v) = 1$ for any central vertex v of G. Thus, by the same lemma, we have $\mathrm{ID}(G) \geq n$.

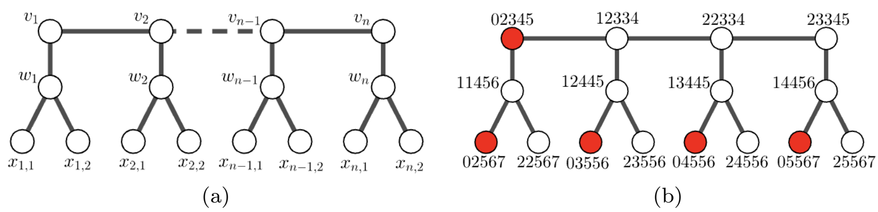


Figure 6. (a) Notations for the vertices of the uniform lobster graph $L(n, 1, 2)$; (b) An ID-coloring for the uniform lobster graph $L(4, 1, 2)$; shown beside each vertex is its multiset code.

We will proceed using the vertex notations as shown in Fig. 6(a). Suppose there is an ID-coloring c_0 of G in which only n vertices are red. Note that for each $i \in$

$\{1,2,\ldots,n\}$, the vertices $x_{i,1}$ and $x_{i,2}$ are twins; thus, by Proposition 1, we must have $c_0(x_{i,1}) \neq c_0(x_{i,2})$. We may then assume that the n red vertices under c_0 are $x_{1,1}, x_{2,1}, \ldots, x_{n,1}$. In this case, however, $M(v_1) = M(v_n)$, which is a contradiction. Thus, an ID-coloring of G in which only n vertices are colored red cannot exist; that is, $\mathrm{ID}(G) \geq n+1$.

Let c be a red-white coloring of G in which the vertices $v_1, x_{1,1}, x_{2,1}, \ldots, x_{n,1}$ are red while the others are white. An example for the case $n = 4$ is illustrated in Fig. 6(b). To complete the proof, we need to show that c is an ID-coloring of G. Let u, y be distinct vertices of G; in each of the following cases, we show that $M(u) \neq M(y)$.

Case 1. Suppose $u = v_i$ and $y = v_j$, where $1 \leq i < j \leq n$. Since v_1 is the only red central vertex, we may assume further that $2 \leq i < j \leq n$. Note that $\max M(v_i) \neq \max M(v_j)$ if and only if $i+j \neq n+1$. Thus, we are left to consider the case when $i = n+1-j$. In this case, symmetry implies that $M(v_i) \setminus \{d(v_i, v_1)\} = M(v_j) \setminus \{d(v_j, v_1)\}$. Moreover, since $d(v_i, v_1) < d(v_j, v_1)$, it follows that $M(v_i) \neq M(v_j)$ as well.

Case 2. Suppose $u = w_i$ and $y = w_j$, where $1 \leq i < j \leq n$. Similar to Case 1, note that $\max M(w_i) \neq \max M(w_j)$ if and only if $i+j \neq n+1$. The proof then proceeds similarly.

Case 3. Suppose $u = x_{i_1,i_2}$ and $y = x_{j_1,j_2}$, where $1 \leq i_1 \leq j_1 \leq n$ and $i_2, j_2 \in \{1,2\}$. The desired conclusion follows immediately if u and y are of different colors. Thus, we may assume that $i_1 < j_1$ and $i_2 = j_2$. We further assume that $i_2 = j_2 = 1$ as the proof for $i_2 = j_2 = 2$ is the same. Similar to the previous two cases, note that $\max M(x_{i_1,1}) \neq \max M(x_{j_1,1})$ if and only if $i_1 + j_1 \neq n+1$. The proof then proceeds similarly as in Case 1.

Case 4. Suppose $u = v_i$ and $y = w_j$, where $i, j \in \{1, 2, ..., n\}$. Note that $1 \in M(w_j)$ for any j. Fix an arbitrary j. If $i \neq 2$, then $1 \notin M(v_i)$, which implies that $M(v_i) \neq M(w_j)$. If $i = 2$, then $M(v_2) \neq M(w_j)$ since $M(v_2)$ has 2 copies of 3 while $M(w_j)$ has only at most one.

