Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-24
https://doi.org/10.22049/cco.2025.30515.2510

CCO
Commun. Comb. Optim.

Research Article

Uncertainty in inverse data envelopment analysis: A novel approach for CO_2 emission efficiency

Jafar Pourmahmoud*, **Sima Aliabadi**†, **Reza Farzipoor Saen**‡, **Alireza Ghaffari-Hadigheh**§

Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
*pourmahmoud@azaruniv.ac.ir
†simaaliabadi20@gmail.com
‡farzipour@yahoo.com
§hadigheha@azaruniv.ac.ir

Received: 14 April 2025; Accepted: 24 August 2025
Published Online: 2 October 2025

Abstract: Industries are increasingly relying on analytical approaches for performance evaluation and decision-making. Consequently, they must invest suitable resources at the right time for the appropriate engagements. Inverse Data Envelopment Analysis is a post-DEA sensitivity analysis method designed to tackle resource allocation. The primary objective of Inverse DEA is to determine the optimal input and/or output levels for each decision-making unit under varying conditions to achieve a specified efficiency target. Traditional inverse DEA models require precise data on the inputs and outputs of Decision-Making Units. However, in many scenarios, such as system flexibility, social and cultural contexts information may be indeterminate. In these cases, experts' opinions are used to model uncertainty. Uncertainty theory, a branch of mathematics, logically deals with degrees of belief. This paper aims to develop an InvDEA model incorporating uncertainty theory. We assume that inputs and outputs of decision-making units are based on experts' belief degrees. An input-oriented model is developed, and several properties are proven. To demonstrate the model is performance, we employ a case study involving CO_2 emission data from OPEC countries.

Keywords: inverse data envelopment analysis, uncertainty, multi objective programming, efficiency.

AMS Subject classification: 90C08, 90C70

1. Introduction

1.1. DEA Foundations and Evolution

Data Envelopment Analysis (DEA) is a mathematical programming tool used to analyze the relative efficiency of Decision-Making Units (DMUs). DEA was initially

* *Corresponding Author*

developed by Charnes et al. [5]. Their model serves as the foundation for various DEA models based on constant returns to scale. Banker et al. [4] argued that returns to scale can vary under different circumstances. They extended the CCR model and introduced the BCC model, which accounts for Variable Returns to Scale (VRS). Numerous applications have demonstrated that DEA is a valuable tool for evaluating the efficiency of various systems [2, 10, 30].

1.2. Inverse DEA: DEA-Based Resource Optimization

Inverse Data Envelopment Analysis (InvDEA), first introduced by Wei et al. [33] and further developed by Yan et al. [35], addresses a key research question in project assessment systems: identifying the optimal input and output levels for each DMU to achieve a specific efficiency target amidst input or output perturbations. Inspired by inverse optimization, InvDEA enables traditional DEA to tackle a variety of inverse challenges. Recently, the popularity of InvDEA approaches has grown significantly. Thereupon, several linear and nonlinear InvDEA models have been proposed to meet research needs. For instance, Hadi-Vencheh et al. [15] proposed a method to handle multi-objective InvDEA problems. Letworasirikul et al. [20] proposed the InvDEA model under the assumption of variable returns to scale, known as the Inverse BCC model.Ghiyasi et al. [12] identified some shortcomings in the Lertworasikirul model and revisited the inverse variable returns to scale DEA model based on production technology characteristics, providing a simpler proof.
The InvDEA method has attracted significant attention due to its broad applications across various sectors, including business, supply chain management, agriculture, and more. A documented use of InvDEA is in resource allocation, sensitivity analysis, and risk assessment related to input and output changes within sustainable supply chains. For instance, Moghaddas et al. [26] presented a network InvDEA model to evaluate supply chain sustainability, illustrating the benefits of applying InvDEA models in supply chain management and sustainability contexts. Younesi et al. [37] introduced an slack-based InvDEA model for interval data, offering decision-makers additional tools to analyze potential mergers and acquisitions by extending InvDEA applications to different data types. Amin and Ibn-Boamah [3] developed a strategic business partnership framework within an InvDEA context to help decision-makers enhance competitiveness through strategic alliances and partnerships.
The application of the InvDEA approach to address CO_2 emissions has garnered considerable interest in DEA literature. Emrouznejad et al. [11] proposed a novel method based on InvDEA and conducted a three-stage analysis to meet CO_2 emission targets. Wegener and Amin [32] suggested a new InvDEA technique for optimizing greenhouse gas emissions. Zhang et al. [40] used an advanced InvDEA model to examine CO_2 emission distribution in the Chinese construction industry. Recently, Emrouznejad et al. [8] provided a thorough review of the origins, development, and future potential of InvDEA.

1.3. Critical Limitation, Existing Solutions, and Their Shortcomings

A critical limitation of traditional InvDEA is its reliance on deterministic data. In many instances, input and output quantities are not precisely known. Typically, nondeterministic data is treated as statistical, with randomness managed using probability theory. Researches in stochastic DEA are directed in three ways. One approach develops DEA models to handle observed deviations from the frontier as random variations. Another designs DEA models to manage random noise. A third perspective views the Production Possibility Set (PPS) as a random PPS [27]. Gholami et al. [13] explored incorporating stochastic data into InvDEA to address resource allocation and investment analysis challenges.

When data uncertainty stems from ambiguity rather than randomness, fuzzy logic offers an alternative. The use of fuzzy theory, introduced by Zadeh [39], has gained significant interest in DEA literature for addressing inherent ambiguity. Emrouznejad et al. [9] conducted an in-depth examination of employing fuzzy techniques in DEA. Although significant findings have emerged regarding Fuzzy DEA models, they are not without limitations. Some models simplify to linear optimization problems only if fuzzy numbers are assumed to be trapezoidal [16]. Additionally, some models may exhibit unbounded optimal values, and many Fuzzy DEA models are both computationally inefficient and costly [14]. This issue is prevalent across most Fuzzy DEA models [31]. Moreover, using fuzzy methods to address nondeterministic issues may lead to inconsistencies and contradictions in certain situations. In the sequel, we discuss this problem further.

Consider an industrial system with multiple branches across various cities. A DEA model can be utilized to evaluate the system's performance. One of the system's outputs might be CO_2 emission, with scores ranging from 60 to 100. If the expert considers the output as a fuzzy variable with the following membership function, the possibility of having "emission CO_2 level is being 80" is equal to one.

$$\mu(x) = \begin{cases} (x-60)/20 & 60 \leq x < 80 \\ (100-x)/20 & 80 \leq x \leq 100. \end{cases}$$

However, there is almost no confidence in the precise value of 80 for customer satisfaction level, and it is difficult for anyone to accept that the exact amount of 80 for customer satisfaction level is accurate. The main difference between fuzzy logic based on possibility theory [38] and uncertainty theory [25] is that the latter assumes two events are independent. But, in the former it is not important whether they are independent or not. when two events are not independent, the measure of their union is often greater than the maximum of the measures of individual events. According to this fact, the human brain does not behave in a fuzzy way. Consequently, it is not always appropriate to use the fuzzy theory for dealing with ambiguity in inputs and outputs since they cannot be measured by the possibility measure. Furthermore, probability theory would not be able to produce practical conclusions when experts' opinions are required.

Consider the expert is %90 sure that CO_2 emissions affects the environment. He is %90 sure that it affects social response, as well. Let us assume that these factors are independent. Based on probability theory the impact of both of them is %81. If there are 10 independent factors, the impact of them altogether is $(0.9)^{10} \approx 0.3$. Since we believe that the impact of one of them is %90, this result is nonsense. Based on uncertainty theory, the impact of these 10 factors is %90 which is an acceptable outcome. The uncertainty theory book [25] might be consulted for more discussion.

1.4. Uncertainty Theory as a Targeted Solution

Uncertainty theory is a mathematical framework that aims to model human opinions. This theory can be applied in four different scenarios:

1. It can be used to make predictions when there is no available sample or during emergencies like war, floods, earthquakes, or a pandemic. In such situations, historical data may not provide accurate information.

2. It can be employed to analyze the past in cases where specific measurements are inaccessible, such as carbon emissions or social benefits.

