Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-19
https://doi.org/10.22049/cco.2025.29249.1909

CCO
Commun. Comb. Optim.

Research Article

Super spanning connectivity of the cartesian product of complete graphs

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Received: 6 December 2023; Accepted: 16 July 2025
Published Online: 20 July 2025

Abstract: Let G be a graph and s be an integer. *A s-container* $C(x, y)$ of G between two vertices x and y is a set of s internally vertex disjoint x, y-paths. A s-container $C(x, y)$ is *a s^*-container* if $V(C(x, y)) = V(G)$, where $V(C(x, y))$ is the set of vertices incident with some paths in $C(x, y)$. Then G is *s^*-connected* if there exists a s^*-container between any two distinct vertices of G. *The spanning connectivity* $\kappa^*(G)$ of G is the largest integer k such that G is s^*-connected for any s with $1 \leq s \leq k$. Further, G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of G. In this paper, we show that the n-th cartesian product of complete graph K_t $(t \geq 3)$ is super spanning connected. Our results, in some sense, extended a previous result in *[Shih et al., One-to-one disjoint path covers on k-ary n-cubes, Theoret. Comput. Sci. (2011)]*.

Keywords: cartesian product, complete graph, connectivity, spanning connectivity.

AMS Subject classification: 05C40

1. Introduction

In today's telecommunication networks, the construction of vertex disjoint paths between a pair of distinct vertices in a network has been an important subject [8, 16]. The vertex disjoint paths are used to speed up the transfer of a large amount of data by splitting the data over several vertex disjoint communication paths [7]. Additional benefits of adopting such a vertex disjoint routing scheme are the enhanced robustness to vertex failures and congestion, and the enhanced capability of load balancing [16].

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Table 1. Previous known results and our results on $\kappa^*(G)$

Graph G	Conditions	n	$\kappa^*(G)$	Authors
Pancake graph P_n	$n \neq 3$	$n \geq 1$	$n-1$	Lin et al. (2005) [13]
(n,k)-star graph $S_{n,k}$	$n-k \geq 2$	$n \geq 3$	$n-1$	Hsu et al. (2006) [9]
Burnt pancake graph B_n	$n \neq 2$	$n \geq 1$	n	Chin et al. (2009) [6]
Folded hypercube FQ_n	n is an even integer	$n \geq 2$	$n+1$	Chang et al. (2009) [3]
Enhanced hypercube $Q_{n,m}$	m is an even integer	$n \geq m \geq 2$	$n+1$	Chang et al. (2009) [3]
k-ary n-cube Q_n^k	$k \geq 3$ is an odd integer	$n \geq 2$	$2n$	Shih et al. (2011) [18]
Non-bipartite torus $T(k_1,k_2,\ldots,k_n)$	$k_i \geq 3$	$n \geq 2$	$2n$	Li et al. (2015) [11]
Alternating group graph AG_n		$n \geq 3$	$2n-4$	You et al. (2015) [24]
DCell with n-port switches $D_{k,n}$	$k \geq 0$ and $D_{k,n} \neq D_{1,2}$	$n \geq 2$	$n+k-1$	Wang et al. (2016) [20]
Arrangement graph $A_{n,k}$	$n-k \geq 2$	$n \geq 4$	$k(n-k)$	Li et al. (2017) [12]
WK-recursive network $K(n,t)$	$t \geq 1$	$n \geq 4$	$n-1$	You et al. (2018) [23]
Split-star network S_n^2		$n \geq 4$	$2n-3$	Li et al. (2021) [10]
Folded divide-and-swap cube $FDSC_n$	$d \geq 1$	$n = 2^d$	$d+2$	You et al. (2023) [25]
Cartesian product of complete graphs K_t^n	$t \geq 3$	$n \geq 1$	$n(t-1)$	**Current authors**

Table 2. Previous known results on $\kappa^{*l}(G)$

Graph G	Conditions	n	$\kappa^{*l}(G)$	Authors
Hypercube Q_n		$n \geq 1$	n	Chang et al. (2004) [2]
Star graph S_n	$n \neq 3$	$n \geq 1$	$n-1$	Lin et al. (2005) [13]
Bipartite hypercube-like graph B'_n		$n \geq 1$	n	Lin et al. (2007) [15]
Folded hypercube FQ_n	n is an odd integer	$n \geq 1$	$n+1$	Chang et al. (2009) [3]
Enhanced hypercube $Q_{n,m}$	m is an odd integer	$n \geq m \geq 2$	$n+1$	Chang et al. (2009) [3]
k-ary n-cube Q_n^k	$k \geq 4$ is an even integer	$n \geq 2$	$2n$	Shih et al. (2011) [18]
Hypercube Q_n		$n \geq 1$	n	Wang et al. (2019) [19]

Recent progress of the study of disjoint paths in a variety of networks can be found in the literature [10, 19, 25]. In this article, we further request that the set of vertex disjoint paths between any given pair of distinct vertices is a cover of the network. Studies about disjoint path covers of some networks or graphs can be found in the literature [4, 9, 14, 15, 17]. Below, following [13], we use terminology k^*-container instead of disjoint path cover.

A *k-container* $C(u,v)$ between two vertices u and v of a graph G is a set of k internal vertex disjoint paths joining u to v, i.e., $C(u,v) = \{P_1, P_2, \ldots, P_k\}$. Let $V(C(u,v))$ to denote the union of the vertices of these paths, i.e., $V(C(u,v)) = V(P_1) \cup V(P_2) \cup \cdots \cup V(P_k)$. A k-container $C(u,v)$ is a *k^*-container* if $V(C(u,v)) = V(G)$. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct vertices. *The spanning connectivity* $\kappa^*(G)$ of a graph G is the largest integer k such that for any integer m with $1 \leq m \leq k$ and for any $u, v \in V(G)$ with $u \neq v$, G has an m^*-container between u and v [5]. Further, G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of G. By definition, it is not difficult to see that the spanning connectivity of a graph is a common generalization of connectivity and hamiltonicity. We summarize some recent results on the spanning connectivity of well-known graphs and networks in Table 1.

A counter part of the spanning connectivity in bipartite graphs is spanning laceability. A bipartite graph T is *k^*-laceable* if there exists a k^*-container between any two

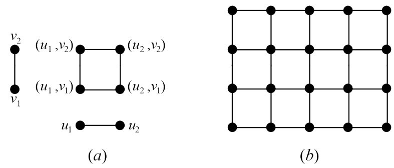

Figure 1. **(a) The cartesian product $K_2 \,\square\, K_2$, and (b) the (5×4)-grid**

vertices from different partite sets of T. *The spanning laceability* $\kappa^{*l}(T)$ of a bipartite graph T is the largest integer k such that T is i^*-laceable for any i with $1 \leq i \leq k$. Further, T is *super spanning laceable* if $\kappa^{*l}(T) = \kappa(T)$. We list some recent results of the spanning laeability of bipartite graphs in Table 2.

The cartesian product of simple graphs G and H is the graph $G \,\square\, H$ whose vertex set is $V(G)\times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. Thus, for each edge u_1u_2 of G and each edge v_1v_2 of H, there are four edges in $G \,\square\, H$, namely $(u_1, v_1)(u_2, v_1)$, $(u_1, v_2)(u_2, v_2)$, $(u_1, v_1)(u_1, v_2)$, and $(u_2, v_1)(u_2, v_2)$ (see Figure 1.(a)); the notation used for the cartesian product reflects this fact. More generally, the cartesian product $P_m \,\square\, P_n$ of two paths is the $(m \times n)$*-grid*. An example is shown in Figure 1.(b).