Case 5. Suppose $u = w_i$ and $y = x_{j_1,j_2}$, where $i, j_1 \in \{1, 2, ..., n\}$ and $j_2 \in \{1,2\}$. Since u is white, we only need to consider the case where y is white; that is, we may assume that $j_2 = 2$. Consequently, $1 \notin M(y) = M(x_{j_1,2})$ for any j_1 while $1 \in M(u) = M(w_i)$ for any i. The desired conclusion follows.

Case 6. Suppose $u = v_i$ and $y = x_{j_1,j_2}$, where $i, j_1 \in \{1, 2, ..., n\}$ and $j_2 \in \{1,2\}$. If $j_1 \neq 2$, then $M(v_i) \neq M(x_{j_1,j_2})$ since $3 \in M(v_i)$ while $3 \notin M(x_{j_1,j_2})$. On the other hand, if $j_1 = 2$, then the desired conclusion also follows since $\max M(x_{j_1,j_2}) = n+2 > n+1 \geq \max M(v_i)$. $\square$

Proposition 6. *The uniform lobster graph $L(n,2,1)$ is an ID-graph for any integer $n \geq 2$; moreover,*

$$\mathrm{ID}(L(n,2,1)) = \begin{cases} 3, & \text{if } n = 2, \\ n, & \text{if } n \geq 3. \end{cases}$$

Proof. Let $n \geq 2$ and $G = L(n, 2, 1)$. Then at each central vertex v of G, there are exactly two branches, both of which are of P_3-type; thus, $s_2(v) = 2$ and $s_1(v) = s_3(v) = 0$. Using the notation in Lemma 2, we have $r(v) = 1$ for any central vertex v of G. Thus, by the same lemma, we have $\mathrm{ID}(G) \geq n$.

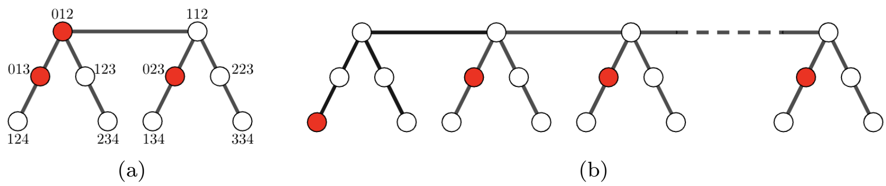


Figure 7. ID-colorings for (a) $L(2, 2, 1)$, (b) $L(n, 2, 1)$, where $n \geq 3$; in (a), shown beside each vertex is its multiset code.

When $n = 2$, Proposition 2 implies that $\mathrm{ID}(L(2, 2, 1)) \geq 3$. In Fig. 7(a), an ID-coloring of $L(2, 1, 1)$ with 3 red vertices is presented. Therefore, $\mathrm{ID}(L(2, 2, 1)) = 3$.

For the general case where $n \geq 3$, a red-white coloring of $L(n, 2, 1)$ with n red vertices is shown. Using a similar approach as in the proof of Proposition 5, it can be shown that this red-white coloring is an ID-coloring of $L(n, 2, 1)$. $\square$

Proposition 7 ([1]). *For any integer $n \geq 2$, the uniform lobster graph $L(n, 2, 2)$ is an ID-graph and* $\mathrm{ID}(L(n, 2, 2)) = 3n + 1$.

Proof. Let $n \geq 2$ and $G = L(n, 2, 2)$. Then at each central vertex v of G, there are exactly two branches, both of which are of $K_{1,3}$-type; thus, $s_3(v) = 2$ and $s_1(v) = s_2(v) = 0$. Using the notation in Lemma 2, we have $r(v) = 3$ for any central vertex v of G. Thus, by the same lemma, we have $\mathrm{ID}(G) \geq 3n$.

Let c_0 be any ID-coloring of G. For each central vertex v, Proposition 1 and Claim 2A in Lemma 1 imply that the two branches at v, excluding v itself, must have exactly 3 red vertices as shown in Fig. 8. Thus, there are exactly $3n$ red non-central vertices. If no central vertex is red, then the multiset codes of the endvertices of the central path will be equal; thus, there must be a $(3n + 1)$th red vertex. Hence, $\mathrm{ID}(L(n, 2, 2)) \geq 3n + 1$.

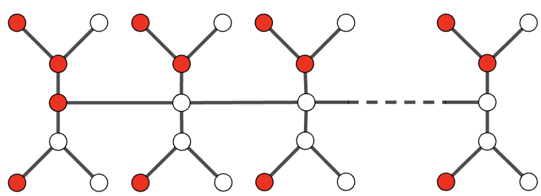


Figure 8. ID-coloring of $L(n, 2, 2)$, $n \geq 2$, with $3n + 1$ red vertices.