3. It can be used to model certain concepts, like "emission CO_2" or "amount of greenhouse gases" which are ambiguous in human language.

4. It can be utilized to model dynamic systems with continuous-time noise, such as stock prices.

Several researchers in the field of DEA have recently applied this theory. Lio and Liu [23] introduced an uncertain CCR model that incorporates uncertain variables for both inputs and outputs. They calculated the expected value of these uncertain variables and proposed a counterpart crisp model. Pourmahmoud and Bagheri [28] developed a basic two-stage model to account for uncertainty in the network structure. During the COVID-19 pandemic, probabilistic statistics might not function accurately due to the lack of comparable situations. Pourmahmoud and Bagheri [29] used an uncertain model to evaluate healthcare system performance during the COVID-19 outbreak.

1.5. Contributions: Bridging Methodology and Practice

Given the significance of InvDEA in the literature, this paper concentrates on efficiency analysis in scenarios where some data are based on belief degrees. The model is input-oriented and employs VRS technology. As an example, we examine the performance of OPEC countries considering CO_2 emissions as an uncertain variable. With the rapid growth of the global economy, many countries face challenges due to rising carbon dioxide emissions, leading to global climate change [17] .Unlike traditional DEA's deterministic models, this approach explicitly incorporates expert opinions, critical for OPEC nations where emission data lacks precision but climate commitments demand rigor. We focus on OPEC nations because of their dual role as energy security anchors and laggards in climate action. The group's struggle to balance oil revenues with emission reductions makes it an ideal testing ground for uncertain InvDEA's policy-prescriptive capabilities.

In light of the increasing emphasis on efficiency, scholars have shown interest in optimization methods. The DEA method is widely used in studying CO_2 emissions allocation due to its simplicity and practicality. Cui and Li [7] introduced a novel virtual frontier for evaluating transportation carbon efficiency. Kwon et al. [19] utilized a two-stage model to assess the technical efficiency of reducing CO_2 emissions. Yang et al. [36] applied the ZSG-DEA model to study carbon emissions allocation across Chinese provinces. Li et al. [21] employed an slack-based model with undesirable outputs to measure industrial total-factor CO_2 emission efficiency.

It is essential to note that CO_2 emissions allocation in the DEA model involves adjusting inputs and outputs. Using the InvDEA model for CO_2 emission allocation can enhance accuracy regarding efficiency. However, carbon emissions cannot be measured precisely. In such cases, the proposed model can be applicable. In summary, and to the best of our knowledge, this study makes the following contributions.

1. To assess the sustainability of environmental management system, we propose a novel InvDEA model with based on uncertainty theory. specifically, in this model CO_2 emissions are considered as an uncertain undesirable output that should be determined by experts' opinions, and incorporated into the evaluation of the environmental efficiency of OPEC countries.this model is not only methodologically innovative but also serves as a paratical tool for recommending emission reduction policies in oil-producing countries.

2. The OPEC member countries were selected as the case study for this research due to their significant contribution to global CO_2 emissions and their strong economic dependence on fossil fule production and exports. this dependency creates a critical challenge in balancing economic growth with environmental sustainability. Therefore, assessing their environmental efficiency particularly by considering undesirable outputs such as CO_2 emissions is of great relevance. Moreover, availability of reliable data and diversity of economic structures among OPEC countries provide a suitable context for evaluation and benchmarking within the proposed InvDEA framework.

3. The proposed uncertain InvDEA model precisely identifies the necessary adjustments in inputs and undesirable outputs such as CO_2 emissions to enhance environmental efficiency, providing a robust foundation for policymakers to formulate data-driven strategies aimed at emission reduction, optimal resource allocation, and the advancement of environmental sustainability. Additionally, this model enables OPEC policymakers to balance GDP growth with CO_2 targets by quantifying input adjustments, and its sensitivity to belief degrees helps prevent over- or under-investment in emission controls.

4. We have validated this approach using a case study.

This paper is structured as follows: Section 2 reviews fundamental concepts of InvDEA and uncertainty theory. Section 3 presents the uncertain InvDEA and some theorems are proved. Practical use of the model is illustrated in Section 4, followed by conclusions in Section 5 and a discussion on future research.

2. Preliminary concepts

Since sensitivity analysis is crucial in optimization problems, a variation of the DEA model known as the InvDEA model has been introduced. The InvDEA method is a kind of multi-objective programming with several objective functions, each defining a specific goal to attain. In the first subsection, basic concepts of the InvDEA are reviewed. Decisions in reality are based on non-determinant situations. Axiomatic mathematics incorporates the concept of uncertainty theory to model human belief. The second subsection delves into the fundamental principles of this theory. Uncertain theory has been applied in DEA literature by some researches. In the third subsection the uncertain BCC model is presented in a nutshell.

2.1. Inverse Data Envelopment Analysis

Assume DMU_j $(j = 1, 2, \ldots, n)$ consumes input vector $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ to produce desirable output vector $Y_j^D = (y_{1j}^D, y_{2j}^D, \ldots, y_{sj}^D)$ and undesirable output $Y_j^{ND} = (y_{1j}^{ND}, y_{2j}^{ND}, \ldots, y_{s'j}^{ND})$. For the evaluated DMU_k $k \in \{1, 2, \ldots, n\}$ the BCC model is as follows.

$$
\begin{aligned}
\min \quad & \theta_k \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_k x_{ik}, i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^D \geq y_{rk}^D, r = 1, 2, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j y_{r'j}^{ND} \leq y_{r'k}^{ND}, r' = 1, 2, \ldots, s', \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, j = 1, 2, \ldots, n
\end{aligned}
\tag{2.1}
$$

Definition 1. The DMU_k is (weak) efficient when the optimal value of the model 1 is equal to one. InvDEA models aim to address queries such as: if DMU_k $k \in \{1, 2, \ldots, n\}$ alters its output to $\beta_k = y_k + \Delta y_k$, how much input is needed to maintain DMU_k is relative efficiency? The DEA literature introduces the following Multiple Objectives Linear Programming (MOLP) model for calculating this required input.

$$
\begin{aligned}
\min \quad & (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk}) = (x_{1k} + \Delta x_1, x_{2k} + \Delta x_2, \ldots, x_{mk} + \Delta x_m) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_k^* \alpha_{ik}, i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{D} \geq \beta_{rk}, r = 1, 2, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j y_{r'j}^{ND} \leq y_{r'k}^{ND}, r' = 1, 2, \ldots, s', \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, j = 1, 2, \ldots, n
\end{aligned}
\tag{2.2}
$$

where θ_k^* represents the optimal value of the BCC model and $\alpha_i = x_{ik} + \Delta x_i$ ($i = 1, 2, \ldots, m$) represents the required inputs to guarantee the unchanged DMU_k in relative efficiency. Lertworasirikul et al. [20] utilized Model (2.1) to introduce an InvDEA model assuming VRS technology. Initially, they addressed the InvDEA Model as a non-linear program and then transitioned to the MOLP Model (2.1) due to the complexity of solving a non-linear problem. Their main theorem and its proof have significant flaws. Ghiyasi [12] adjusted the inverse BCC model of Lertworasirikul et al. [20] by expanding the normal PPS of T to T', which includes all current DMUs and a perturbed DMU with new input and output values. Consequently, they utilized the following model to assess the relative efficiency of the perturbed DMU_k.

$$
\begin{aligned}
\min \quad & \theta_{k'} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_{k'} \alpha_{ik} \quad i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{D} \geq \beta_{rk} \quad r = 1, 2, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j y_{r'j}^{ND} \leq y_{r'k}^{ND} \quad r' = 1, 2, \ldots, s', \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \ j = 1, 2, \ldots, n
\end{aligned}
\tag{2.3}
$$

Definition 2. Let $(\lambda, \alpha) = (\lambda_1, \lambda_2, \ldots, \lambda_n; \alpha_1, \alpha_2, \ldots, \alpha_m)$ represent a feasible solution for Model (2.3). If there is not any feasible solution such as $(\bar{\lambda}, \bar{\alpha})$ for this model where $\bar{\alpha}_i < \alpha_i$, $i = 1, 2, \ldots, m$, then (λ, α) is considered a weakly efficient solution for the model.