Many famous interconnection networks are constructed by cartesian product. The n*-dimensional hypercube* Q_n is defined as the cartesian product of n complete graphs, i.e., $Q_n = K_2 \,\square\, K_2 \,\square \cdots \square\, K_2$, and the 3*-ary* n*-cube* Q_n^3 is also defined as the cartesian product of n complete graphs, i.e., $Q_n^3 = K_3 \,\square\, K_3 \,\square \cdots \square\, K_3$. The super spanning laceability of the n-dimensional hypercube Q_n has been studied in literature [2]. To be specific, Q_n is super spanning laceable for any positive integer n. The spanning connectivity of Q_n^3 has been studied in [18], and has been proved that the spanning connectivity of Q_n^3 is $2n$. In this paper, we further study the spanning connectivity of the cartesian product of complete graphs K_t for $t \geq 3$.

The rest of this article is organized as follows. In Section 2, the basic structures of the cartesian product of complete graphs K_t ($t \geq 3$) will be introduced. In Section 3, the main result of the paper will be given. Finally, the conclusions of this paper will be given in Section 4.

2. Preliminaries

For the graph definition and notation we basically follow [1]. The *sets of vertices and edges* of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If u, v are vertices of a graph G such that there is an edge $e = uv \in E(G)$ between u and v, then we say that the vertices u and v are *adjacent* in G. A *path* P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, \ldots, v_k\rangle$, where each pair of consecutive vertices are connected by an edge. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_0\rangle$. We also write the path $P = \langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, v_1, \ldots, v_i\rangle\langle v_{i+1}, \ldots, v_k\rangle$ or $\langle v_0, v_1, \ldots, v_{i-1}, Q, v_{j+1}, \ldots, v_k\rangle$, where Q denotes the

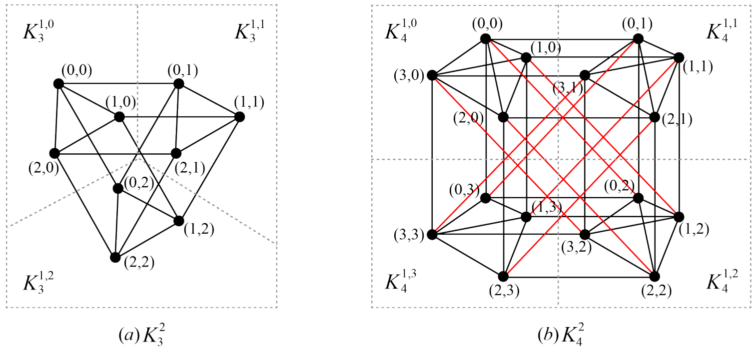

Figure 2. **Two graphs K_3^2 and K_4^2**

path $\langle v_i, v_{i+1}, \dots, v_j \rangle$. The *length* of a path P is the number of edges in P. We use $d_G(u, v)$ to denote the length of the shortest path between two vertices u and v in G. If there is no path connecting u and v, we set $d_G(u, v) := \infty$. A path is a *hamiltonian path* of a graph G if its vertices span the vertex set of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle.

For a faulty subset of vertics F, $G - F$ represents the subgraph of G derived from $V(G)-F$. Let k be an nonnegative integer. A graph G is *k-fault-tolerant hamiltonian* (abbreviated as *k-hamiltonian*) if $G - F$ is hamiltonian for every F with $|F| \le k$. A graph G is *k-fault-tolerant hamiltonian connected* (abbreviated as *k-hamiltonian connected*) if $G - F$ is hamiltonian connected for every F with $|F| \le k$.

For a graph G, its *line graph* $L(G)$ is a graph whose vertex set is edge set of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G.

The *n-th cartesian product of complete graph* K_t is denoted by K_t^n. A vertex $u \in V(K_t^n)$ is represented by $(u(0), u(1), \dots, u(n-1))$, where $0 \le u(i) \le t-1$. Then two vertices u and v in K_t^n are adjacent if and only if $|u(i) - v(i)| \ne 0$ for some i and $u(j) = v(j)$ for any $0 \le j \le n-1$ with $j \ne i$. Two graphs K_3^2 and K_4^2 are shown in Figure 2.

From the definition of K_t^n and the property of the cartesian product, we get that the connectivity of K_t^n is $n(t-1)$. It is shown that K_t^n is vertex-symmetric [21]. This means that given any two distinct vertices v and v' of K_t^n, there is an automorphism of K_t^n mapping v to v'. Note that each vertex of K_t^n is represented by a n-bit tuple. We will call the dth-bit *the dth dimension*. We can partition K_t^n over dimension d by fixing the dth element of any vertex tuple at some value a for every $a \in \{0, 1, 2, \dots, t-1\}$. This results in t copies of K_t^{n-1}, denoted by $K_t^{n-1,0}$, $K_t^{n-1,1}$, $K_t^{n-1,2}$, ... , $K_t^{n-1,t-1}$, with corresponding vertices in $K_t^{n-1,0}$, $K_t^{n-1,1}$, $K_t^{n-1,2}$, ... , $K_t^{n-1,t-1}$ joined in a

complete graph of order t (in dimension d).
In this article, we always partition K_t^n over the n-th dimension by letting $V(K_t^{n-1,j}) = \{(v(0), v(1), v(2), \ldots, j) \mid 0 \leq v(i) \leq t-1,\ 0 \leq i \leq n-2\}$ for $0 \leq j \leq t-1$. See Figure 2 for an illustration. Given a vertex $x = (x(0), x(1), \ldots, x(n-1)) \in V(K_t^n)$, the symbol $x^j = (x(0), x(1), x(2), \ldots, j)$, where $0 \leq j \leq t-1$, is defined to be the vertex corresponding to x in $K_t^{n-1,j}$ for simplicity. If $P = \langle x_1, x_2, \ldots, x_k\rangle$ is a path contained within $K_t^{n-1,i}$, then $P^j = \langle x_1^j, x_2^j, \ldots, x_k^j\rangle$ is a corresponding path contained within $K_t^{n-1,j}$.

Theorem 1. *[22] For $n \geq 2$, Q_n^3 is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected.*

3. The main result

In this section, we will derive our main result i.e., Theorem 2, using mathematical induction. We first prove the following three lemmas for later use.

Lemma 1. *K_t^n is 1^*-connected for $t \geq 3$.*

Proof. We shall prove the lemma by mathematical induction on n. It is worthy of noting that $\kappa^*(K_t) = t-1$. Thus, K_t is 1^*-connected, and so the lemma holds for $n = 1$. As the induction hypothesis, we assume that K_t^{n-1} is 1^*-connected for $n \geq 2$. Note that K_t^n is vertex-symmetric. Thus given two distinct vertices $u, v \in V(K_t^n)$, without loss of generality, we set $u = (0, 0, \ldots, 0) \in V(K_t^{n-1,0})$. We consider the following cases pertaining to the parity of t.
Case 1. t is odd.
Case 1.1. $v \in V(K_t^{n-1,0})$.
By induction hypothesis, $K_t^{n-1,j}$ is 1^*-connected for every $0 \leq j \leq t-1$. Let $xv \in E(K_t^{n-1,0})$ such that $x \neq u$. Then we construct hamiltonian paths $\langle u, L_0, x, v\rangle$, $\langle u^{t-1}, L_{t-1},$
$v^{t-1}\rangle$ and $\langle x^j, T^j, u^j\rangle$ in $K_t^{n-1,0}$, $K_t^{n-1,t-1}$ and $K_t^{n-1,j}$ for $1 \leq j \leq t-2$, respectively. By concatenating these paths, we construct a hamiltonian path

$$H = \langle u, L_0, x\rangle\langle x^1, T^1, u^1\rangle\langle u^2, (T^2)^{-1}, x^2\rangle \cdots \langle x^{t-2}, T^{t-2}, u^{t-2}\rangle\langle u^{t-1}, L_{t-1}, v^{t-1}\rangle\langle v\rangle$$

between u and v in K_t^n (Figure 3 (a)).
Case 1.2. $v \notin V(K_t^{n-1,0})$.
By the symmetry, we may suppose $v \in V(K_t^{n-1,1})$. Take $x \in V(K_t^{n-1,0})$ such that $x \neq u$. Further, choose $y \in V(K_t^{n-1,1})$ such that $y \neq v$ and $y^0 \neq x$. Using induction hypothesis, we construct hamiltonian paths $\langle u^j, T^j, x^j\rangle$ in $K_t^{n-1,j}$ for all $0 \leq j \leq t-2$ except for $j = 1$. By similar way, we also construct hamiltonian paths