We are left to show that G has an ID-coloring with $3n + 1$ red vertices. For this, it can be shown, using a similar approach as in the proof for Proposition 6, that the red-white coloring of $L(n, 2, 2)$, $n \geq 2$, in Fig. 8 is an ID-coloring. The complete details of the proof can also be found in [1]. □

Proposition 8. *For any integer $n \geq 2$, the uniform lobster graph $L(n, 3, 1)$ is an ID-graph and $\mathrm{ID}(L(n, 3, 1)) = 2n + 1$.*

Proof. Let $n \geq 2$ and $G = L(n, 3, 1)$. Then at each central vertex v of G, there are exactly three branches, all of which are of P_3-type; thus, $s_2(v) = 3$ and $s_1(v) = s_3(v) = 0$. Using the notation in Lemma 2, we have $r(v) = 2$ for any central vertex v of G. Thus, by the same lemma, we have $\mathrm{ID}(G) \geq 2n$. However, similar to previous proofs, it can be verified that G cannot have an ID-coloring with exactly $2n$ red vertices; thus, $\mathrm{ID}(G) \geq 2n + 1$. Finally, it can be verified that the red-white coloring in Fig. 9 is an ID-coloring of G. This completes the proof.

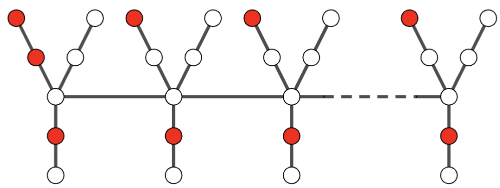


Figure 9. ID-coloring of $L(n, 3, 1)$, $n \geq 2$, with $2n + 1$ red vertices.

□

The remaining family of uniform lobsters consists of $L(n, 4, 1)$, where $n \geq 2$. For this family, each central vertex has four branches, all of which are of P_3-type. Before we present the result for this family, we consider the special case $n = 3$; i.e., $L(3, 4, 1)$.

Let v be a central vertex of $L(3, 4, 1)$. Suppose the four branches at v are vw_1x_1, vw_2, x_2, vw_3x_3, and vw_4x_4. For any ID-coloring c of $L(3, 4, 1)$, Claim 2B in Lemma 1 implies that the ordered pairs $(c(w_1), c(x_1))$, $(c(w_2), c(x_2))$, $(c(w_3), c(x_3))$, and $(c(w_4), c(x_4))$ must be pairwise distinct. Without loss of generality, we may assume that

$$\begin{array}{ll} (c(w_1), c(x_1)) = (\text{red}, \text{red}), & (c(w_2), c(x_2)) = (\text{white}, \text{white}) \\ (c(w_3), c(x_3)) = (\text{white}, \text{red}), & (c(w_4), c(x_4)) = (\text{red}, \text{white}). \end{array}$$

As the choice of v is arbitrary, the same coloring can be assumed for the branches at other central vertices. This means there are exactly 12 non-central red vertices in any ID-coloring of $L(3, 4, 1)$. Clearly, the multiset codes of the endvertices of the central path are going to be equal if (a) no central vertex is red, or (b) only the middle central vertex is red, or (c) all three central vertices are red, or (d) both end vertices of the central path are red. Up to symmetry, only two possibilities are left and these are shown in Fig. 10. And we see that both possibilities do not yield ID-colorings as well. Therefore, $L(3, 4, 1)$ does not have an ID-coloring; that is, it is not an ID-graph.

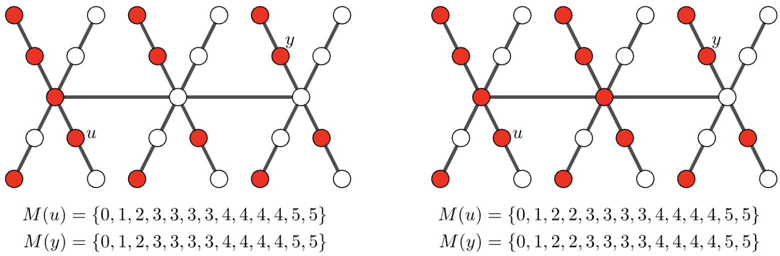


Figure 10. Red-white colorings of $L(3, 4, 1)$ that are not ID because $M(u) = M(y)$