Theorem 1. *[12] If a perturbation is made to DMU_k from (x_k, y_k) to (α_k, β_k), and (λ_k, α_k) is a weakly efficient solution for Model (2.3), then the scores of efficiency for all DMUs remain the same after perturbation.*

2.2. Uncertainty theory

As mentioned before, expert opinion should provide information in some situations resulting in a potential indeterminacy. Several concepts have been proposed to explore this indeterminacy. Liu [24] proposed an uncertain measure with four axioms to manage belief degrees. An uncertainty space is defined as a triplet (Γ, L, M), where Γ is a nonempty set, L is a σ-algebra over Γ, and M is a set function. Each member of L is referred to as an uncertain event.

Definition 3. [24] An uncertain variable is a measurable function ξ that maps from an uncertainty space (Γ, L, M) to the real number set such that $\{\xi \in B\}$ is an uncertain event for any Borel set B of real numbers. Furthermore, the uncertainty distribution Φ of an uncertain variable ξ is defined as

$$\Phi(x) = M\{\xi \leq x\}, \quad \forall x \in \Re.$$

The uncertainty distribution is considered regular if it is a continuous and strictly increasing function satisfying $0 < \Phi(x) < 1$, $\lim_{x\to-\infty} \Phi(x) = 0$, and $\lim_{x\to\infty} \Phi(x) = 1$.

Inverse uncertainty distribution of ξ is the inverse function of Φ denoted by Φ^{-1}. Moreover, expected value of ξ is defined by $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$ when Φ is regular. Some uncertainty distributions are discussed in the literature. Linear uncertain variable distribution is one of them defined as

$$\Phi(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b, \end{cases}$$

where a and b are integers with $a < b$. Inverse uncertainty distribution of linear uncertain variable is $\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$.
Following theorem determines the inverse uncertainty distribution and expected value of a function involving independent uncertain variables. This theorem forms the basis for our main results.

Theorem 2. *[25] Suppose $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then*

1. *$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with the inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right).$$

2. *Expected value of ξ is*

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right) d\alpha.$$

2.3. Uncertain BCC model with undesirable outputs

Suppose DMU_j $(j = 1, 2, \ldots, n)$ consumes uncertain inputs $\tilde{X}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{mj})$ to produce uncertain desirable output $\tilde{Y}_j^D = (\tilde{y}_{1j}^D, \tilde{y}_{2j}^D, \ldots, \tilde{y}_{sj}^D)$ and uncertain undesirable output $\tilde{Y}_j^{ND} = (\tilde{y}_{1_j}^{ND}, \tilde{y}_{2j}^{ND}, \ldots, \tilde{y}_{s'j}^{ND})$. To assess the relative efficiency of $DMU_k,\ k \in \{1, 2, \ldots, n\}$ under VRS technology the following model is considered

$$
\begin{aligned}
\min \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j E(\tilde{x}_{ij}) - \theta E(\tilde{x}_{ik}) \le 0 \quad i = 1, 2, \ldots, m, \\
& -\sum_{j=1}^{n} \lambda_j E(\tilde{y}_{rj}^D) + E(\tilde{y}_{rk}^D) \le 0 \quad r = 1, 2, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j E(\tilde{y}_{r'j}^{ND}) - E(\tilde{y}_{r'k}^{ND}) \le 0 \quad r' = 1, 2, \ldots, s', \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0,\ j = 1, 2, \ldots, n
\end{aligned}
\tag{2.4}
$$

In situations where experts' beliefs are needed such as system flexibility or the known CO_2 emission case, traditional InvDEA models cannot provide accurate results. To address this gap in the literature, an uncertain InvDEA (UInvDEA) model will be introduced in the sequel. To transfer Model (2.4) into uncertain version, the following theorem is presented.

Theorem 3. *Suppose $\tilde{X}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{mj})$, $\tilde{Y}_j^D = (\tilde{y}_{1_j}^D, \tilde{y}_{2j}^D, \ldots, \tilde{y}_{sj}^D)$, and $\tilde{Y}_j^{ND} = (\tilde{y}_{1_j}^{ND}, \tilde{y}_{2j}^{ND}, \ldots, \tilde{y}_{s'j}^{ND})$ are independent uncertain input, desirable output, and undesirable output variables respectively with regular distribution $\Phi_j = (\Phi_{1j}, \Phi_{2j}, \ldots, \Phi_{mj})$, $\Psi_j = (\Psi_{1j}, \Psi_{2j}, \ldots, \Psi_{mj})$, and $\Gamma_j = (\Gamma_{1j}, \Gamma_{2j}, \ldots, \Gamma_{mj})$ for inputs, desirable outputs, and undesirable outputs. The crisp form of Model 3 is as follow.*

$$
\begin{aligned}
\min \quad & \theta \\
s.t. \quad & \int_0^1 \left[\sum_{j=1}^{n} \lambda_j \Phi_{ij}^{-1}(\alpha) - \theta \Phi_{ik}^{-1}(1-\alpha) \right] d\alpha \le 0 \ i = 1, 2, \ldots, m, \ldots \\
& \int_0^1 \left[-\sum_{j=1}^{n} \lambda_j \Psi_{rj}^{-1}(1-\alpha) + \Psi_{rk}^{-1}(\alpha) \right] d\alpha \le 0 \ r = 1, 2, \ldots, s, \\
& \int_0^1 \left[\sum_{j=1}^{n} \lambda_j \Gamma_{r'j}^{-1}(\alpha) - \Gamma_{r'k}^{-1}(1-\alpha) \right] d\alpha \le 0 \ r' = 1, 2, \ldots, s', \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0,\ j = 1, 2, \ldots n.
\end{aligned}
\tag{2.5}
$$

Proof. See [29].

3. Uncertain InvDEA model

In this section we propose UInvDEA model in which inputs and outputs are provided by an expert. Suppose DMU_k, $k \in \{1, 2, \ldots, n\}$ perturbs its uncertain desirable output $\tilde{Y}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, \ldots, \tilde{y}_{sj})$ to $\tilde{\beta}_k = \tilde{Y}^D{}_k + \Delta\tilde{Y}^D$. The InvDEA models determine the required uncertain input level to maintain the previous efficiency despite the mentioned perturbation. The following model achieves this goal.

$$
\begin{aligned}
\min \quad & \theta_{k'} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j E(\tilde{x}_{ij}) - \theta_{k'} E(\tilde{\alpha}_{ik}) \leq 0 \quad i = 1, 2, \ldots, m \\
& -\sum_{j=1}^{n} \lambda_j E(\tilde{y}_{rj}) + E(\tilde{\beta}_{rk}) \leq 0 \quad r = 1, 2, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j E(\tilde{y}^{ND}_{r'j}) - E(\tilde{y}^{ND}_{r'k}) \leq 0 \;\; r' = 1, 2, \ldots, s', \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0, \; j = 1, 2, \ldots, n
\end{aligned}
\tag{3.1}
$$

where $\theta_{k'}$ is the efficiency of $DMU_{k'}$ whose input and output levels are increased. If θ^*_k (the optimal value of Model (2.4)) is equal to $\theta_{k'}$ (the optimal value of Model (3.1)), the efficiency of DMU_k remains unchanged. That is $\text{Eff}(\tilde{x}_k, \tilde{y}_k) = \text{Eff}(\tilde{\alpha}_k, \tilde{\beta}_k)$. To find the input levels of $DMU_{k'}$, the following Uncertain MOLP (UMOLP) is proposed.