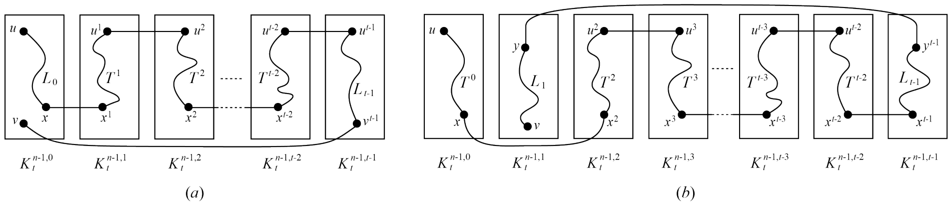

Figure 3. **An illustration for Case 1.1 and Case 1.2 of Lemma 1**

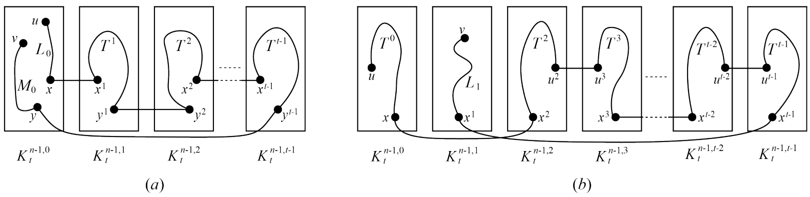

Figure 4. **An illustration for Case 2.1 and Case 2.2 of Lemma 1**

$\langle y, L_1, v\rangle$ and $\langle x^{t-1}, L_{t-1}, y^{t-1}\rangle$ in $K_t^{n-1,1}$ and $K_t^{n-1,t-1}$, respectively. Merging these paths, we construct a hamiltonian path

$$H = \langle u, T^0, x\rangle\langle x^2, (T^2)^{-1}, u^2\rangle\langle u^3, T^3, x^3\rangle \cdots \langle x^{t-3}, (T^{t-3})^{-1}, u^{t-3}\rangle\langle u^{t-2}, T^{t-2}, x^{t-2}\rangle \langle x^{t-1}, L_{t-1}, y^{t-1}\rangle\langle y, L_1, v\rangle$$

between u and v in K_t^n (Figure 3 (b)).

Case 2. t is even.

Case 2.1. $v \in V(K_t^{n-1,0})$.

By induction hypothesis, $K_t^{n-1,j}$ is 1^*-connected for every $0 \le j \le t-1$. Let $xy \in E(K_t^{n-1,0})$ such that $x \ne u, v$ and $y \ne u, v$. Then we have hamiltonian paths $\langle u, L_0, x, y,$
$M_0, v\rangle$ and $\langle x^j, T^j, y^j\rangle$ in $K_t^{n-1,0}$ and $K_t^{n-1,j}$ for $1 \le j \le t-1$, respectively. Combining these paths, we construct a hamiltonian path between u and v in K_t^n as following (Figure 4 (a)):

$$\langle u, L_0, x\rangle\langle x^1, T^1, y^1\rangle\langle y^2, (T^2)^{-1}, x^2\rangle \cdots \langle x^{t-1}, T^{t-1}, y^{t-1}\rangle\langle y, M_0, v\rangle.$$

Case 2.2. $v \notin V(K_t^{n-1,0})$.

Without loss of generality, we may suppose $v \in V(K_t^{n-1,1})$. Choose $x \in V(K_t^{n-1,0})$ such that $x \ne u$ and $x^1 \ne v$. By induction hypothesis, we have hamiltonian paths $\langle u, T^0, x\rangle$, $\langle x^1, L_1, v\rangle$ and $\langle u^j, T^j, x^j\rangle$ in $K_t^{n-1,0}, K_t^{n-1,1}$ and $K_t^{n-1,j}$ for $2 \le j \le t-1$, respectively. Then we construct a hamiltonian path between u and v in K_t^n as following (Figure 4 (b)):

$$\langle u, T^0, x\rangle\langle x^2, (T^2)^{-1}, u^2\rangle\langle u^3, T^3, x^3\rangle \cdots \langle x^{t-2}, (T^{t-2})^{-1}, u^{t-2}\rangle\langle u^{t-1}, T^{t-1}, x^{t-1}\rangle\langle x^1, L_1, v\rangle.$$

Combining Cases 1 and 2, we infer that K_t^n is 1^*-connected. □

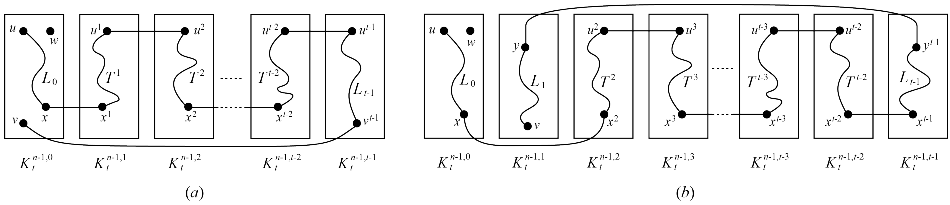

Figure 5. **An illustration for Case 1 and Case 2 of Lemma 2**

Lemma 2. *For any odd integer $t \geq 5$, K_t^n is 1-hamiltonian connected.*

Proof. It is easy to see that $\kappa^*(K_t - \{w\}) = \kappa^*(K_{t-1}) = t-2$ for any $w \in V(K_t)$, thus K_t is 1-hamiltonian connected. This means that the result holds for $n = 1$. As the induction hypothesis, we assume that K_t^{n-1} is 1-hamiltonian connected for $n \geq 2$ and $t \geq 5$. We need to prove that $K_t^n - \{w\}$ is 1^*-connected for any vertex $w \in V(K_t^n)$. Because K_t^n is vertex-symmetric, without loss of generality, we set $w = (0, 0, \ldots, 0) \in V(K_t^{n-1,0})$. Let $u, v \in V(K_t^n)$. According to the position of u and v, we have the following three situations.

Case 1. $u, v \in V(K_t^{n-1,0})$.

By induction hypothesis, we construct a hamiltonian path $\langle u, L_0, x, v\rangle$ of $K_t^{n-1,0} - \{w\}$. By Lemma 1, we construct hamiltonian paths $\langle u^{t-1}, L_{t-1}, v^{t-1}\rangle$ and $\langle u^j, T^j, x^j\rangle$ in $K_t^{n-1,t-1}$ and in $K_t^{n-1,j}$ for all $1 \leq j \leq t-2$, respectively. Using these paths, we construct a hamiltonian path in $K_t^n - \{w\}$ between u and v as following (Figure 5 (a)):

$$H = \langle u, L_0, x\rangle\langle x^1, (T^1)^{-1}, u^1\rangle\langle u^2, T^2, x^2\rangle \cdots \langle x^{t-2}, (T^{t-2})^{-1}, u^{t-2}\rangle\langle u^{t-1}, L_{t-1}, v^{t-1}\rangle\langle v\rangle.$$

Case 2. $u \in V(K_t^{n-1,0}), v \notin V(K_t^{n-1,0})$.

