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Research Article

Algebraic structures of Fibonacci matrices over ring

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Abstract: In this paper we have developed some algebraic structures for the set Fibonacci matrices over initial value spaces ring and field and shown that set of all Fibonacci matrices forms a ring or field (coined as Fibonacci Ring or Fibonacci Field) in either cases. We also investigated those structures over $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ and found that over $\mathbb{Q}$ it forms a Fibonacci Field but over $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ it is a Fibonacci Ring. Finally we have introduced a new concept of f-inverse initial value along with that of f-congruent equivalence class and demonstrated graphically which leads a wide scope of future work.

Keywords: algebraic structure, field theory, Fibonacci sequence, Fibonacci matrix.

AMS Subject classification: 08A05, 12E99, 11B39

1. Introduction

The theories and properties of an algebraic structure provide several ideas to solve some long standing problems algebraically. In addition, matrix representation of a problem ease to apply properties of linear algebra. In this article, algebraic structures on set of Fibonacci matrices over a ring and over a field have been introduced, which may be helpful to enrich the mathematical tools and to encounter problems on Number theory and Combinatorics.

There are several articles and books has been published on the properties and applications of Fibonacci, Lucas and Pell numbers [4, 5]. In [2], Eugeni & Mascella has given direct solutions of second order recurrence relations from a generalization of Pascal's Triangle. Further many authors have studied about the properties and applications of Fibonacci Q-Matrix and generalized Q-Matrix which include Lucas Matrix and others[3, 6]. Fozi M. and Dannan [1] established some properties for Fibonacci and Pell numbers in which Fibonacci Q-type matrices has been used as a tool. Sia J. Ying et.al. [9] has investigated some algebraic properties of generalized Fibonacci sequence by employing two different matrix methods. Using linear recurrence relations

in $GL(d;\mathbb{C})$ combinatorial form of some sequences has been defined and investigated its properties[10]. Shou-Qiang Shen et.al.[7] has constructed circulant matrices and obtained the existence of their inverses and obtained the pseudo-inverse of generalized Fibonacci matrix in [8].
Most of the researchers worked on Fibonacci matrices with initial value $(0, 1)$ or $(2, 1)$ (Lucas). There are several applications of Fibonacci matrix. Specially the inverse of such matrices has been used in cryptography with these special initial values [6, 8]. Koshy [5], in his book given an idea of generalized Fibonacci sequence with arbitrary initial values.
In this paper some algebraic structures of Fibonacci matrices with arbitrary initial value has been developed. In section 2 the second order Fibonacci sequence over a ring and f-congruent relation over the ring has been introduced. However in section 3 the algebraic structure of second order Fibonacci matrix over both a ring or a field has been developed and coined Fibonacci Ring and Fibonacci Field over an initial value space. Finally in section 4 a new concept of inverse initial value has been introduced and investigated its properties along with location of inverse elements demonstrated graphically followed by the conclusion of the paper in section 5.

2. The Fibonacci Sequence over a Ring

The Fibonacci sequence $\{f_{(a,b)}(n)\}$ over a ring R is given by

$$f_{(a,b)}(n+2) = f_{(a,b)}(n+1) + f_{(a,b)}(n) \qquad n \geq 0, \tag{2.1}$$

with initial values $f_{(a,b)}(0) = a$ and $f_{(a,b)}(1) = b$, where both $a, b \in R$.
The sequence $\{f_{(a,b)}(n)\}$ can be extended in negative direction also using the equation (2.1) rewritten as

$$f_{(a,b)}(n) = f_{(a,b)}(n+2) - f_{(a,b)}(n+1) \qquad n < 0, \tag{2.2}$$

with initial values $f_{(a,b)}(0) = a$ and $f_{(a,b)}(1) = b$, where both $a, b \in R$.

In particular the sequences $\{f_{(0,1)}(n)\}$ and $\{f_{(2,1)}(n)\}$ over $\mathbb{Z}$ coincide with the standard Fibonacci and Lucus sequences respectively. Both the sequences are also extended in negative directions using corresponding equation (2.2). However to generate the sequences in negative directions for Fibonacci and Lucus Sequence also be used the relations $f_{(0,1)}(-n) = (-1)^{n+1} f_{(0,1)}(n)$ and $f_{(2,1)}(-n) = (-1)^n f_{(2,1)}(n)$ for $n \in \mathbb{N}$ respectively.

Further we have introduced a relation on the initial value space R as follows and observed that it is an equivalence relation.

Definition 1. An ordered pair of initial values $(a_1, b_1) \in R^2$ is said to be f-congruent to the ordered pair of initial values $(a_2, b_2) \in R^2$ if there exists an integer r such that $f_{(a_1,b_1)}(r) = a_2$ and $f_{(a_1,b_1)}(r+1) = b_2$.