Proposition 9. *Let $n \geq 2$ be an integer. The uniform lobster graph $L(n, 4, 1)$ is an ID-graph and $\mathrm{ID}(L(n, 4, 1)) = 4n + 1$ if and only if $n \neq 3$.*

Proof. The preceding discussion covers the case when $n = 3$. Let $n \geq 2$ with $n \neq 3$ and let $G = L(n, 4, 1)$. Then as previously mentioned, at each central vertex v of G, there are exactly four branches, all of which are of P_3-type; thus, $s_2(v) = 4$ and $s_1(v) = s_3(v) = 0$. Using the notation in Lemma 2, we have $r(v) = 4$ for any central vertex v of G. Thus, by the same lemma, we have $\mathrm{ID}(G) \geq 4n$. However, similar to previous proofs, it can be verified that G cannot have an ID-coloring with exactly $4n$ red vertices; thus, $\mathrm{ID}(G) \geq 4n + 1$. Finally, it can be verified that the red-white coloring in Fig. 11 is an ID-coloring of G. This completes the proof.

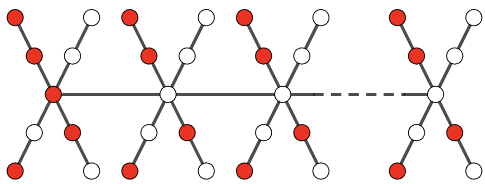


Figure 11. ID-coloring of $L(n, 4, 1)$, $n \geq 2$ and $n \neq 3$, with $4n + 1$ red vertices.

□

Combining Corollary 2 and Propositions 4-9, we obtain the following characterization of uniform ID-lobsters.

Theorem 3. *Let n, a, b be positive integers such that $n \geq 2$. The uniform lobster graph $L(n, a, b)$ is an ID-graph if and only if $(a, b) \in \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (4, 1)\}$ and $(n, a, b) \neq (3, 4, 1)$. Moreover, if $L(n, a, b)$ is an ID-lobster, then*

$$\mathrm{ID}(L(n, a, b)) = \begin{cases} 1, & \text{if } (n, a, b) = (2, 1, 1), \\ 3, & \text{if } (a, b) = (1, 1) \text{ and } n \geq 3; \text{ or } (n, a, b) = (2, 2, 1), \\ n, & \text{if } (a, b) = (2, 1) \text{ and } n \geq 3, \\ n + 1, & \text{if } (a, b) = (1, 2), \\ 2n + 1, & \text{if } (a, b) = (3, 1), \\ 3n + 1, & \text{if } (a, b) = (2, 2), \\ 4n + 1, & \text{if } (a, b) = (4, 1) \text{ and } n \neq 3. \end{cases}$$

4. Conclusion and Future Direction

The equivalent notions of ID-colorings and multiset resolving sets have been previously studied in relation to trees [3, 7], particularly caterpillars [3, 8] and lobsters [3]. These previous works focused on characterizing ID-caterpillars and ID-lobsters and on the identification numbers of some ID-caterpillars. In this paper, we extended the study on these topics by focusing on the identification numbers of ID-lobsters. In Section 2, we established a sharp lower bound for the identification number of all ID-lobsters and determined the identification number of any ID-lobster with only one central vertex. We then focused on a general family of lobsters called uniform lobsters. For positive integers n, a, b with $n \geq 2$, the uniform lobster $L(n, a, b)$ is the lobster graph that has exactly n central vertices, each of which is adjacent to exactly a distance-1 vertices while each distance-1 vertex is adjacent to exactly b distance-2 vertices. In Section 3, we characterized and determined the identification numbers of all uniform ID-lobsters. Particularly, we showed that the uniform lobster graph $L(n, a, b)$ is an ID-graph if and only if $(a, b) \in \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (4, 1)\}$ and $(n, a, b) \neq (3, 4, 1)$. Moreover, Propositions 4-9 provide the identification numbers of these uniform ID-lobsters. For future work, we propose the following interesting problems:

(a) To determine the identification numbers of all ID-lobsters; develop a general procedure for constructing an ID-coloring of any ID-lobster with the minimum possible number of red vertices;

(b) To study identification colorings of other graph families such as regular graphs or higher-dimensional grids or toroidal graphs.

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Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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