$$
\begin{aligned}
\min \quad & (\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k}, \ldots, \tilde{\alpha}_{mk}) = (\tilde{x}_{1k} + \Delta\tilde{x}_1, \tilde{x}_{2k} + \Delta\tilde{x}_2, \ldots, \tilde{x}_{mk} + \Delta\tilde{x}_m) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \theta^*_k \tilde{\alpha}_{ik} \leq 0 i = 1, 2, \ldots, m, \\
& -\sum_{j=1}^{n} \lambda_j \tilde{y}^D_{rj} + \tilde{\beta}_{rk} \leq 0 \quad r = 1, 2, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}^{ND}_{r'j} - \tilde{y}^{ND}_{r'k} \leq 0 \;\; r' = 1, 2, \ldots, s', \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \; j = 1, 2, \ldots, n.
\end{aligned}
\tag{3.2}
$$

where $(\lambda, \tilde{\alpha})$ is the vector of variables and θ^*_k is a constant value representing the optimal value of Model (2.4). Assume that all inputs are assessed, signed fixed weights corresponding to their priorities. Let w_i be the weight for the i-th input. The weighted sum model is proposed as follow.

$$
\begin{aligned}
\min \quad & E(\textstyle\sum_{i=1}^{m} w_i\tilde{\alpha}_i) \\
s.t. \quad & E(\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij} - \theta_k^*\tilde{\alpha}_i) \leq 0 \quad i=1,2,\ldots,m, \\
& E(-\sum_{j=1}^{n}\lambda_j\tilde{y}_{rj}^{D} + \tilde{\beta}_{rk}) \leq 0 \quad r=1,2,\ldots,s, \\
& E(\sum_{j=1}^{n}\lambda_j\tilde{y}_{r'j}^{ND} - \tilde{y}_{r'k}^{ND}) \leq 0 \;\; r'=1,2,\ldots,s', \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \geq 0, \; j=1,2,\ldots,n.
\end{aligned}
\tag{3.3}
$$

Observe that Model (3.3) is based on the expected values of uncertain inequalities. To obtain a clear representation of the model, following theorem is proposed..

Theorem 4. *Let* $\tilde{X}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{mj})$, $\tilde{Y}_j^D = (\tilde{y}_{1j}^D, \tilde{y}_{2j}^D, \ldots, \tilde{y}_{sj}^D)$, $\tilde{Y}_j^{ND} = (\tilde{y}_{1j}^{ND}, \tilde{y}_{2j}^{ND}, \ldots, \tilde{y}_{s'j}^{ND})$, $\tilde{\alpha}_k = (\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k}, \ldots, \tilde{\alpha}_{mk})$, *and* $\tilde{\beta}_k = (\tilde{\beta}_{1k}, \tilde{\beta}_{2k}, \ldots, \tilde{\beta}_{sk})$ *be independent uncertain input, desirable output, undesirable output variables, increasing the level of inputs, and increasing the level of outputs respectively with the regular distribution of* $\Phi_j = (\Phi_{1j}, \Phi_{2j}, \ldots, \Phi_{mj})$, $\Psi_j = (\Psi_{1j}, \Psi_{2j}, \ldots, \Psi_{mj})$, $\Gamma_j = (\Gamma_{1j}, \Gamma_{2j}, \ldots, \Gamma_{mj})$, $\Upsilon_k = (\Upsilon_{1k}, \Upsilon_{2k}, \ldots, \Upsilon_{mk})$, *and* $Z_k = (Z_{1k}, Z_{2k}, \ldots, Z_{sk})$. *The equivalent of Model (3.3) is as follow.*

$$
\begin{aligned}
\min \quad & \int_0^1 \left[\sum_{i=1}^{m} w_i\Upsilon_{ik}^{-1}(\alpha)\right] d\alpha \\
s.t. \quad & \int_0^1 \left[\sum_{j=1}^{n}\lambda_j\Phi_{ij}^{-1}(\alpha) - \theta_k^*\Upsilon_{ik}^{-1}(1-\alpha)\right] \leq 0 \;\; i=1,2,\ldots,m, \\
& \int_0^1 \left[-\sum_{j=1}^{n}\lambda_j\Psi_{rj}^{-1}(1-\alpha) + Z_{rk}^{-1}(\alpha))\right] \leq 0 \;\; r=1,2,\ldots,s, \\
& \int_0^1 \left[\sum_{j=1}^{n}\lambda_j\Gamma_{r'j}^{-1}(\alpha) - \Gamma_{r'k}^{-1}(1-\alpha)\right] \leq 0 \;\; r'=1,2,\ldots,s', \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \geq 0, \; j=1,2,\ldots,n.
\end{aligned}
\tag{3.4}
$$

Proof. For the sake of simplicity in notation, we consider the following notation.

$$\begin{aligned} f_1(\tilde{\alpha}_i) &= \sum_{i=1}^{m} w_i \tilde{\alpha}_i \\ f_2(\lambda_j, \tilde{x}_{ij}, \tilde{\alpha}_i) &= \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \theta_k^* \tilde{\alpha}_i \\ f_3(\lambda_j, \tilde{y}_{rj}^D, \tilde{\beta}_{rk}) &= -\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}^D + \tilde{\beta}_{rk} \\ f_4(\lambda_j, \tilde{y}_{r'j}^{ND}, \tilde{y}_{r'k}^{ND}) &= \sum_{j=1}^{n} \lambda_j \tilde{y}_{r'j}^{ND} - \tilde{y}_{r'k}^{ND}. \end{aligned}$$

According to Theorem 2-1, the inverse uncertainty distributions of these uncertain variables are

$$\begin{aligned} f_1^{-1}(\alpha) &= \sum_{i=1}^{m} w_i \Upsilon_{ik}^{-1}(\alpha) \\ f_2^{-1}(\alpha) &= \sum_{j=1}^{n} \lambda_j \Phi_{ij}^{-1}(\alpha) - \theta_k^* \Upsilon_{ik}^{-1}(1-\alpha) \\ f_3^{-1}(\alpha) &= -\sum_{j=1}^{n} \lambda_j \Psi_{rj}^{-1}(1-\alpha) + Z_{rk}^{-1}(\alpha)) \\ f_4^{-1}(\alpha) &= \sum_{j=1}^{n} \lambda_j \Gamma_{r'j}^{-1}(\alpha) - \Gamma_{r'k}^{-1}(1-\alpha). \end{aligned}$$

Furthermore, based on Theorem 2-2, we have the following relationships.

$$\begin{aligned} E(f_1) &= \int_0^1 \left[\sum_{i=1}^{m} w_i \Upsilon_{ik}^{-1}(\alpha) \right] d\alpha \\ E(f_2) &= \int_0^1 \left[\sum_{j=1}^{n} \lambda_j \Phi_{ij}^{-1}(\alpha) - \theta_k^* \Upsilon_{ik}^{-1}(1-\alpha) \right] d\alpha \quad i = 1, 2, \ldots, m \\ E(f_3) &= \int_0^1 \left[-\sum_{j=1}^{n} \lambda_j \Psi_{rj}^{-1}(1-\alpha) + Z_{rk}^{-1}(\alpha)) \right] d\alpha \quad r = 1, 2, \ldots, s, \\ E(f_4) &= \int_0^1 \left[\sum_{j=1}^{n} \lambda_j \Gamma_{r'j}^{-1}(\alpha) - \Gamma_{r'k}^{-1}(1-\alpha) \right] d\alpha \quad r' = 1, 2, \ldots, s' \end{aligned}$$

The claim follows immediately. □

Definition 4. Feasible solution $\Delta = (\lambda^*, \tilde{\alpha}^*_{1k}, \tilde{\alpha}^*_{2k}, \ldots, \tilde{\alpha}^*_{mk})$ is called weak uncertain Pareto-optimal solution of the problem UMOLP, provided that no other feasible solution like $\Delta' = (\lambda, \tilde{\alpha}_{1k}, \tilde{\alpha}_{2k}, \ldots, \tilde{\alpha}_{mk})$ exists such that

$$E(\tilde{\alpha}_{ik} - \tilde{\alpha}^*_{ik}) < 0, \ i = 1, 2, \ldots, m.$$

4. Applications

In recent years, global greenhouse gas emissions have been rising. CO_2 emissions from the construction sector reached their highest level in 2018 and continue to increase. Thus, it is crucial to balance the reduction of CO_2 emissions with the promotion of sustainable economic development. DEA, being one of the most effective methods for assessing CO_2 emission efficiency, has many related studies in the literature. Moreover, InvDEA enables decision-makers to incorporate targets or policies into production and resource allocation. However, most research does not consider CO_2 emissions as a non-deterministic variable, despite its uncertain nature according to experts opinion. This section applies the proposed model to evaluate the efficiency of OPEC countries, considering CO_2 emissions as an uncertain undesirable output. Most studies have focused either on domestic analyses such as provinces in China or international ones that have only studied developed countries. In this study, we examine OPEC member countries as representatives of developing countries. Firstly, we begin by introducing the evaluation criteria that have been extracted. Based on the analysis provided, Labor, Capital, Energy consumption are typically considered as input measures, GDP as a desirable output, and CO_2 emissions as undesirable output. Previous researches are reviewed in Table 1 to guarantee the rationality of the selected indices.