Theorem 1. *The f-congruent relation is an equivalence relation on R^2.*

Proof. For any initial value $(a, b) \in R^2$ we have $f_{(a,b)}(0) = a$ and $f_{(a,b)}(1) = b$. So f-congruent is reflexive.
Let the ordered pair of initial values $(a_1, b_1) \in R^2$ is f-congruent to $(a_2, b_2) \in R^2$. Then there exists an integer r such that $f_{(a_1,b_1)}(r) = a_2$ and $f_{(a_1,b_1)}(r+1) = b_2$. So a_2 and b_2 are the rth and $(r+1)$th term in the Fibonacci sequence $\{f_{(a_1,b_1)}(n)\}$ respectively. In other words a_1 and b_1 are the $-r$th and $(-r+1)$th term in the Fibonacci sequence $\{f_{(a_2,b_2)}(n)\}$ respectively i.e. $f_{(a_2,b_2)}(-r) = a_1$ and $f_{(a_2,b_2)}(-r+1) = b_1$.
Finally if $(a_1, b_1) \in R^2$ is f-congruent to $(a_2, b_2) \in R^2$ and $(a_2, b_2) \in R^2$ is f-congruent to $(a_3, b_3) \in R^2$, there exists integers r and s such that $f_{(a_1,b_1)}(r) = a_2$, $f_{(a_1,b_1)}(r+1) = b_2$ and $f_{(a_2,b_2)}(s) = a_3$, $f_{(a_2,b_2)}(s+1) = b_3$. Therefore $f_{(a_1,b_1)}(r+s) = a_3$, $f_{(a_1,b_1)}(r+s+1) = b_3$ therefore (a_1, b_1) is f-congruent to (a_3, b_3). Hence f-congruent is an equivalence relation. □

Observations:- Beyond this result we have the following observations on the terms of Fibonacci sequence over ring R.

(O1.) $f_{(ka,kb)}(n) = kf_{(a,b)}(n)$ for all $n \in \mathbb{Z}$ and $k \in R$.

(O2.) $f_{(0,0)}(n) = 0$ for all $n \in \mathbb{Z}$.

(O3.) $f_{(-a,-b)}(n) = -f_{(a,b)}(n)$ for all $n \in \mathbb{Z}$.

(O4.) $f_{(a,b)}(n) + f_{(c,d)}(n) = f_{(a+c,b+d)}(n)$ for all $n \in \mathbb{Z}$ and $a, b, c, d \in R$.

The observations (O1.) and (O4.) can be proved easily using mathematical induction on n. However (O2.) and (O3.) are the particular cases of (O1.) for $k = 0$ and $k = -1$ respectively.

3. Algebraic Structure Of Fibonacci Matrix Over a Ring or a Field

3.1. The Fibonacci Matrix Over a Ring

The nth Fibonacci matrix is a 2×2 matrix denoted by $Q^{(n)}_{(a,b)}$-matrix over a ring R and defined as $Q^{(n)}_{(a,b)} = \begin{bmatrix} f_{(a,b)}(n+1) & f_{(a,b)}(n) \\ f_{(a,b)}(n) & f_{(a,b)}(n-1) \end{bmatrix}$ for $n \in \mathbb{Z}$, where the entries $f_{(a,b)}(n)$ are obtained from the Fibonacci sequences defined in (2.1)and (2.2) with initial values $f_{(a,b)}(0) = a \in R$ and $f_{(a,b)}(1) = b \in R$.

Throughout the paper $Q^{(n)}_{(a,b)}$ is the matrix obtained by the above definition and $Q^{n}_{(a,b)}$ is the matrix obtained by n times multiplication of $Q_{(a,b)}$.

Definition 2. The set $\mathcal{Q}(R)$ of all second order Fibonacci matrices over a ring R i.e. $\mathcal{Q}(R) = \{Q^{(n)}_{(a,b)} : n \in \mathbb{Z} \ \& \ (a,b) \in R^2\}$ is said to be a Fibonacci Ring over R if it is a ring under usual matrix additions and multiplications .

The set $\mathcal{Q}(R)$ has following algebraic properties under usual matrix additions and multiplications.

Theorem 2. *For any ring R, the set $\mathcal{Q}(R)$ is a Fibonacci Ring over R.*

Proof. To prove the result, first we shall prove the following claim.
Claim:- A 2×2 matrix A over R belongs to $\mathcal{Q}(R)$ if and only if it is of the form $A = \begin{bmatrix} p+q & q \\ q & p \end{bmatrix}$, where $p, q \in R$
Proof of the claim:- Let us consider a 2×2 matrix A over R belongs to $\mathcal{Q}(R)$. Then by definition of $\mathcal{Q}(R)$ there exists some $n \in \mathbb{Z}$ & $a, b \in R$ for which

$$\begin{aligned} A &= Q^{(n)}_{(a,b)} \\ &= \begin{bmatrix} f_{(a,b)}(n+1) & f_{(a,b)}(n) \\ f_{(a,b)}(n) & f_{(a,b)}(n-1) \end{bmatrix} \\ &= \begin{bmatrix} f_{(a,b)}(n) + f_{(a,b)}(n-1) & f_{(a,b)}(n) \\ f_{(a,b)}(n) & f_{(a,b)}(n-1) \end{bmatrix}. \end{aligned}$$