Without loss of generality, we may suppose $v \in V(K_t^{n-1,1})$. Choose $x \in V(K_t^{n-1,0} - \{w\})$ such that $x \neq u$, and choose $y \in K_t^{n-1,1}$ such that $y \neq v$ and $y^0 \neq x$. By the induction hypothesis, we construct a hamiltonian path $\langle u, L_0, x\rangle$ of $K_t^{n-1,0} - \{w\}$, and construct a hamiltonian path $\langle u^j, T^j, x^j\rangle$ of $K_t^{n-1,j}$ for all $2 \leq j \leq t-2$, respectively. Similarly, we also construct hamiltonian paths $\langle y, L_1, v\rangle$ and $\langle x^{t-1}, L_{t-1}, y^{t-1}\rangle$ of $K_t^{n-1,1}$ and $K_t^{n-1,t-1}$, respectively. Then we construct a hamiltonian path between u and v in $K_t^n - \{w\}$ as following (Figure 5 (b)):

$$\begin{aligned} H = &\langle u, L_0, x\rangle\langle x^2, (T^2)^{-1}, u^2\rangle\langle u^3, T^3, x^3\rangle \cdots \langle x^{t-3}, (T^{t-3})^{-1}, u^{t-3}\rangle\langle u^{t-2}, T^{t-2}, x^{t-2}\rangle \\ &\langle x^{t-1}, L_{t-1}, y^{t-1}\rangle\langle y, L_1, v\rangle. \end{aligned}$$

Case 3. $u, v \notin V(K_t^{n-1,0})$.

Based on the structure of K_t^n, we further consider following subcases.

Case 3.1 $u, v \in V(K_t^{n-1,1})$.

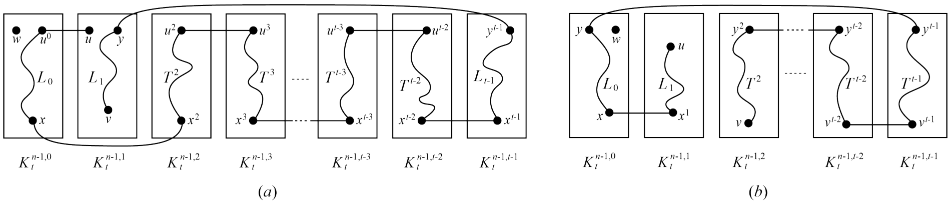

Figure 6. **An illustration for Case 3.1 and Case 3.2 of Lemma 2**

Without loss of generality, we may suppose $u^0 \neq w$. Choose $x \in V(K_t^{n-1,0} - \{w\})$ such that $x \neq u^0$, and choose $uy \in E(K_t^{n-1,1})$ such that $y \neq v$ and $y^0 \neq x$. By the induction hypothesis, construct a hamiltonian path $\langle u^0, L_0, x\rangle$ of $K_t^{n-1,0} - \{w\}$, and construct a hamiltonian path $\langle u^j, T^j, x^j\rangle$ of $K_t^{n-1,j}$ for all $2 \leq j \leq t-2$, respectively. Similarly, construct hamiltonian paths $\langle u, y, L_1, v\rangle$ and $\langle x^{t-1}, L_{t-1}, y^{t-1}\rangle$ of $K_t^{n-1,1}$ and $K_t^{n-1,t-1}$, respectively. Then we construct a hamiltonian path between u and v in $K_t^n - \{w\}$ as following (Figure 6 (a)):

$$\begin{aligned}H =&\langle u\rangle\langle u^0, L_0, x\rangle\langle x^2, (T^2)^{-1}, u^2\rangle\langle u^3, T^3, x^3\rangle \cdots \langle x^{t-3}, (T^{t-3})^{-1}, u^{t-3}\rangle\langle u^{t-2}, T^{t-2}, x^{t-2}\rangle \\ &\langle x^{t-1}, L_{t-1}, y^{t-1}\rangle\langle y, L_1, v\rangle.\end{aligned}$$

Case 3.2 $u \in V(K_t^{n-1,1}),\ v \in V(K_t^{n-1,2})$.
Choose two distinct vertices $x, y \in V(K_t^{n-1,0} - \{w\})$ such that $x \neq u^0$ and $y \neq v^0$. By the induction hypothesis, construct hamiltonian paths $\langle x, L_0, y\rangle$, $\langle u, L_1, x^1\rangle$ and $\langle v^j, T^j, y^j\rangle$ of $K_t^{n-1,0} - \{w\}$, $K_t^{n-1,1}$ and $K_t^{n-1,j}$ for all $2 \leq j \leq t-1$, respectively. As a result, we construct a hamiltonian path between u and v in $K_t^n - \{w\}$ as following (Figure 6 (b)):

$$H =\langle u, L_1, x^1\rangle\langle x, L_0, y\rangle\langle y^{t-1}, (T^{t-1})^{-1}, v^{t-1}\rangle\langle v^{t-2}, T^{t-2}, y^{t-2}\rangle \cdots \langle y^2, (T^2)^{-1}, v\rangle.$$

□

Lemma 3. $\kappa^*(K_t^2) = 2(t-1)$ *for* $t \geq 3$.

Proof. By Lemma 1, K_t^2 is 1^*-connected. In the following, we just need to prove that K_t^2 has a s^*-container between any two vertices in K_t^2 for all $2 \leq s \leq 2(t-1)$. Because K_t^2 is vertex-symmetric, without loss of generality, let $u = (0, 0)$. Apparently, $u \in V(K_t^{1,0})$. Depending on the parity of t, we divide into the following two cases.
Case 1. t is odd.
Case 1.1. $v \in V(K_t^{1,0})$.
Note that $K_t^{1,j}$ is isomorphic to K_t for all $0 \leq j \leq t-1$. Thus, $K_t^{1,j}$ has an m^*-container between any two vertices for all $1 \leq m \leq t-1$. Hence, there exist m vertex disjoint paths $\{P_i\}_{i=0}^{m-1}$ between u and v, whose union covers all vertices of $K_t^{1,0}$. Since $K_t^{1,j}$ is hamiltonian connected, we set hamiltonian paths $\langle u^j, T^j, v^j\rangle$ in $K_t^{1,j}$ for

all $1 \leq j \leq t-1$. Further, we reconstruct hamiltonian paths $\langle u^{t-2}, L_{t-2}, y^{t-2}\rangle$ and $\langle y^{t-1}, L_{t-1}, v^{t-1}\rangle$ respectively in $K_t^{1,t-2}$ and in $K_t^{1,t-1}$, where $y \in K_t^{1,0}$ and $y \neq u, v$. Let

$$\begin{aligned} P_m =&\langle u\rangle\langle u^1, T^1, v^1\rangle\langle v^2, (T^2)^{-1}, u^2\rangle \cdots \langle u^{t-4}, T^{t-4}, v^{t-4}\rangle\langle v^{t-3}, (T^{t-3})^{-1}, u^{t-3}\rangle \\ &\langle u^{t-2}, L_{t-2}, y^{t-2}\rangle\langle y^{t-1}, L_{t-1}, v^{t-1}\rangle\langle v\rangle. \end{aligned}$$

Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^m$ between u and v whose union covers all the vertices of K_t^2. This means K_t^2 has a s^*-container between u and v for all $2 \leq s \leq t$. We can construct remaining s^*-containers for $t \leq s \leq 2t-2$ as following way. Set

$$P_m = \langle u, u^1, T^1, v^1, v\rangle, P_{m+1} = \langle u, u^2, T^2, v^2, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v\rangle.$$

Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v, whose union covers all the vertices of K_t^2. As a result, K_t^2 has a s^*-container between u and v for all $t \leq s \leq 2t-2$.