Table 1. **Indices of evaluation performed in previous researches**

Refs.	Year	Inputs	Outputs
[41]	2018	Energy, Labor, Capital, Equipment	CO_2 emissions, Gross output value, Total profits
[18]	2018	Energy consumption, Labor, Capital	CO_2 emission, GDP
[22]	2019	Energy consumption, Labor, Asset	CO_2 emission, Gross Industrial output value
[11]	2020	Energy consumption Labor, Capital	CO_2 emission, GDP
[6]	2021	Energy consumption, Capital, Urban employment	CO_2 emission, GDP
[34]	2021	Energy consumption, Labor, Capital stock	CO_2 emission, GDP

4.1. Results and discussions

The data for these indices, corresponding to all sixteen OPEC member countries, were obtained from [1]. We noted that the original data were dispersed across multiple sources. For the purposes of this research, we have compiled them into a single, consolidated table (Table 2). This study models the CO_2 emission index as an uncertain variable with a linear distribution, based on expert opinions. The distribution's interval is defined by adding and subtracting three times the data's standard deviation from the reported crisp value. Results are shown in Table 3. We apply the data in crisp model and our proposed model. Table 4 reports the results of two models.

Table 2. **Data of OPEC countries**

No.	Country	Inputs			Undesirable Output	Desirable Output
		Population	Energy Consumption	Capital	CO_2 Emission	GDP
1	Iran	89524247	39151	2745216449	7.8	15461
2	Iraq	44070556	14915	20074823489	4	9199
3	Saudi Arabia	32175352	82945	829230000000	18.2	50188
4	UAE	10242085	143594	289662252000	25.8	74918
5	Algeria	45477391	16351	8800869647	3.9	11198
6	Gabon	2430752	8010	3690152270	2.4	13940
7	Guinea	14055137	1282	11547837727	0.4	2699
8	Malaysia	34695494	39227	297815552000	8.6	28384
9	Kazakh	20034612	42235	5998817993	14	26093
10	Russia	145579890	59650	21537121900	11.4	27450
11	Bahrain	1533459	161111	3290717700	25.7	51855
12	Mexican	128613113	17828	5206736471	4	20255
13	Sudan	49383343	2317	133482900000	0.5	3571
14	Oman	4730227	91101	9644067100	15.7	35337
15	Rep. of Azerbaijan	10295307	20302	6442070800	3.7	15094
16	Rep. of Congo	6035107	2348	1201896000	1.2	3670

By comparing the results, the efficiency of DMUs in the crisp model is **better** than in the uncertain model. As it is seen, eleven DMUs are introduced efficient in the crisp model, while the uncertain model introduces ten DMUs as efficient countries. In addition, the technical efficiency of DMUs 2, 5, and 10 is very weak. However, the efficiency of these DMUs in the crisp model is not too weak. We aim to explore the relationship between inputs, desirable output, undesirable output, and efficiency. It is crucial to note that reducing CO_2 emission may not be feasible in the short term. Therefore, we explore three scenarios for CO_2 reduction. Throughout the progression from the first to the third scenario, we progressively implemented stricter environmental policies regarding the production of undesirable outputs, specifically CO_2 in this study. In the first scenario, there were no limitations on undesirable outputs. The second scenario introduced a lower growth rate for undesirable outputs

Table 3. **Uncertain data with linear distribution.**

No.	Country	Inputs			Undesirable Output CO_2 Emission	Desirable Output GDP
		Population	Energy Consumption	Capital		
1	Iran	89524247	39151	2745216449	L(7.7072,7.8928)	15461
2	Iraq	44070556	14915	20074823489	L(3.5692,4.4308)	9199
3	Saudi Arabia	32175352	82945	829230000000	L(14.670,21.7001)	50188
4	UAE	10242085	143594	289662252000	L(21.6733,29.9267)	74918
5	Algeria	45477391	16351	8800869647	L(3.7407,4.0593)	11198
6	Gabon	2430752	8010	3690152270	L(2.3036,2.4964)	13940
7	Guinea	14055137	1282	11547837727	L(0.3036,0.4964)	2699
8	Malaysia	34695494	39227	297815552000	L(7.7515,9.4485)	28384
9	Kazakh	20034612	42235	5998817993	L(12.0658,15.9342)	26093
10	Russia	145579890	59650	21537121900	L(9.8666,12.9334)	27450
11	Bahrain	1533459	161111	3290717700	L(22.9825,28.4175)	51855
12	Mexican	128613113	17828	5206736471	L(3.2525,4.7475)	20255
13	Sudan	49383343	2317	133482900000	L(0,1)	3571
14	Oman	4730227	91101	9644067100	L(12.6537,18.7463)	35337
15	Rep. of Azerbaijan	10295307	20302	6442070800	L(3.4823,3.9177)	15094
16	Rep. of Congo	6035107	2348	1201896000	L(1.1271,1.2729)	3670

Table 4. **The results of efficiency in crisp and uncertain models.**

No.	Country	Crisp Model	Uncertain Model
1	Iran	0.9064	0.7384
2	Iraq	0.8046	0.2334
3	Saudi Arabia	1.000	0.9042
4	UAE	1.000	1.0000
5	Algeria	0.7417	0.3769
6	Gabon	1.0000	1.0000
7	Guinea	1.0000	1.0000
8	Malaysia	1.0000	0.9010
9	Kazakh	1.0000	1.0000
10	Russia	0.5673	0.2882
11	Bahrain	1.0000	1.0000
12	Mexican	1.0000	1.0000
13	Sudan	1.0000	1.0000
14	Oman	1.0000	1.0000
15	Rep. of Azerbaijan	0.5768	1.0000
16	Rep. of Congo	1.0000	1.0000

relative to desirable outputs, while the third scenario maintained undesirable outputs at a constant level while allowing for an increase in desirable outputs.
The first scenario maintains the same growth rate for both desirable and undesirable outputs. The second scenario allows for a smaller increase in undesirable output compared to desirable output. The third scenario keeps undesirable output constant while increasing desirable output. The result of the first step of the first scenario is reported in Table 5. We initially increased both outputs by 15 percent. Required inputs are listed in the third column (in the crisp model) and the sixth column (in

Table 5. **Step two of the first scenario, expanding both outputs in the same manner**

No.	15 percent expanding in crisp model			15 percent expanding in uncertain model		
	θ	Δx	θ'	θ	Δx	θ'
1	0.9064	0.2329	0.9064	0.7384	0.3931	0.7384
2	0.8046	0.1709	0.8046	0.2334	0.7479	0.2334
3	1.0000	0.6581	0.0152	0.9042	1.0304	0.9042
4	1.0000	1.0240	0.2689	1.0000	1.7808	1.0000
5	0.7417	0.1846	0.7417	0.3769	0.5610	0.3769
6	1.0000	0.1756	1.0000	1.0000	0.1334	1.0000
7	1.0000	0.0748	1.0000	1.0000	0.4650	1.0000
8	1.0000	0.3412	1.0000	0.9010	1.2545	0.9010
9	1.0000	0.4712	1.0000	1.0000	0.5289	1.0000
10	0.5673	0.6069	0.5673	0.2882	0.6886	0.2882
11	1.0000	1.1336	0.1500	1.0000	0.5117	1.0000
12	1.0000	0.7813	0.0942	1.0000	0.7125	1.0000
13	1.0000	0.1079	1.0000	1.0000	0.3787	1.0000
14	1.0000	0.5619	1.0000	1.0000	0.6574	1.0000
15	0.5768	0.2356	0.5768	1.0000	0.2835	1.0000
16	1.0000	0.8034	1.0000	1.0000	0.1310	1.0000

the uncertain model) of Table 5. Comparison of results indicates that a 15 percent increase in output requires increased inputs of all countries in both models. According to the result, DMU_{11} is the most demanding country in the crisp model. However, DMU_4 is the most demanding country regarding the uncertain model. On the other hand, in the crisp model, DMU_7 and DMU_{16} are the least demanding countries, with an average increase of approximately 0.08 percent while in the uncertain model, DMU_6 and DMU_{12} are the least demanding countries, with an average increase of approximately 0.13 percent.