Conversely if the matrix $A = \begin{bmatrix} p+q & q \\ q & p \end{bmatrix}$, where $p, q \in R$ then it can be written as $A = Q^{(1)}_{(p,q)} \in \mathcal{Q}(R)$. This completes the proof of the claim.
So the set $\mathcal{Q}(R)$ is the collection of all 2×2 matrices of the form $\begin{bmatrix} p+q & q \\ q & p \end{bmatrix}$ over R.
For any two matrices $\begin{bmatrix} p_1+q_1 & q_1 \\ q_1 & p_1 \end{bmatrix}$ and $\begin{bmatrix} p_2+q_2 & q_2 \\ q_2 & p_2 \end{bmatrix}$ in $\mathcal{Q}(R)$ we have

$$\begin{bmatrix} p_1+q_1 & q_1 \\ q_1 & p_1 \end{bmatrix} + \begin{bmatrix} p_2+q_2 & q_2 \\ q_2 & p_2 \end{bmatrix} = \begin{bmatrix} (p_1+p_2)+(q_1+q_2) & q_1+q_2 \\ q_1+q_2 & p_1+p_2 \end{bmatrix} \in \mathcal{Q}(R)$$

and

$$\begin{bmatrix} p_1+q_1 & q_1 \\ q_1 & p_1 \end{bmatrix} \begin{bmatrix} p_2+q_2 & q_2 \\ q_2 & p_2 \end{bmatrix} = \begin{bmatrix} (p_1q_2+q_1p_2+q_1q_2)+(q_1q_2+p_1p_2) & p_1q_2+q_1p_2+q_1q_2 \\ p_1q_2+q_1p_2+q_1q_2 & q_1q_2+p_1p_2 \end{bmatrix} \in \mathcal{Q}(R).$$

Also $Q^{(1)}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{Q}(R)$. We know that usual matrix addition is associative and commutative. Also matrix multiplication is associative and both the distributive laws hold. Hence $\mathcal{Q}(R)$ is a ring under matrix addition and multiplication. □

Theorem 3. *The set $\mathcal{Q}(R)$ is a commutative Fibonacci Ring over R with identity if and only if R is a commutative ring with identity.*

Proof. Since R is a commutative ring with identity, consider 0 and 1 are zero and identity element of R respectively. Then we have $Q^{(0)}_{(0,1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{Q}(R)$. Also we have

$$\begin{bmatrix} p_1+q_1 & q_1 \\ q_1 & p_1 \end{bmatrix} \begin{bmatrix} p_2+q_2 & q_2 \\ q_2 & p_2 \end{bmatrix} = \begin{bmatrix} p_2+q_2 & q_2 \\ q_2 & p_2 \end{bmatrix} \begin{bmatrix} p_1+q_1 & q_1 \\ q_1 & p_1 \end{bmatrix}$$
$$= \begin{bmatrix} (p_1q_2+q_1p_2+q_1q_2)+(q_1q_2+p_1p_2) & p_1q_2+q_1p_2+q_1q_2 \\ p_1q_2+q_1p_2+q_1q_2 & q_1q_2+p_1p_2 \end{bmatrix} \in \mathcal{Q}(R).$$

Hence $\mathcal{Q}(R)$ is a commutative ring with identity under matrix addition and multiplication.

Conversely for any $p, q \in R$ we have both $\begin{bmatrix} p+q & q \\ q & p \end{bmatrix}, \begin{bmatrix} p+q & p \\ p & q \end{bmatrix} \in \mathcal{Q}(R)$. Since $\mathcal{Q}(R)$ is a commutative ring with identity we have

$$\begin{bmatrix} p+q & q \\ q & p \end{bmatrix} \begin{bmatrix} p+q & p \\ p & q \end{bmatrix} = \begin{bmatrix} p+q & p \\ p & q \end{bmatrix} \begin{bmatrix} p+q & q \\ q & p \end{bmatrix}$$

implying that $(p+q)^2 + qp = (p+q)^2 + pq$ and so $qp = pq$.

Also $Q^{(0)}_{(0,1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{Q}(R)$ lead to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p+q & q \\ q & p \end{bmatrix} = \begin{bmatrix} p+q & q \\ q & p \end{bmatrix}$ Thus 1 is the identity element of R. Hence R is a commutative ring with identity. □

Example 1. $\mathcal{Q}(\mathbb{Z})$ is a commutative Fibonacci Ring over $\mathbb{Z}$ with identity as $\mathbb{Z}$ is a commutative ring with identity.

Theorem 4. *Let R be a commutative ring with identity with 0 and 1 are zero and identity element of R respectively and $Q^{(0)}_{(0,1)} = Q^{0}_{(0,1)}$. Then for any $(a,b) \in R^2$*

(1) $Q^{(n)}_{(a,b)} = Q^{n}_{(0,1)} Q^{(0)}_{(a,b)} = Q^{(0)}_{(a,b)} Q^{n}_{(0,1)}$; $n \geq 0$, where $Q^{(1)}_{(0,1)} = Q_{(0,1)}$ and

(2) $Q^{(-n)}_{(a,b)} = Q^{-n}_{(0,1)} Q^{(0)}_{(a,b)} = Q^{(0)}_{(a,b)} Q^{-n}_{(0,1)}$; $n \geq 0$, where $Q^{(-1)}_{(0,1)} = Q^{-1}_{(0,1)}$.