Case 1.2. $v \notin V(K_t^{1,0})$.

Without loss of generality, we may assume $v \in V(K_t^{1,1})$. Let $u = u^0$ be in $K_t^{1,0}$ and $v = v^1$ in $K_t^{1,1}$. We have the following two subcases.

Case 1.2.1. $d_{K_t^2}(u, v) = 1$.

Then $u^1 = v$. Choose $w \in V(K_t^{1,0})$ such that $w \neq u$. Note again that $K_t^{1,j}$ is isomorphic to K_t for all $0 \leq j \leq t-1$, thus there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and w^j in $K_t^{1,j}$ whose union covers all the vertices of $K_t^{1,j}$ for all $1 \leq m \leq t-1$. For convenience, we express P_i^j as $P_i^j = \langle u^j, R_i^j, w^j\rangle$. Choose a neighbor y_i^j of u^j in $\{P_i^j\}_{i=0}^{m-1}$. Then let $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^3, \ldots, (R_i^{t-1})^{-1}, y_i^1, v\rangle$ for all $0 \leq i \leq m-2$, where R_i^j is nonempty for all $0 \leq i \leq m-2$. Further, let

$$P_{m-1} = \langle u, R_{m-1}^0, w\rangle\langle w^2, (R_{m-1}^2)^{-1}\rangle\langle R_{m-1}^3, w^3\rangle \cdots \langle w^{t-1}, (R_{m-1}^{t-1})^{-1}\rangle\langle y_{m-1}^1, Q', v\rangle,$$

where $\langle y_{m-1}^1, Q', v\rangle$ is a hamiltonian path between y_{m-1}^1 and v in $K_t^{1,1} - \{y_0^1, y_1^1, \ldots, y_{m-2}^1\}$. Besides, we construct $P_m = \langle u, u^2, u^3, \ldots, u^{t-2}, u^{t-1}, v\rangle$. Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^m$ between u and v whose union covers all the vertices of K_t^2. Thus K_t^2 has a s^*-container between u and v for all $2 \leq s \leq t$. For $t \leq s \leq 2t-2$, we find $t-1$ paths $P_m = \langle u, v\rangle, P_{m+1} = \langle u, u^2, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, v\rangle$. Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v, whose union covers all the vertices of K_t^2. This implies that K_t^2 has a s^*-container between u and v for all $t \leq s \leq 2t-2$.

Case 1.2.2. $d_{K_t^2}(u, v) \geq 2$.

It is the same as above, there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and v^j in $K_t^{1,j}$ whose union covers all the vertices of $K_t^{1,j}$ for all $1 \leq m \leq t-1$ and

$0 \le j \le t-1$. Let $P_i^j = \langle u^j, R_i^j, v^j \rangle$. Then, we find $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^1, v \rangle$, where R_i^j is nonempty for all $0 \le i \le m-2$. We also construct

$$\begin{aligned} P_{m-1} =& \langle u, R_{m-1}^0, v^0 \rangle \langle v^2, (R_{m-1}^2)^{-1}, u^2 \rangle \langle u^3, T^3, v^3 \rangle \langle v^4, (T^4)^{-1}, u^4 \rangle \cdots \langle u^{t-2}, T^{t-2}, v^{t-2} \rangle \\ & \langle v^{t-1}, (T^{t-1})^{-1}, u^{t-1} \rangle \langle u^1, R_{m-1}^1, v \rangle, \end{aligned}$$

where $\langle u^j, T^j, v^j \rangle$ is a hamiltonion path of $K_t^{1,j}$ for every $3 \le j \le t-1$. Hence, there exist m vertex disjoint paths $\{P_i\}_{i=0}^{m-1}$ between u and v, whose union covers all the vertices of K_t^2. Thus K_t^2 has a s^*-container between u and v for all $1 \le s \le t-1$.
The P_{m-1} is decomposed into t paths $P_{m-1} = \langle u, R_{m-1}^0, v^0, v \rangle, P_m = \langle u, u^1, R_{m-1}^1, v \rangle$,
$P_{m+1} = \langle u, u^2, R_{m-1}^2, v^2, v \rangle, P_{m+2} = \langle u, u^3, T^3, v^3, v \rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v \rangle$. Hence, there exist $m-1+t$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v, whose union covers all the vertices of K_t^2. Therefore, K_t^2 has a s^*-container between u and v for all $t \le s \le 2t-2$.

Case 2. t is even.
The case where t is even is slightly different from the case where t is odd.

Case 2.1. $v \in V(K_t^{1,0})$.
Obviously, there exist m vertex disjoint paths $\{P_i\}_{i=0}^{m-1}$ between u and v whose union covers all the vertices of $K_t^{1,0}$. Set $\langle u^j, T^j, v^j \rangle$ is a hamiltonion path of $K_t^{1,j}$ for every $1 \le j \le t-1$. Let

$$P_m = \langle u \rangle \langle u^1, T^1, v^1 \rangle \langle v^2, (T^2)^{-1}, u^2 \rangle \cdots \langle u^{t-1}, T^{t-1}, v^{t-1} \rangle \langle v \rangle.$$

Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^{m}$ between u and v whose union covers all the vertices of K_t^2. Then K_t^2 has a s^*-container for all $2 \le s \le t$. We can construct s^*-containers between u and v for $t \le s \le 2t-2$ as following way: we construct

$$P_m = \langle u, u^1, T^1, v^1, v \rangle, P_{m+1} = \langle u, u^2, T^2, v^2, v \rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v \rangle.$$

Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^2. Then K_t^2 has a s^*-container between u and v for all $t \le s \le 2t-2$.

Case 2.2. $v \notin V(K_t^{1,0})$.
Without loss of generality, we may suppose $v \in V(K_t^{1,1})$. We have the following two subcases.

Case 2.2.1. $d_{K_t^2}(u,v) = 1$.
Then $u^1 = v$. Choose $w \in V(K_t^{1,0})$. Note again that $K_t^{1,j}$ is isomorphic to K_t for all $0 \le j \le t-1$, thus there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and w^j in $K_t^{1,j}$, whose union covers all the vertices of $K_t^{1,j}$ for all $1 \le m \le t-1$. We express

P_i^j as $P_i^j = \langle u^j, R_i^j, w^j\rangle$. Let $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^3, \ldots, (R_i^{t-2})^{-1}, R_i^{t-1}, (R_i^1)^{-1}, v\rangle$ for all $0 \le i \le m-2$, where R_i^j is nonempty for all $0 \le i \le m-2$, and

$$P_{m-1} = \langle u, R_{m-1}^0, w^0\rangle\langle w^2, (R_{m-1}^2)^{-1}\rangle\langle R_{m-1}^3, w^3\rangle \cdots \langle w^{t-2}, (R_{m-1}^{t-2})^{-1}\rangle\langle R_{m-1}^{t-1}, w^{t-1}\rangle \langle w^1, (R_{m-1}^1)^{-1}, v\rangle.$$