In the next step of the first scenario, we increase both outputs by 25 percent and examine the required input for each DMU. This information is presented in the third column (in the crisp model) and the sixth column (in the uncertain model) of Table 6. Similar to the first step in this scenario, the recommendation is to utilize more inputs for all countries. Here, the most demanding country in the crisp model is related to DMU_{11} and in the uncertain model is DMU_8. The least demanding countries in the crisp model are DMU_7 and DMU_{16} with growth of 0.15 and 0.13, respectively. However, the least demanding countries in our model are DMU_6 and DMU_{12} with increases of approximately 0.23 and 0.31 percent, respectively. We can see that these countries were also crucial in the last step of the scenario.

In the second scenario, we increase both outputs but at different rates, and then assess the required input levels. Specifically, we raise the undesirable output by a smaller percentage compared to the desirable output. The findings of this analysis are shown in Tables 7 and 8. In the initial phase, we increase the desirable output by 15 percent while keeping the undesirable output unchanged, as detailed in the third

Table 6. **Step two of the first scenario, expanding both outputs in the same manner**

No.	25 percent expanding in crisp model			25 percent expanding in uncertain model		
	θ	Δx	θ'	θ	Δx	θ'
1	0.9064	0.3222	0.9064	0.7384	0.5738	0.7384
2	0.8046	0.2345	0.8046	0.2334	1.3194	0.2334
3	1.0000	0.6029	0.1152	0.9042	1.4654	0.9042
4	1.0000	0.1130	0.4482	1.0000	1.7808	1.0000
5	0.7417	0.2551	0.7417	0.3769	0.9150	0.3769
6	1.0000	0.2696	1.0000	1.0000	0.2297	1.0000
7	1.0000	0.1510	1.0000	1.0000	0.6592	1.0000
8	1.0000	0.4104	1.0000	0.9010	2.0850	0.9010
9	1.0000	0.6145	1.0000	1.0000	0.6624	1.0000
10	0.5673	0.8174	0.5673	0.2882	2.1515	0.2882
11	1.0000	0.2226	0.2500	1.0000	0.6452	1.0000
12	1.0000	0.7141	0.1570	1.0000	1.6779	1.0000
13	1.0000	0.2020	1.0000	1.0000	0.5121	1.0000
14	1.0000	0.7052	1.0000	1.0000	0.7908	1.0000
15	0.5768	0.3618	0.5768	1.0000	0.5789	1.0000
16	1.0000	0.1347	1.0000	1.0000	0.3075	1.0000

column (in the crisp model) and sixth column (in the uncertain model) of Table 7. Similar to the first step of the first scenario, there is a recommendation to increase inputs. However, there is less increase in DMU_{17} and more increase in DMUs 12, 15, and 16 in the uncertain model. However, the crisp model recommends increasing **inputs the ratio of growth is not the same as before**. The most demanding country is DMU_{11} with 0.97 percent increase and the least demanding country is DMU_{16} with 0.05 percent increase.

In the second step, shown in the third and sixth columns of Table 8, we increase the desirable output by 25 percent and the undesirable output by 15 percent. The outcome is quite similar to the first step, meaning that there is an overall increase in input consumption. In this step, DMU_{11} is introduced as the most demanding country and DMU_{16} as the least demanding country by the crisp model. The uncertain model however identifies DMU_8 and DMU_{10} as the most demanding countries with approximately 2.8 and 2.15 percent increase in inputs. In addition, the least demanding country is DMU_6 with a 0.27 percent increase in its inputs.

The third scenario examines the required input by increasing the desirable output by varying percentages while maintaining the undesirable output constant throughout the scenario. The expansion rates for the desirable output are set at 15 and 25 percent. Results are depicted in Tables 9 and 10. It is important to note that this scenario is more stringent than the previous one. In the **earlier** scenario, there was a smaller increase in undesirable outputs compared to desirable outputs, whereas, in this scenario, the undesirable output remains unchanged. In this analysis, our model

Table 7. **First step of the second scenario, expanding both outputs at different rates**

No.	15 percent expanding desirable output in crisp model			15 percent expanding desirable output in uncertain model		
	θ	Δx	θ'	θ	Δx	θ'
1	0.9064	0.1560	0.9064	0.7384	0.3931	0.7384
2	0.8046	0.0843	0.8046	0.2334	0.7479	0.2334
3	1.0000	0.7710	0.0654	0.9042	1.0304	0.9042
4	1.0000	0.8841	0.1500	1.0000	1.7808	1.0000
5	0.7417	0.0907	0.7417	0.3769	0.5610	0.3769
6	1.0000	0.5653	1.0000	1.0000	0.1334	1.0000
7	1.0000	0.5960	0.4476	1.0000	0.1738	1.0000
8	1.0000	0.1781	1.0000	0.9010	1.2545	0.9010
9	1.0000	0.3224	1.0000	1.0000	0.5289	1.0000
10	0.5673	0.4551	0.5673	0.2882	1.6886	0.2882
11	1.0000	0.9735	1.0000	1.0000	0.5117	1.0000
12	1.0000	0.8822	0.1500	1.0000	1.7644	1.0000
13	1.0000	0.5207	0.0472	1.0000	0.3787	1.0000
14	1.0000	0.4153	1.0000	1.0000	0.6574	1.0000
15	0.5768	0.1150	0.5768	1.0000	0.3336	1.0000
16	1.0000	0.0501	1.0000	1.0000	0.1502	1.0000

Table 8. **Second step of the second scenario, expanding both outputs at different rates**

No.	25 percent expanding desirable output and 15 percent expanding undesirable output in crisp model			25 percent expanding desirable output and 15 percent expanding undesirable output in uncertain model		
	θ	Δx	θ'	θ	Δx	θ'
1	0.9064	0.2588	0.9064	0.7384	0.5738	0.7384
2	0.8046	0.1771	0.8046	0.2334	1.3194	0.2334
3	1.0000	0.6702	0.1096	0.9042	1.4654	0.9042
4	1.0000	1.0198	0.2500	1.0000	1.7808	1.0000
5	0.7417	0.1936	0.7417	0.3769	0.0189	0.3769
6	1.0000	0.3248	1.0000	1.0000	0.2735	1.0000
7	1.0000	0.1196	1.0000	1.0000	0.5647	1.0000
8	1.0000	0.2515	1.0000	0.9010	2.0850	0.9010
9	1.0000	0.4795	1.0000	1.0000	0.6624	1.0000
10	0.5673	0.6847	0.5673	0.2882	2.1515	0.2882
11	1.0000	1.1162	0.1494	1.0000	0.6452	1.0000
12	1.0000	0.7813	0.1942	1.0000	1.7125	1.0000
13	1.0000	0.2741	1.0000	1.0000	0.5121	1.0000
14	1.0000	0.5724	1.0000	1.0000	0.7908	1.0000
15	0.5768	0.2969	0.5768	1.0000	0.6403	1.0000
16	1.0000	0.0882	1.0000	1.0000	0.3689	1.0000

identifies DMU_4 and DMU_{12} as the countries with the highest input demand in the first step and DMU_8 and DMU_{10} in the second step. The countries with the lowest input demand in the first step are DMUs 6, 7, and 12 and in the second step are

DMU_6 and DMU_7 with 0.36 percent increase. In the crisp model, DMU_{11} is the country with the highest input demand in the first step and DMU_7 in the second step with approximately 0.97 percent. The country with the lowest input demand in the first step is DMU_{16} and in the second step is DMU_2 with a 0.05 and 0.09 percent increase, respectively.