(3) In particular $Q^{(n)}_{(0,1)} = Q^{n}_{(0,1)}$ for all $n \in \mathbb{Z}$

Furthermore the class of matrices $\mathcal{F} = \{Q^{(n)}_{(0,1)} : n \in \mathbb{Z}\}$ together with matrix multiplication is isomorphic to the group $(\mathbb{Z}, +)$.

Proof. We have $Q^{(1)}_{(0,1)} = Q_{(0,1)}$ and $Q^{(0)}_{(0,1)} = Q^{0}_{(0,1)} = I_2$. By the definition of $Q^{(n)}_{(a,b)}$, we have

$$\begin{aligned} Q_{(0,1)}Q^{(n)}_{(a,b)} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{(a,b)}(n+1) & f_{(a,b)}(n) \\ f_{(a,b)}(n) & f_{(a,b)}(n-1) \end{bmatrix} \\ &= \begin{bmatrix} f_{(a,b)}(n+1) + f_{(a,b)}(n) & f_{(a,b)}(n) + f_{(a,b)}(n-1) \\ f_{(a,b)}(n+1) & f_{(a,b)}(n) \end{bmatrix} \\ &= \begin{bmatrix} f_{(a,b)}(n+2) & f_{(a,b)}(n+1) \\ f_{(a,b)}(n+1) & f_{(a,b)}(n) \end{bmatrix} \\ &= Q^{(n+1)}_{(a,b)}. \end{aligned}$$

And by post multiplying we get $Q^{(n)}_{(a,b)}Q_{(0,1)} = Q^{(n+1)}_{(a,b)}$.
Therefore we have a first order recurrence relation

$$Q^{(n+1)}_{(a,b)} = Q_{(0,1)}Q^{(n)}_{(a,b)} = Q^{(n)}_{(a,b)}Q_{(0,1)}; \quad n > 0, Q^{(0)}_{(a,b)} \quad = \begin{bmatrix} b & a \\ a & b-a \end{bmatrix} \tag{3.1}$$

The solution of equation (3.1) is given by

$$Q^{(n)}_{(a,b)} = Q^{n}_{(0,1)}Q^{(0)}_{(a,b)} = Q^{(0)}_{(a,b)}Q^{n}_{(0,1)}; \quad n \geq 0 \tag{3.2}$$

In particular $Q^{(n)}_{(0,1)} = Q^{n}_{(0,1)}; \quad n \geq 0$ as $Q^{(0)}_{(0,1)} = I_2$.
Similarly as $Q^{(-1)}_{(0,1)} = Q^{-1}_{(0,1)}$ we have

$$\begin{aligned} Q^{(-1)}_{(0,1)}Q^{(-n)}_{(a,b)} &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_{(a,b)}(-n+1) & f_{(a,b)}(-n) \\ f_{(a,b)}(-n) & f_{(a,b)}(-n-1) \end{bmatrix} \\ &= \begin{bmatrix} f_{(a,b)}(-n) & f_{(a,b)}(-n-1) \\ f_{(a,b)}(-n+1) - f_{(a,b)}(-n) & f_{(a,b)}(-n) - f_{(a,b)}(-n-1) \end{bmatrix} \\ &= \begin{bmatrix} f_{(a,b)}(-n) & f_{(a,b)}(-(n+1)) \\ f_{(a,b)}(-(n+1)) & f_{(a,b)}(-(n+2)) \end{bmatrix} \\ &= Q^{(-(n+1))}_{(a,b)}. \end{aligned}$$

And by post multiplying we get $Q^{(-n)}_{(a,b)}Q^{(-1)}_{(0,1)} = Q^{(-(n+1))}_{(a,b)}$.
Therefore we have another first order recurrence relation for negative direction

$$Q^{(-(n+1))}_{(a,b)} = Q^{-1}_{(0,1)}Q^{(-n)}_{(a,b)} = Q^{(-n)}_{(a,b)}Q^{-1}_{(0,1)}; \quad n \geq 0, Q^{(0)}_{(a,b)} \quad = \begin{bmatrix} b & a \\ a & b-a \end{bmatrix}. \tag{3.3}$$

The solution of equation (3.3) is given by

$$Q^{(-n)}_{(a,b)} = Q^{-n}_{(0,1)}Q^{(0)}_{(a,b)}; \quad n \geq 0. \tag{3.4}$$

In particular $Q^{(-n)}_{(0,1)} = Q^{-n}_{(0,1)};\quad n \geq 0$ as $Q^{(0)}_{(0,1)} = I_2$. Therefore $Q^{(n)}_{(0,1)} = Q^{n}_{(0,1)}$ for all $n \in \mathbb{Z}$.
Let us consider an one-one onto mapping $g : \mathbb{Z} \to \mathcal{F}$ defined as $g(n) = Q^{(n)}_{(0,1)}$ where 0 and 1 are zero and identity elements of R. For any $m, n \in \mathbb{Z}$ we have
$g(m+n) = Q^{(m+n)}_{(0,1)} = Q^{m+n}_{(0,1)} = Q^{m}_{(0,1)}Q^{n}_{(0,1)} = Q^{(m)}_{(0,1)}Q^{(n)}_{(0,1)} = g(m)g(n)$
Hence $\mathbb{Z}$ is isomorphic to $\mathcal{F}$. Also $g(0) = Q^{(0)}_{(0,1)} = I_2$ and
$g(1) = Q^{(1)}_{(0,1)} = Q_{(0,1)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $Q^{(-1)}_{(0,1)} = Q^{-1}_{(0,1)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. □