Besides, we construct $P_m = \langle u, u^2, u^3, \ldots, u^{t-1}, v\rangle$. Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^m$ between u and v whose union covers all the vertices of K_t^2. Then K_t^2 has a s^*-container between u and v for all $2 \le s \le t$.
For $t \le s \le 2t-2$, we find $t-1$ paths $P_m = \langle u, v\rangle, P_{m+1} = \langle u, u^2, v\rangle, P_{m+2} = \langle u, u^3, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, v\rangle$. Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^2. This implies K_t^2 has a s^*-container between u and v for all $t \le s \le 2t-2$.
Case 2.2.2. $d_{K_t^2}(u, v) \ge 2$.
It is the same as above, there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and v^j in $K_t^{1,j}$ whose union covers all the vertices of $K_t^{1,j}$ for all $1 \le m \le t-1$ and $0 \le j \le t-1$. Let $P_i^j = \langle u^j, R_i^j, v^j\rangle$. Then, we find $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^1, v\rangle$ and $P_{m-1} = \langle u, R_{m-1}^0, v^0, v\rangle$, where R_i^j is nonempty for all $0 \le i \le m-2$. We also construct

$$P_m = \langle u, u^2, R_{m-1}^2, v^2\rangle\langle v^3, (T^3)^{-1}, u^3\rangle\langle u^4, T^4, v^4\rangle \cdots \langle v^{t-1}, (T^{t-1})^{-1}, u^{t-1}\rangle\langle u^1, R_{m-1}^1, v\rangle,$$

where $\langle u^j, T^j, v^j\rangle$ is a hamiltonion path of $K_t^{1,j}$ for all $3 \le j \le t-1$. Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^m$ between u and v, whose union covers all the vertices of K_t^2. Then K_t^2 has a s^*-container between u and v for all $2 \le s \le t$. Note that the P_m is divided into $t-1$ paths $P_m = \langle u, u^1, R_{m-1}^1, v\rangle, P_{m+1} = \langle u, u^2, R_{m-1}^2, v^2, v\rangle, P_{m+2} = \langle u, u^3, T^3, v^3, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v\rangle$. Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^2. Therefore, K_t^2 has a s^*-container between u and v for all $t \le s \le 2t-2$. □

Theorem 2. $\kappa^*(K_t^n) = n(t-1)$, $t \ge 3$.

Proof. We prove this theorem by inductino on n. Since the complete graph K_t is super spanning connected for $t \ge 3$, $\kappa^*(K_t) = t-1$. According to Lemma 3, we have $\kappa^*(K_t^2) = 2(t-1)$. These imply that the theorem holds for $n = 1, 2$. As the induction hypothesis, for $n \ge 3$, we assume that K_t^{n-1} has an m^*-container between any two vertices in K_t^{n-1} for all $1 \le m \le (n-1)(t-1)$. Let $u, v \in V(K_t^n)$ be two distinct vertices. Because K_t^n is vertex-symmetric, without loss of generality, we set $u = (0, 0, \ldots, 0) \in V(K_t^{n-1,0})$. By Lemma 1, K_t^n is 1^*-connected, so we just need to prove K_t^n is s^*-connected for all $2 \le s \le m+t-1$. Depending on the parity of t, we divide into the following two cases.
Case 1. t is odd.

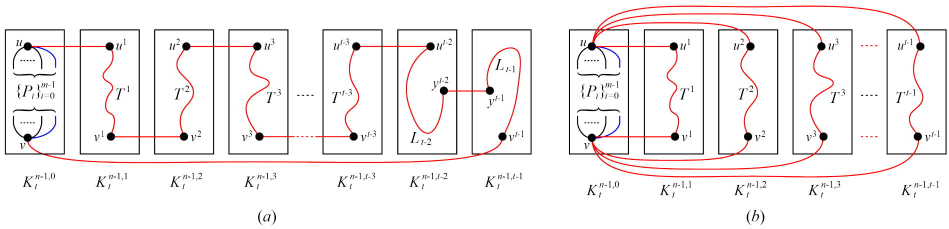

Figure 7. **An illustration for Case 1.1 of Theorem 2**

Case 1.1. $v \in V(K_t^{n-1,0})$.
Note that $K_t^{n-1,j}$ is isomorphic to K_t^{n-1} for all $0 \le j \le t-1$. Thus, $K_t^{n-1,j}$ is m^*-connected for all $1 \le m \le (n-1)(t-1)$. Hence, there exist m vertex disjoint paths $\{P_i\}_{i=0}^{m-1}$ between u and v, whose union covers all the vertices of $K_t^{n-1,0}$. Set hamiltonian paths $\langle u^j, T^j, v^j\rangle$ in $K_t^{n-1,j}$ for all $1 \le j \le t-1$. We respectively reconstruct hamiltonian paths $\langle u^{t-2}, L_{t-2}, y^{t-2}\rangle$ and $\langle y^{t-1}, L_{t-1}, v^{t-1}\rangle$ in $K_t^{n-1,t-2}$ and in $K_t^{n-1,t-1}$, where $y \in V(K_t^{n-1,0})$, $y \ne u, v$. Let

$$\begin{aligned}P_m =&\langle u\rangle\langle u^1, T^1, v^1\rangle\langle v^2, (T^2)^{-1}, u^2\rangle\langle u^3, T^3, v^3\rangle \cdots \langle v^{t-3}, (T^{t-3})^{-1}, u^{t-3}\rangle\\ &\langle u^{t-2}, L_{t-2}, y^{t-2}\rangle\langle y^{t-1}, L_{t-1}, v^{t-1}\rangle\langle v\rangle.\end{aligned}$$

Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^{m}$ between u and v, whose union covers all the vertices of K_t^n. Then K_t^n is s^*-connected for all $2 \le s \le (n-1)(t-1)+1$. In the following, we construct s^*-containers for $t \le s \le n(t-1)$ between u and v as following way. Let

$$P_m = \langle u, u^1, T^1, v^1, v\rangle, P_{m+1} = \langle u, u^2, T^2, v^2, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v\rangle.$$

Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^n. Then K_t^n has a s^*-container between u and v for all $t \le s \le n(t-1)$. Since $(n-1)(t-1)+1-t = (n-2)(t-1) > 0$, we have that K_t^n has a s^*-container between u and v for all $1 \le s \le n(t-1)$ (Figure 7).
Case 1.2. $v \notin V(K_t^{n-1,0})$.
Without loss of generality, we may assume that $v \in V(K_t^{n-1,1})$. Let $u = u^0$ be in $K_t^{n-1,0}$ and $v = v^1$ in $K_t^{n-1,1}$. We have the following two subcases.
Case 1.2.1. $d_{K_t^n}(u, v) = 1$.
Then $u^1 = v$. Choose $w \in V(K_t^{n-1,0})$ such that $w \ne u$ and $d_{K_t^n}(u, w) \ge 2$. Note again that $K_t^{n-1,j}$ is isomorphic to K_t^{n-1} for every $0 \le j \le t-1$, thus there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and w^j in $K_t^{n-1,j}$ whose union covers all the vertices of $K_t^{n-1,j}$ for all $1 \le m \le (n-1)(t-1)$. We express P_i^j as $P_i^j = \langle u^j, R_i^j, w^j\rangle$. Let $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^3, (R_i^4)^{-1}, \ldots, R_i^{t-2}, (R_i^1)^{-1}, v\rangle$ for all $0 \le i \le m-2$. Choose a neighbor y_i^j of w^j in P_i^j. By Theorem 1 and Lemma 2, we construct a hamiltonian