Table 9. **First step of the third scenario, expanding only desirable output**

No.	15 percent expanding only desirable output in crisp model			15 percent expanding only desirable output in uncertain model		
	θ	Δx	θ'	θ	Δx	θ'
1	0.9064	0.1560	0.9064	0.7384	0.3931	0.7384
2	0.8046	0.0843	0.8046	0.2334	0.7479	0.2334
3	1.0000	0.7710	0.0654	0.9042	1.0304	0.9042
4	1.0000	0.8841	0.1500	1.0000	1.7808	1.0000
5	0.7417	0.0907	0.7417	0.3769	0.5610	0.3769
6	1.0000	0.5653	1.0000	1.0000	0.1334	1.0000
7	1.0000	0.5960	0.4476	1.0000	0.1738	1.0000
8	1.0000	0.1781	1.0000	0.9010	1.2545	0.9010
9	1.0000	0.3224	1.0000	1.0000	0.5289	1.0000
10	0.5673	0.4551	0.5673	0.2882	1.6886	0.2882
11	1.0000	0.9735	1.0000	1.0000	0.5117	1.0000
12	1.0000	0.8822	0.1500	1.0000	1.7644	1.0000
13	1.0000	0.5207	0.0472	1.0000	0.3787	1.0000
14	1.0000	0.4153	1.0000	1.0000	0.6574	1.0000
15	0.5768	0.1150	0.5768	1.0000	0.3336	1.0000
16	1.0000	0.0501	1.0000	1.0000	0.1502	1.0000

4.2. Managerial implications

The proposed method provides various implications for environmental managers and decision makers. Reducing CO_2 emission and trying to reduce greenhouse gas emissions is the most important way to promote sustainable development of humanity. Applying the proposed UInvDEA method can enable managers to identify inefficient resources. Therefore, environmental managers and decision makers can identify the challenge and take corrective measures. Uncertainty is another key challenge for managers in the face of disasters. To reduce the risks, managers should not only apply suitable approaches, they should also consider changes. The proposed method in this study is a convenient tool for dealing with uncertainty. The obtained results show how the efficiency of environmental changes according to different scenario undesirable output. This, in turn, assists environmental managers and decision makers with better planning and resource allocation.

Table 10. **Second step of the third scenario, expanding only desirable output**

No.	25 percent expanding only desirable output in crisp model			25 percent expanding only desirable output in uncertain model		
	θ	Δx	θ'	θ	Δx	θ'
1	0.9064	0.1968	0.9064	0.7384	0.5738	0.7384
2	0.8046	0.0912	0.8046	0.2334	1.4126	0.2334
3	1.0000	0.7710	0.1654	0.9042	1.4654	0.9042
4	1.0000	0.8904	0.2500	1.0000	1.7808	1.0000
5	0.7417	0.1180	0.7417	0.3769	1.2635	0.3769
6	1.0000	0.7344	0.0260	1.0000	0.3657	1.0000
7	1.0000	0.9688	1.0000	1.0000	0.3657	1.0000
8	1.0000	0.1524	1.0000	0.9010	2.0850	0.9010
9	1.0000	0.3710	1.0000	1.0000	0.6624	1.0000
10	0.5673	0.5549	0.5673	0.2882	2.1515	0.2882
11	1.0000	0.9559	1.0000	1.0000	0.6452	1.0000
12	1.0000	0.8822	0.2500	1.0000	1.7644	1.0000
13	1.0000	0.6464	0.0787	1.0000	0.5121	1.0000
14	1.0000	0.4595	1.0000	1.0000	0.7908	1.0000
15	0.5768	0.2771	0.5768	1.0000	0.7325	1.0000
16	1.0000	0.2595	1.0000	1.0000	0.4611	1.0000

5. Conclusion

Industries must allocate suitable resources for effective engagement. InvDEA serves as a post-DEA sensitivity analysis designed to address resource allocation issues. This method operates under the assumption that factors are deterministic, but there are many instances where this assumption does not hold true. To effectively manage data uncertainty as indicated by experts, we have introduced an uncertain InvDEA approach that incorporates undesirable outputs. In this approach, uncertain variables are replaced with their expected values. The resulting model can be converted into an equivalent linear model and solved using a linear solver. To demonstrate the applicability of this model, we conducted a case study assessing the efficiency of OPEC countries while considering CO_2 emissions as an undesirable output. We utilized the developed model to evaluate the efficiency of OPEC countries in the presence of CO_2 emissions as an uncertain undesirable output. We designed and analyzed three scenarios to create a gradual CO_2 reduction strategy for the industrial sector of OPEC countries. While improving environmental efficiency may incur costs, it is a crucial concern for the future that warrants increased attention. The proposed model could also be applied to other contexts involving uncertainty.

Among various uncertainty-handling methods, substituting uncertain variables with their expected values offers a practical solution, enabling planners to utilize deterministic frameworks. This approach proves particularly suitable when quantity variations are relatively minor. By conducting sensitivity and uncertainty analyses on the expected-value model, researchers can quantify and evaluate the influence of uncertainty on the efficiency measurements. Future research directions could explore confidence level approaches to yield more precise outcomes. Furthermore, in this study, we have considered a black box system. However, different network structures can be studied in future work.

Statements and Declarations

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript. The authors have no relevant financial or non-financial interests to disclose. All authors contributed to the study conception and design and commented on previous versions of the manuscript. All authors read and approved the final manuscript. This research did not generate or analyze any datasets.

References

[1] *Our world in data*, https://ourworldindata.org/.

[2] M. Ahmadkhanlou Gharakhanlo, G. Tohidi, N. Azarmir Shotorbani, S. Razavyan, and R. Abbasi, *Cost, Revenue and profit efficiency in multi-period network system: A DEA-R based approach*, Commun. Comb. Optim. **9** (2024), no. 4, 725–746.
https://doi.org/10.22049/cco.2023.28179.1474.

[3] G.R. Amin and M.I. Boamah, *Modeling business partnerships: A data envelopment analysis approach*, Eur. J. Oper. Res. **305** (2023), no. 1, 329–337.
https://doi.org/10.1016/j.ejor.2022.05.036.

[4] R.D. Banker, A. Charnes, and W.W. Cooper, *Some models for estimating technical and scale inefficiencies in data envelopment analysis*, Manag. Sci. **30** (1984), no. 9, 1078–1092.

[5] A. Charnes, W.W. Cooper, and E. Rhodes, *Measuring the efficiency of decision making units*, European journal of operational research **2** (1978), no. 6, 429–444.
https://doi.org/10.1016/0377-2217(78)90138-8.

[6] L. Chen, Y. Gao, M.J. Li, Y.M. Wang, and L.H. Liao, *A new inverse data envelopment analysis approach to achieve China's road transportation safety objectives*, Saf. Sci. **142** (2021), 105362.
https://doi.org/10.1016/j.ssci.2021.105362.

[7] Q. Cui and Y. Li, *An empirical study on the influencing factors of transportation carbon efficiency: Evidences from fifteen countries*, Appl. Energy **141** (2015), 209–217.
https://doi.org/10.1016/j.apenergy.2014.12.040.

[8] A. Emrouznejad, G.R. Amin, M. Ghiyasi, and M. Michali, *A review of inverse data envelopment analysis: origins, development and future directions*, IMA J. Manag. Math. **34** (2023), no. 3, 421–440.
https://doi.org/10.1093/imaman/dpad006.

[9] A. Emrouznejad, M. Tavana, and A. Hatami-Marbini, *The state of the art in fuzzy data envelopment analysis*, Performance Measurement with Fuzzy Data

Envelopment Analysis (A. Emrouznejad and M. Tavana, eds.), Springer Berlin Heidelberg, Berlin, Heidelberg, 2014, pp. 1–45.

[10] A. Emrouznejad and G.l. Yang, *A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016*, Socio-Econ. Plan. Sci. **61** (2018), 4–8. https://doi.org/10.1016/j.seps.2017.01.008.