3.2. The Fibonacci Matrix Over a Field

Let us consider the set $\mathcal{Q}(\mathbb{F})$ of all second order Fibonacci matrices over a field $\mathbb{F}$ i.e. $\mathcal{Q}(\mathbb{F}) = \{Q^{(n)}_{(a,b)} : n \in \mathbb{Z}\ \&\ (a,b) \in \mathbb{F}\}$.

Definition 3. The set $\mathcal{Q}(\mathbb{F})$ of all second order Fibonacci matrices over a field $\mathbb{F}$ i.e. $\mathcal{Q}(\mathbb{F}) = \{Q^{(n)}_{(a,b)} : n \in \mathbb{Z}\ \&\ (a,b) \in \mathbb{F}^2\}$ is said to be a Fibonacci Field over $\mathbb{F}$ if it is a field under usual matrix additions and multiplications .

Being $\mathbb{F}$ is a commutative ring with identity from theorem 3 it is clear that the set $\mathcal{Q}(\mathbb{F})$ is a commutative ring with identity under usual matrix additions and multiplications. $\mathcal{Q}(\mathbb{F})$ is a field if each non-zero element is invertible. The following theorem states the necessary and sufficient condition for existence of inverse of each non-zero element of $\mathcal{Q}(\mathbb{F})$.

Theorem 5. *The set $\mathcal{Q}(\mathbb{F}^*)$ is a Fibonacci Field over $\mathbb{F}^*$ if and only if $\mathbb{F}^*$ is a field in which there is no non-trivial solutions of the equation $a^2 + ab - b^2 = 0$ for a and b i.e $a^2 + ab - b^2 \neq 0$ for all $a, b \in \mathbb{F}^*$ and $a \neq 0$ or $b \neq 0$.*

Proof. Let $\mathcal{Q}(\mathbb{F}^*)$ is a field. So each non-zero element is invertible i.e. for any $Q^{(n)}_{(a,b)}$ with $a \neq 0$ or $b \neq 0$, $\det\left(Q^{(n)}_{(a,b)}\right) \neq 0$. From theorem 4 we have $Q^{(\pm n)}_{(a,b)} = Q^{\pm n}_{(0,1)}Q^{(0)}_{(a,b)}$ for $n \geq 0$. Therefore,

$$\begin{aligned}
\det(Q^{(\pm n)}_{(a,b)}) &= \det(Q^{\pm n}_{(0,1)}) \det(Q^{(0)}_{(a,b)}) \\
&= (\det(Q_{(0,1)}))^{\pm n} \det(Q^{(0)}_{(a,b)}) \\
&= \left(\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)\right)^{\pm n} \det\left(\begin{bmatrix} b & a \\ a & b-a \end{bmatrix}\right) \\
&= (-1)^{\pm n}(b^2 - ab - a^2) \\
&= (-1)^{\pm n+1}(a^2 + ab - b^2).
\end{aligned}$$

Thus

$$\det(Q^{(n)}_{(a,b)}) = (-1)^{\pm n+1}(a^2 + ab - b^2) \tag{3.5}$$

So $a^2+ab-b^2 \neq 0$ for all $a, b \in \mathbb{F}^*$ and $a \neq 0$ or $b \neq 0$.
Conversely let $\mathbb{F}^*$ is a field in which there is no non-trivial solutions of the equation $a^2+ab-b^2=0$ for a and b. From theorem 3 it is clear that $\mathcal{Q}(\mathbb{F}^*)$ is a commutative ring with identity. Now it is enough to show that each non-zero element of $\mathcal{Q}(\mathbb{F}^*)$ is invertible with respect to multiplication. Let $Q^{(n)}_{(a,b)}$ be a non-zero element in $\mathcal{Q}(\mathbb{F}^*)$. Then either $a \neq 0$ or $b \neq 0$, otherwise by definition $a=b=0$ implies $Q^{(n)}_{(a,b)} = Q^{(n)}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So from equation (3.5), $Q^{(n)}_{(a,b)}$ is invertible in $\mathcal{Q}(\mathbb{F}^*)$ for all $n \in \mathbb{Z}$. Hence $\mathcal{Q}(\mathbb{F}^*)$ is a field. □

Corollary 1. *The set $\mathcal{Q}(\mathbb{Q})$ is Fibonacci Field over $\mathbb{Q}$, the set of all rational numbers. However both $\mathcal{Q}(\mathbb{R})$ and $\mathcal{Q}(\mathbb{C})$ are not Fibonacci Fields over $\mathbb{R}$ and $\mathbb{C}$ respectively.*