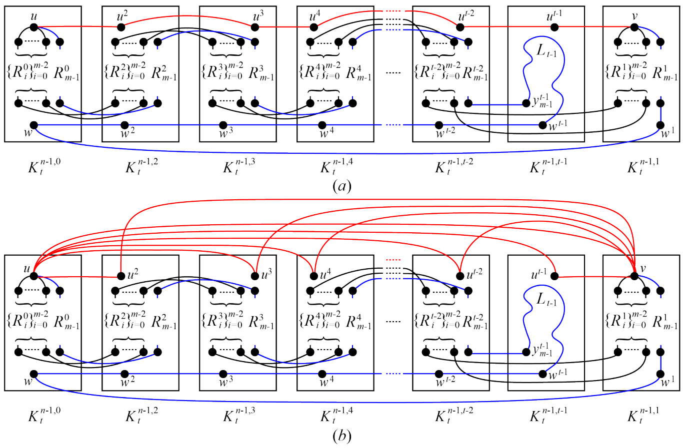

Figure 8. **An illustration for Case 1.2.1 of Theorem 2**

path $\langle y_{m-1}^{t-1}, L_{t-1}, w^{t-1}\rangle$ of $K_t^{n-1,t-1} - \{u^{t-1}\}$. And construct

$$\begin{aligned} P_{m-1} =& \langle u, R_{m-1}^0, (R_{m-1}^2)^{-1}, R_{m-1}^3, (R_{m-1}^4)^{-1}, \ldots, R_{m-1}^{t-2}\rangle\langle y_{m-1}^{t-1}, L_{t-1}, w^{t-1}\rangle \\ & \langle w^{t-2}, w^{t-3}, \ldots, w^2, w\rangle\langle w^1, (R_{m-1}^1)^{-1}, v\rangle. \end{aligned}$$

Besides, we construct $P_m = \langle u, u^2, u^3, \ldots, u^{t-2}, u^{t-1}, v\rangle$. Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^m$ between u and v whose union covers all the vertices of K_t^n. Therefore, K_t^n has a s^*-container between u and v for all $2 \le s \le (n-1)(t-1)+1$. For $t \le s \le n(t-1)$, we find $t-1$ paths $P_m = \langle u, v\rangle, P_{m+1} = \langle u, u^2, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, v\rangle$. Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^n. This implies that K_t^n has a s^*-container between u and v for every s with $t \le s \le n(t-1)$. Because $(n-1)(t-1)+1-t = (n-2)(t-1) > 0$, thus K_t^n has a s^*-container between u and v for every s with $1 \le s \le n(t-1)$ (Figure 8).

Case 1.2.2. $d_{K_t^n}(u, v) \ge 2$.

It is the same as above, there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and v^j in $K_t^{n-1,j}$ whose union covers all the vertices of $K_t^{n-1,j}$ for all $1 \le m \le (n-1)(t-1)$ and $0 \le j \le t-1$. Let $P_i^j = \langle u^j, R_i^j, v^j\rangle$. Then, we find $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^1, v\rangle$, where R_i^j is nonempty for every i with $0 \le i \le m-2$. We also construct a hamiltonion path

$$\begin{aligned} P_{m-1} =& \langle u, R_{m-1}^0, v^0\rangle\langle v^2, (R_{m-1}^2)^{-1}, u^2\rangle\langle u^3, T^3, v^3\rangle\langle v^4, (T^4)^{-1}, u^4\rangle \cdots \langle u^{t-2}, T^{t-2}, v^{t-2}\rangle \\ & \langle v^{t-1}, (T^{t-1})^{-1}, u^{t-1}\rangle\langle u^1, R_{m-1}^1, v\rangle, \end{aligned}$$

where $\langle u^j, T^j, v^j\rangle$ of $K_t^{n-1,j}$ for every j with $3 \le j \le t-1$. Hence, there exist m vertex disjoint paths $\{P_i\}_{i=0}^{m-1}$ between u and v, whose union covers all the vertices of K_t^n. Then K_t^n has a s^*-container for all $1 \le s \le (n-1)(t-1)$. The P_{m-1}

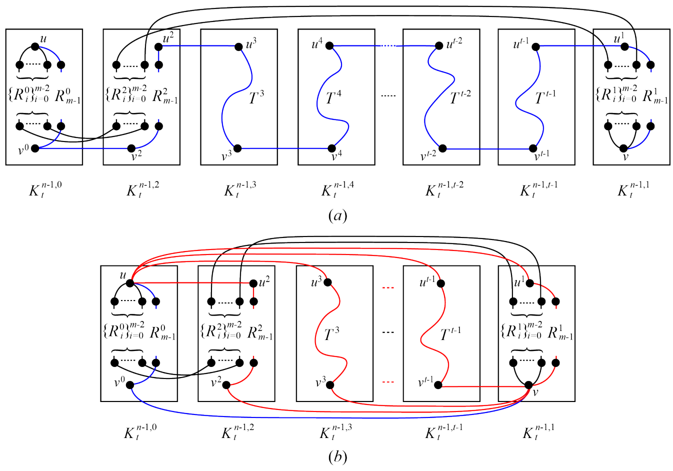

Figure 9. **An illustration for Case 1.2.2 of Theorem 2**

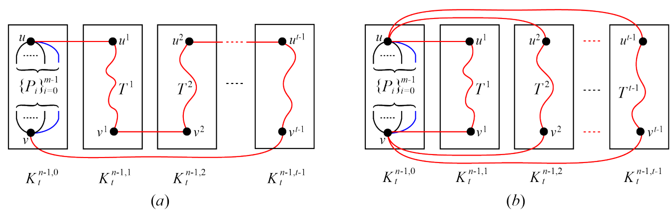

Figure 10. **An illustration for Case 2.1 of Theorem 2**

is divided into t paths $P_{m-1} = \langle u, R_{m-1}^0, v^0, v\rangle, P_m = \langle u, u^1, R_{m-1}^1, v\rangle, P_{m+1} = \langle u, u^2, R_{m-1}^2, v^2, v\rangle, P_{m+2} = \langle u, u^3, T^3, v^3, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v\rangle$. Hence, there exist $m-1+t$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^n. Then K_t^n has a s^*-container between u and v for every $t \le s \le n(t-1)$. Since $(n-1)(t-1)+1-t = (n-2)(t-1) > 0$, then K_t^n has a s^*-container between u and v for every $1 \le s \le n(t-1)$ (Figure 9).

Case 2. t is even.

The case where t is even is slightly different from the case where t is odd.

Case 2.1. $v \in V(K_t^{n-1,0})$.

Obviously, there exist m vertex disjoint paths $\{P_i\}_{i=0}^{m-1}$ between u and v whose union covers all the vertices of $K_t^{n-1,0}$. Set a hamiltonion path $\langle u^j, T^j, v^j\rangle$ of $K_t^{n-1,j}$ for evey $1 \le j \le t-1$. Let

$$P_m = \langle u\rangle\langle u^1, T^1, v^1\rangle\langle v^2, (T^2)^{-1}, u^2\rangle \cdots \langle u^{t-1}, T^{t-1}, v^{t-1}\rangle\langle v\rangle.$$

Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^{m}$ between u and v whose union covers all the vertices of K_t^n. Then K_t^n has a s^*-container between u and v for every

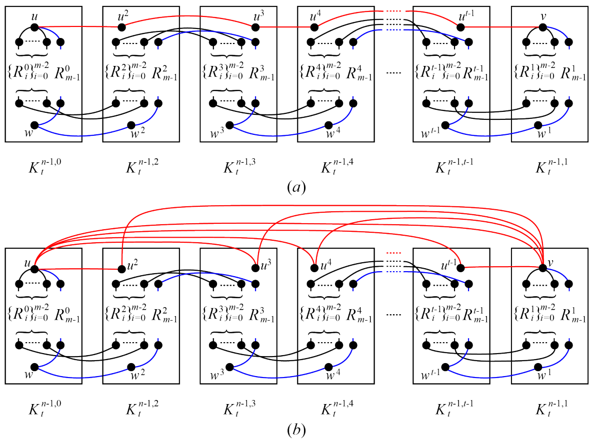

Figure 11. **An illustration for Case 2.2.1 of Theorem 2**

$2 \leq s \leq (n-1)(t-1)+1$. We can construct s^*-containers between u and v for $t \leq s \leq n(t-1)$ as following way. Let

$$P_m = \langle u, u^1, T^1, v^1, v\rangle, P_{m+1} = \langle u, u^2, T^2, v^2, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v\rangle.$$

Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^n. Then K_t^n has a s^*-container between u and v for every $t \leq s \leq n(t-1)$. Because $(n-1)(t-1)+1-t = (n-2)(t-1) > 0$, then K_t^n has a s^*-container between u and v for every $1 \leq s \leq n(t-1)$ (Figure 10).