[11] A. Emrouznejad, G.l. Yang, and G.R. Amin, *A novel inverse DEA model with application to allocate the CO2 emissions quota to different regions in chinese manufacturing industries*, J. Oper. Res. Soc. **70** (2019), no. 7, 1079–1090. https://doi.org/10.1080/01605682.2018.1489344.

[12] M. Ghiyasi, *On inverse DEA model: The case of variable returns to scale*, Comput. Ind. Eng. **87** (2015), 407–409. https://doi.org/10.1016/j.cie.2015.05.018.

[13] A. Ghomi, S. Ghobadi, M.H. Behzadi, and M. Rostamy-Malkhalifeh, *Inverse data envelopment analysis with stochastic data*, RAIRO Oper. Res. **55** (2021), no. 5, 2739–2762. https://doi.org/10.1051/ro/2021135.

[14] P. Guo and H. Tanaka, *Decision making based on fuzzy data envelopment analysis*, Intelligent Decision and Policy Making Support Systems (Da Ruan, F. Hardeman, and K. van der Meer, eds.), Springer Berlin Heidelberg, Berlin, Heidelberg, 2008, pp. 39–54.

[15] A. Hadi-Vencheh, A.A. Foroughi, and M. Soleimani-damaneh, *A DEA model for resource allocation*, Econ. Model. **25** (2008), no. 5, 983–993. https://doi.org/10.1016/j.econmod.2008.01.003.

[16] A. Hatami-Marbini, A. Emrouznejad, and M. Tavana, *A taxonomy and review of the fuzzy data envelopment analysis literature: two decades in the making*, Eur. J. Oper. Res. **214** (2011), no. 3, 457–472. https://doi.org/10.1016/j.ejor.2011.02.001.

[17] D. Hou, Y. Song, J. Zhang, M. Hou, D. O'Connor, and M. Harclerode, *Climate change mitigation potential of contaminated land redevelopment: A city-level assessment method*, J. Clean. Prod. **171** (2018), 1396–1406. https://doi.org/10.1016/j.jclepro.2017.10.071.

[18] Y. Iftikhar, Z. Wang, B. Zhang, and B. Wang, *Energy and CO_2 emissions efficiency of major economies: A network DEA approach*, Energy **147** (2018), 197–207. https://doi.org/10.1016/j.energy.2018.01.012.

[19] D.S. Kwon, J.H. Cho, and S.Y. Sohn, *Comparison of technology efficiency for CO_2 emissions reduction among European countries based on DEA with decomposed factors*, J. Clean. Prod. **151** (2017), 109–120. https://doi.org/10.1016/j.jclepro.2017.03.065.

[20] S. Lertworasirikul, P. Charnsethikul, and S.C. Fang, *Inverse data envelopment analysis model to preserve relative efficiency values: The case of variable returns to scale*, Comput. Ind. Eng. **61** (2011), no. 4, 1017–1023. https://doi.org/10.1016/j.cie.2011.06.014.

[21] S. Li, H. Diao, L. Wang, and L. Li, *A complete total-factor CO_2 emissions ef-*

ficiency measure and "2030–60 CO_2 emissions targets" for shandong province, china, J. Clean. Prod. **360** (2022), 132230.
https://doi.org/10.1016/j.jclepro.2022.132230.

[22] Y. Li, Y. Wei, X. Zhang, and Y. Tao, *Regional and provincial CO_2 emission reduction task decomposition of China's 2030 carbon emission peak based on the efficiency, equity and synthesizing principles*, Struct. Change Econ. Dyn. **53** (2020), 237–256.
https://doi.org/10.1016/j.strueco.2020.02.007.

[23] W. Lio and B. Liu, *Uncertain data envelopment analysis with imprecisely observed inputs and outputs*, Fuzzy Optim. Decis. Mak. **17** (2018), no. 3, 357–373.
https://doi.org/10.1007/s10700-017-9276-x.

[24] B. Liu, *Uncertainty Theory*, 2007.

[25] B. Liu, *Uncertainty Theory.Berlin:springer*, 2017.

[26] Z. Moghaddas, B.M. Tosarkani, and S. Yousefi, *Resource reallocation for improving sustainable supply chain performance: an inverse data envelopment analysis*, Int. J. Prod. Econ. **252** (2022), 108560.
https://doi.org/10.1016/j.ijpe.2022.108560.

[27] O.B. Olesen and N.C. Petersen, *Stochastic data envelopment analysis–A review*, Eur. J. Oper. Res. **251** (2016), no. 1, 2–21.
https://doi.org/10.1016/j.ejor.2015.07.058.

[28] J. Pourmahmoud and N. Bagheri, *Providing an uncertain model for evaluating the performance of a basic two-stage system*, Soft Comput. **25** (2021), no. 6, 4739–4748.
https://doi.org/10.1007/s00500–020-05481-8.

[29] ———, *Uncertain malmquist productivity index: An application to evaluate healthcare systems during COVID-19 pandemic*, Socio-Econ. Plan. Sci. **87** (2023), 101522.
https://doi.org/10.1016/j.seps.2023.101522.

[30] J. Pourmahmoud and D. Norouzi Bene, *Global Malmquist productivity index for evaluation of multistage series systems with undesirable and non-discretionary data*, Commun. Comb. Optim. **10** (2025), no. 4, 905–931.
https://doi.org/10.22049/cco.2024.29063.1831.

[31] M. Soleimani-Damaneh, G.R. Jahanshahloo, and S. Abbasbandy, *Computational and theoretical pitfalls in some current performance measurement techniques; and a new approach*, Appl. Math. Comput. **181** (2006), no. 2, 1199–1207.
https://doi.org/10.1016/j.amc.2006.01.085.

[32] M. Wegener and G.R. Amin, *Minimizing greenhouse gas emissions using inverse DEA with an application in oil and gas*, Expert Syst. Appl. **122** (2019), 369–375.
https://doi.org/10.1016/j.eswa.2018.12.058.

[33] Q. Wei, J. Zhang, and X. Zhang, *An inverse DEA model for inputs/outputs estimate*, Eur. J. Oper. Res. **121**, no. 1, 151–163.
https://doi.org/10.1016/S0377-2217(99)00007-7.

[34] H. Xiao, Y. Zhou, N. Zhang, D. Wang, Y. Shan, and J. Ren, *CO_2 emission reduction potential in China from combined effects of structural adjustment of economy*

and efficiency improvement, Resour. conserv. recycl. **174** (2021), 105760.
https://doi.org/10.1016/j.resconrec.2021.105760.

[35] H. Yan, Q. Wei, and G. Hao, *DEA models for resource reallocation and production input/output estimation*, Eur. J. Oper. Res. **136** (2002), no. 1, 19–31.
https://doi.org/10.1016/S0377-2217(01)00046-7.

[36] M. Yang, Y. Hou, Q. Ji, and D. Zhang, *Assessment and optimization of provincial CO_2 emission reduction scheme in China: an improved ZSG-DEA approach*, Energy Econ. **91** (2020), 104931.
https://doi.org/10.1016/j.eneco.2020.104931.

[37] A. Younesi, F.H. Lotfi, and M. Arana-Jiménez, *Using slacks-based model to solve inverse DEA with integer intervals for input estimation*, Fuzzy Optim. Decis. Mak. **22** (2023), no. 4, 587–609.
https://doi.org/10.1007/s10700-022-09403-1.

[38] L.A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets Syst. **100** (1978), no. 1, 3–28.
https://doi.org/10.1016/S0165-0114(99)80004-9.

[39] L.A. Zadeh, G.J. Klir, and B. Yuan, *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by lotfi a zadeh.united states*, World scientific, 1996.

[40] J. Zhang, W. Jin, G.l. Yang, H. Li, Y. Ke, and S.P. Philbin, *Optimizing regional allocation of CO_2 emissions considering output under overall efficiency*, Socio-Econ. Plan. Sci. **77** (2021), 101012.
https://doi.org/10.1016/j.seps.2021.101012.

[41] J. Zhang, H. Li, B. Xia, and M. Skitmore, *Impact of environment regulation on the efficiency of regional construction industry: A 3-stage data envelopment analysis (DEA)*, J. Clean. Prod. **200** (2018), 770–780.
https://doi.org/10.1016/j.jclepro.2018.07.189.