Proof. To prove the result its enough to show that the equation $a^2+ab-b^2=0$ has no solution for a and b in $\mathbb{Q}$, the set of rational numbers. Let us consider that it has a solution in $\mathbb{Q}$. So $a^2+ab-b^2=0 \Rightarrow \left(\frac{a}{b}\right)^2 + \frac{a}{b} - 1 = 0 \Rightarrow \frac{a}{b} = \frac{-1\pm\sqrt{5}}{2}$ $\Rightarrow a = b\left(\frac{-1\pm\sqrt{5}}{2}\right) \Rightarrow$ one of a and b is an irrational number. So the equation $a^2+ab-b^2=0$ has no solution for a and b in $\mathbb{Q}$.
Each of $\mathcal{Q}(\mathbb{R})$ and $\mathcal{Q}(\mathbb{C})$ is a commutative ring with identity but not field because the non-zero element $Q^{(n)}_{\left(b\left(\frac{-1\pm\sqrt{5}}{2}\right),b\right)}$ is not invertible in $\mathcal{Q}(\mathbb{R})$ or $\mathcal{Q}(\mathbb{C})$. □

4. f-Inverse Initial Value

In previous section it has been shown that the matrix $Q^{(n)}_{(a,b)}$ over a commutative ring with identity R is invertible if $a^2+ab-b^2 \neq 0$. In this section we find the multiplicative inverse of $Q^{(n)}_{(a,b)}$ over R and its location if exists. For this we introduce the following definition of f-inverse initial value.

Definition 4. Let R be a commutative ring with identity. An ordered pair of initial values $(a_1, b_1) \in R^2$ is said to be f-inverse initial value to the ordered pair of initial values $(a_2, b_2) \in R^2$ and vice-versa if $Q^{(n)}_{(a_1,b_1)} Q^{(-n)}_{(a_2,b_2)} = I_2$ for all $n \in \mathbb{Z}$.
Further the f-congruent class of (a_1, b_1) is said to be the inverse class of the f-congruent class of (a_2, b_2).

Theorem 6. *Let R be a commutative ring with identity. For any ordered pair of initial value $(a,b) \in R^2$ with $a^2+ab-b^2 \neq 0$ the f-inverse initial value of (a,b) is given by $\left(\frac{-a}{b^2-ab-a^2}, \frac{b-a}{b^2-ab-a^2}\right)$.*

Proof. We have

$$Q^{(0)}_{(a,b)} Q^{(0)}_{\left(\frac{-a}{b^2-ab-a^2}, \frac{b-a}{b^2-ab-a^2}\right)} = \begin{bmatrix} b & a \\ a & b-a \end{bmatrix} \begin{bmatrix} \frac{b-a}{b^2-ab-a^2} & \frac{-a}{b^2-ab-a^2} \\ \frac{-a}{b^2-ab-a^2} & \frac{b}{b^2-ab-a^2} \end{bmatrix} = I_2.$$

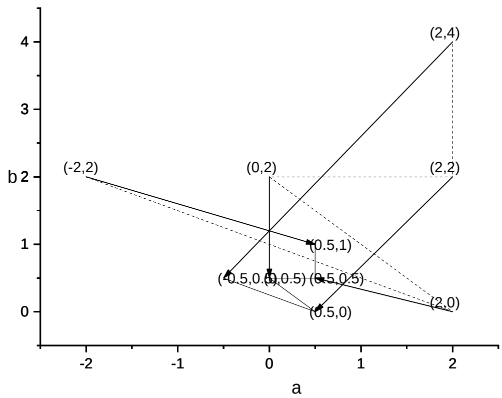

Figure 1. ***f*-inverse initial values and classes.**

Therefore

$$Q^{(n)}_{(a,b)}Q^{(-n)}_{\left(\frac{-a}{b^2-ab-a^2},\frac{b-a}{b^2-ab-a^2}\right)} = Q^{n}_{(0,1)}Q^{(0)}_{(a,b)}Q^{(0)}_{\left(\frac{-a}{b^2-ab-a^2},\frac{b-a}{b^2-ab-a^2}\right)}Q^{-n}_{(0,1)} = I_2.$$

Hence $Q^{(-n)}_{\left(\frac{-a}{b^2-ab-a^2},\frac{b-a}{b^2-ab-a^2}\right)}$ is the multiplicative inverse of $Q^{(n)}_{(a,b)}$ and vice-versa. Also the f-inverse initial value of (a,b) is $\left(\frac{-a}{b^2-ab-a^2},\frac{b-a}{b^2-ab-a^2}\right)$. □

Note-1. The inverse of nth term of the two ended sequence $\left\{Q^{(k)}_{(a,b)}\right\}_{k\in\mathbb{Z}}$ is the $(-n)$th term of the two ended sequence $\left\{Q^{(k)}_{\left(\frac{-a}{b^2-ab-a^2},\frac{b-a}{b^2-ab-a^2}\right)}\right\}_{k\in\mathbb{Z}}$.