Case 2.2. $v \notin V(K_t^{n-1,0})$.

Without loss of generality, we may suppose $v \in V(K_t^{n-1,1})$. We have the following two subcases.

Case 2.2.1. $d_{K_t^n}(u,v) = 1$.

Then $u^1 = v$. Choose $w \in V(K_t^{n-1,0})$ such that $d_{K_t^n}(u,w) \geq 2$. Note again that $K_t^{n-1,j}$ is isomorphic to K_t^{n-1} for every $0 \leq j \leq t-1$, thus there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and w^j in $K_t^{n-1,j}$ whose union covers all the vertices of $K_t^{n-1,j}$ for all $1 \leq m \leq (n-1)(t-1)$. We express P_i^j as $P_i^j = \langle u^j, R_i^j, w^j\rangle$. Let $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^3, (R_i^4)^{-1}, \ldots, R_i^{t-1}, (R_i^1)^{-1}, v\rangle$ for all $0 \leq i \leq m-2$,

$$\begin{aligned}P_{m-1} =&\langle u, R_{m-1}^0, w\rangle\langle w^2, (R_{m-1}^2)^{-1}\rangle\langle R_{m-1}^3, w^3\rangle\langle w^4, (R_{m-1}^4)^{-1}\rangle\cdots\\ &\langle R_{m-1}^{t-1}, w^{t-1}\rangle\langle w^1, (R_{m-1}^1)^{-1}, v\rangle.\end{aligned}$$

Besides, we construct $P_m = \langle u, u^2, u^3, \ldots, u^{t-1}, v\rangle$. Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^m$ between u and v whose union covers all the vertices of K_t^n. Thus K_t^n has a s^*-container between u and v for every $2 \leq s \leq (n-1)(t-1)+1$. For $t \leq s \leq n(t-1)$, we find $t-1$ paths $P_m = \langle u, v\rangle, P_{m+1} = \langle u, u^2, v\rangle, P_{m+2} = \langle u, u^3, v\rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, v\rangle$. Hence, there exist $m+t-1$ vertex disjoint

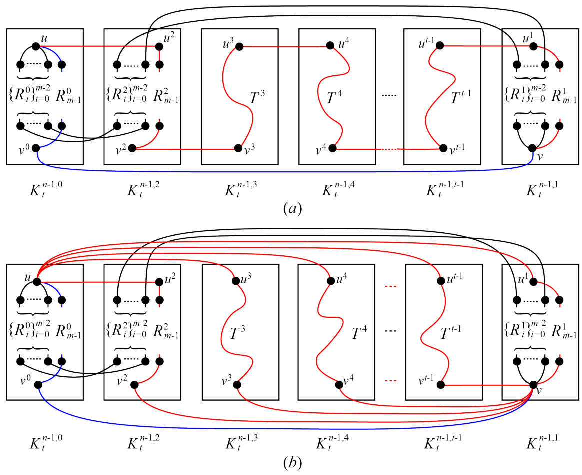

Figure 12. **An illustration for Case 2.2.2 of Theorem 2**

paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^n. This implies that K_t^n has a s^*-container between u and v for every $t \le s \le n(t-1)$. Because $(n-1)(t-1)+1-t = (n-2)(t-1) > 0$, so K_t^n has a s^*-container between u and v for every $1 \le s \le n(t-1)$ (Figure 11).

Case 2.2.2. $d_{K_t^n}(u,v) \ge 2$.

It is the same as above, there exist m vertex disjoint paths $\{P_i^j\}_{i=0}^{m-1}$ between u^j and v^j in $K_t^{n-1,j}$ whose union covers all the vertices of $K_t^{n-1,j}$ for all $1 \le m \le (n-1)(t-1)$ and $0 \le j \le t-1$. Let $P_i^j = \langle u^j, R_i^j, v^j \rangle$. Then, we find $P_i = \langle u, R_i^0, (R_i^2)^{-1}, R_i^1, v \rangle$ and $P_{m-1} = \langle u, R_{m-1}^0, v^0, v \rangle$ where R_i^j is nonempty for every $0 \le i \le m-2$. We also construct a hamiltonion path

$$P_m = \langle u, u^2, R_{m-1}^2, v^2 \rangle \langle v^3, (T^3)^{-1}, u^3 \rangle \langle u^4, T^4, v^4 \rangle \cdots \langle v^{t-1}, (T^{t-1})^{-1}, u^{t-1} \rangle \langle u^1, R_{m-1}^1, v \rangle,$$

where $\langle u^j, T^j, v^j \rangle$ of $K_t^{n-1,j}$ for every $3 \le j \le t-1$. Hence, there exist $m+1$ vertex disjoint paths $\{P_i\}_{i=0}^{m}$ between u and v whose union covers all the vertices of K_t^n. Then K_t^n has a s^*-container between u and v for every $2 \le s \le (n-1)(t-1)+1$. The P_m is divided into $t-1$ paths $P_m = \langle u, u^1, R_{m-1}^1, v \rangle, P_{m+1} = \langle u, u^2, R_{m-1}^2, v^2, v \rangle$, $P_{m+2} = \langle u, u^3, T^3, v^3, v \rangle, \ldots, P_{m+t-2} = \langle u, u^{t-1}, T^{t-1}, v^{t-1}, v \rangle$. Hence, there exist $m+t-1$ vertex disjoint paths $\{P_i\}_{i=0}^{m+t-2}$ between u and v whose union covers all the vertices of K_t^n. Then K_t^n has a s^*-container between u and v for every $t \le s \le n(t-1)$. Because $(n-1)(t-1)+1-t = (n-2)(t-1) > 0$, then K_t^n has a s^*-container between u and v for every $1 \le s \le n(t-1)$ (Figure 12). □

As mentioned in the first part, the 3-ary n-cube Q_n^3 is the n-th cartesian product of K_3. Thus, by Theorem 2, we have the following corollary:

Corollary 1. *([18])* $\kappa^*(Q_n^3) = 2n$.

Note that the line graph of a complete bipartite graph $K_{t,t}$ is isomorphic to the cartesian product of two complete graphs K_t's. Thus, using the main theorem of the paper we derive the following result:

Corollary 2. $\kappa^*(L(K_{t,t})) = 2(t-1)$ *for* $t \geq 3$.

4. Concluding remarks

In this paper, we prove that the spanning connectivity of the n-th cartesian product $K_t^n = K_t \square K_t \square \cdots \square K_t$ of the complete graph $K_t (t \geq 3)$ is the same as its connectivity, i.e., $\kappa^*(K_t^n) = \kappa(K_t^n) = n(t-1)$. Since the spanning connectivity of a graph G is not exceed the connectivity of G, the result is optimal. In the future, we further study the spanning connectivity of the graph $K_{t_1} \square K_{t_2} \square \cdots \square K_{t_n}$, where K_{t_i} is a complete graph with t_i vertices.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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