Graphically in figure 1 the vertices on the dotted curve is generated by the Fibonacci sequence with initial value $(0,2)$ and those on the solid curve is generated by the Fibonacci sequence with initial value $(0,0.5)$. The arrow locates the matrix $Q^{(n)}_{(0,2)}$ at $(f_{(0,2)}(n), f_{(0,2)}(n+1))$ to its inverse $Q^{(-n)}_{(0,0.5)}$ at $(f_{(0,0.5)}(-n), f_{(0,0.5)}(-n+1))$ and vice-versa for all $n\in\mathbb{Z}$.

i.e. the inverse of $Q^{(0)}_{(0,2)} = \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$ at $(0,2)$ is $Q^{(0)}_{(0,0.5)} = \begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix}$ at $(0,0.5)$,

the inverse of $Q^{(1)}_{(0,2)} = \begin{bmatrix}2 & 2\\ 2 & 0\end{bmatrix}$ at $(2,2)$ is $Q^{(-1)}_{(0,0.5)} = \begin{bmatrix}0 & 0.5\\ 0.5 & -0.5\end{bmatrix}$ at $(0.5,0)$,

the inverse of $Q^{(2)}_{(0,2)} = \begin{bmatrix}4 & 2\\ 2 & 2\end{bmatrix}$ at $(2,4)$ is $Q^{(-2)}_{(0,0.5)} = \begin{bmatrix}0.5 & -0.5\\ -0.5 & 1\end{bmatrix}$ at $(-0.5,0.5)$ and vice-versa. So on. Similarly in other direction,

the inverse of $Q^{(-1)}_{(0,2)} = \begin{bmatrix}0 & 2\\ 2 & -2\end{bmatrix}$ at $(2,0)$ is $Q^{(1)}_{(0,0.5)} = \begin{bmatrix}0.5 & 0.5\\ 0.5 & 0\end{bmatrix}$ at $(0.5,0.5)$,

the inverse of $Q^{(-2)}_{(0,2)} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$ at $(-2,2)$ is $Q^{(2)}_{(0,0.5)} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ at $(0.5,1)$ and vice-versa. So on.
Hence the order pair $(0,2) \in \mathbb{R}^2$ is the f-inverse initial value of the order pair $(0,0.5) \in \mathbb{R}^2$. The f-congruent class of vertices of solid curve is the inverse class of the f-congruent class of vertices of the dotted curve in fig. 5.

Note-2. The set $\mathcal{Q}(\mathbb{R})$ is also classified into equivalence classes according to f-congruent classes.

In order to find the self f-inverse initial values in a commutative ring R with identity, an ordered pair $(a,b) \in R^2$ is a self f-inverse initial value iff

$$Q^{(n)}_{(a,b)} Q^{(-n)}_{(a,b)} = I_2 \tag{4.1}$$

for all $n \in \mathbb{Z}$. In particular

$$Q^{(0)}_{(a,b)} Q^{(-0)}_{(a,b)} = Q^{(0)}_{(a,b)} Q^{(0)}_{(a,b)} = \begin{bmatrix} b & a \\ a & b-a \end{bmatrix} \begin{bmatrix} b & a \\ a & b-a \end{bmatrix} = I_2 \tag{4.2}$$

i.e.

$$a^2 + b^2 = 1 \quad \text{and} \quad 2ab - a^2 = 0 \tag{4.3}$$

Remark (1) For the ring $(\mathbb{Z},+,.)$, the initial values $(0,1)$ and $(0,-1)$ are only self f-inverse initial values.

Remark (2) Similarly for the field $(\mathbb{Q},+,.)$, the initial values $(0,1)$ and $(0,-1)$ are only self f-inverse initial values.

Remark (3) But for the field $(\mathbb{R},+,.)$, the initial values $(0,1)$, $(0,-1)$, $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ and $(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$ are self f-inverse initial values.

Remark (4) For the field $(\mathbb{C},+,.)$, let $(a_1 + ia_2, b_1 + ib_2)$ is a self f-inverse initial value then it satisfies the equation (4.3) and then we have equations

$$(a_1^2+b_1^2)-(a_2^2+b_2^2) = 1, \quad a_1a_2+b_1b_2 = 0 \text{ and } (a_1+ia_2)((2b_1-a_1)+i(2b_2-a_2)) = 0 \tag{4.4}$$

Hence the self f-inverse initial values are $(0,1)$, $(0,-1)$, $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ and $(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$.

Graphically in fig.(a) of figure 2 it can be observe that the vertices of the dotted curve represents the Fibonacci sequence with initial value $(0,1)$. The arrow locates the matrix $Q^{(n)}_{(0,1)}$ at $(f_{(0,1)}(n), f_{(0,1)}(n+1))$ to its inverse $Q^{(-n)}_{(0,1)}$ at $(f_{(0,1)}(-n), f_{(0,1)}(-n+1))$ and vice-versa for all $n \in \mathbb{Z}$. Since both the points for each n lies on the same curve, the order pair $(0,1)$ is a self f-inverse initial value.
Similarly the fig.(b), fig.(c) and fig.(d) of figure 2 demonstrate that the order pairs $(0,-1)$, $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ and $(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$ are self f-inverse initial values respectively.

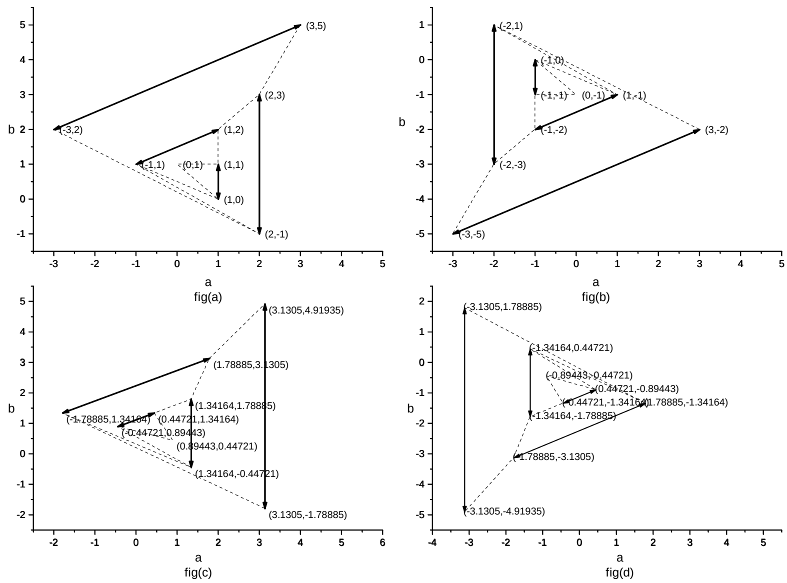

Figure 2. **Self f-inverse initial values and classes.**

Remark (5) From theorem (4) we may also conclude that

$$Q^{(n)}_{(0,-1)} = -Q^{n}_{(0,1)}; \quad n \geq 0 \quad \text{as} \quad Q^{(0)}_{(0,-1)} = -I_2 \tag{4.5}$$

and

$$Q^{(-n)}_{(0,-1)} = -Q^{-n}_{(0,1)}; \quad n \geq 0 \quad \text{as} \quad Q^{(0)}_{(0,-1)} = -I_2 \tag{4.6}$$

5. Conclusion

In the context formation of algebraic structures for the set Fibonacci matrices, the initial value space has a vital role. If the initial value space is a ring (commutative ring with identity) then the set of all Fibonacci matrices over the initial value space form a Fibonacci Ring (commutative ring with identity). However the set of all Fibonacci matrices over a field forms a Fibonacci Field under a certain condition. It has been observed that for the initial space $\mathbb{Q}$, the set of rational numbers it forms a Fibonacci Field but for that of $\mathbb{R}$ or $\mathbb{C}$ it doesn't. The formation of such algebraic structures may lead further applications of Fibonacci sequence and matrix in several areas of mathematics like algebra, combinatorics, coding theory, cryptography etc. Further a new concept of f-inverse initial value as well as inverse f-congruent class of initial values with an equivalence relation f-congruent has been introduced and location of inverse element for each invertible element of the ring of Fibonacci matrices. Finally the self inverse initial values has been obtained for each of the initial value spaces $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ which makes a wide scopes of future work.

Statements and Declarations

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References

[1] F.M. Dannan and Q. Doha, *Fibonacci Q-type matrices and properties of a class of numbers related to the Fibonacci, Lucas and pell numbers*, Commun. Appl. Anal. **9** (2005), no. 2, 247–262.

[2] F. Eugeni and R. Mascella, *A note on generalized Fibonacci numbers*, J. Discrete Math. Sci. Cryptogr. **4** (2001), no. 1, 33–45. https://doi.org/10.1080/09720529.2001.10697917.

[3] H.W. Gould, *A history of the Fibonacci Q-matrix and a higher-dimensional problem*, Fibonacci Quart. **19** (1981), no. 3, 250–257. https://doi.org/10.1080/00150517.1981.12430088.

[4] R.C. Johnson, *Fibonacci Numbers and Matrices*, Durham University, 2009.

[5] T. Koshy, *Fibonacci and Lucas Numbers with Applications, Volume 2*, vol. 2, John Wiley & Sons, 2019.

[6] K. Prasad and H. Mahato, *Cryptography using generalized Fibonacci matrices with Affine-Hill cipher*, J. Discrete Math. Sci. Cryptogr. **25** (2022), no. 8, 2341–2352.
https://doi.org/10.1080/09720529.2020.1838744.

[7] S.Q. Shen, J.M. Cen, and Y. Hao, *On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers*, Appl. Math. Comput. **217** (2011), no. 23, 9790–9797.
https://doi.org/10.1016/j.amc.2011.04.072.

[8] S.Q. Shen and J.J. He, *Moore–Penrose inverse of generalized Fibonacci matrix and its applications*, Int. J. Comput. Math. **93** (2016), no. 10, 1756–1770. https://doi.org/10.1080/00207160.2015.1074189.

[9] J.Y. Sia and C.K. Ho, *Algebraic properties of generalized Fibonacci sequence via matrix methods*, J. Eng. Appl. Sci. **11** (2016), no. 11, 2396–2401.

[10] R.B. Taher and M. Rachidi, *Linear recurrence relations in the algebra of matrices and applications*, Linear Algebra Appl. **330** (2001), no. 1-3, 15–24. https://doi.org/10.1016/S0024-3795(01)00259-2.