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Research Article

A study on the complement graph of the completely separated topological graph

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Abstract: In this paper, we study $\overline{G(\tau)}$, the complement graph of the completely separated topological graph, and its line graph $L(\overline{G(\tau)})$ on a topological space (X, τ). We show that for a discrete topological space (X, τ), $\overline{G(\tau)}$ is Hamiltonian and Eulerian if and only if $|X| \geq 3$, and for any topological space (X, τ) such that $|X| \geq 3$, $e(X\backslash\{p\}) = 2$ for all $p \in X$ if and only if (X, τ) is a discrete space. Also, for any T_1 topological space (X, τ), $dt(\overline{G(\tau)}) = 2$ if and only if X has at least one isolated point. Finally, if (X, τ_X) and (Y, τ_Y) are discrete topological spaces such that $|X| \geq 3$ and $|Y| \geq 3$, then $\overline{G(\tau_X)}$ is isomorphic to $\overline{G(\tau_Y)}$ if and only if X and Y are homeomorphic if and only if $L(\overline{G(\tau_X)})$ is isomorphic to $L(\overline{G(\tau_Y)})$.

Keywords: open set, topological spaces, continuous function, complete graph, dominating set.

AMS Subject classification: 05C25, 05C69, 05C07, 05C12

1. Introduction

Let G be a graph with vertex set $V(G)$ and A, B be two distinct vertices in G. The distance $d(A, B)$ is the length of the shortest path joining them in G, and if no such path exists, we set $d(A, B) = \infty$. The diameter of a graph G is defined as $diam(G) = sup\{d(A, B) : A,\ B \in V(G)\}$. The associate number $e(A)$ of a vertex A is defined as $e(A) = sup\{d(B, A) : B \neq A\}$. A vertex A is a center in G if $e(A) \leq e(B)$ for any vertex B in G. The radius of G is defined to be $\rho(G) = inf\{e(A) : A \in V(G)\}$. A graph G is connected if any two vertices are linked by a path in G; otherwise, G

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is disconnected. A circuit in a graph G is a closed trail of length 3 or more, where no edge is traversed more than once. A circuit that repeats no vertex, except for the first and last, is a cycle. $c(A, B)$ is the length of the shortest cycle containing both A and B. The girth of a graph is the length of its shortest cycle if it exists; otherwise, girth is ∞. A graph in which all vertices are pairwise adjacent is called a complete graph. A complete graph with n vertices is denoted by K_n and a cycle with n vertices is denoted by C_n. The clique number of a graph G is defined as $\omega(G) = sup\{|V(H)| : H \text{ is a complete subgraph of } G\}$, where $V(H)$ is the vertex set of H. A graph G is said to be triangulated (hypertriangulated) if each vertex (edge) is a vertex (an edge) of a triangle. The collection of all vertices of G adjacent to A is called an open neighbourhood of A. It is denoted by $N(A)$. The degree of a vertex A denoted by $deg(A)$, is the number of vertices adjacent to A. The closed neighbourhood $N[A]$ of A is the set $N(A)\bigcup\{A\}$. A vertex A is called simplicial if $N[A]$ induces a complete subgraph of G. A graph is called planar graph if it can be drawn in the plane so that no two of its edges crosses each other. A cycle in a graph G that contains every vertex of G is called a Hamiltonian cycle of G. A Hamiltonian graph is a graph that contains a Hamiltonian cycle. A circuit that contains every edge of G is called an Eulerian circuit. A connected graph that contains an Eulerian circuit is called an Eulerian graph. The complement graph $\overline{G}$ of a graph G is the graph having the same vertex set as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The line graph of G, denoted by $L(G)$, is a graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are incident to a common vertex in G. We denote the set $\mathbb{R}$ equipped with usual topology by $\mathbb{R}_u$. For undefined terms concerning topology and graph theory, we refer the reader to [8] and [7] respectively.

Beck [1] was the first to initiate the idea of studying graphs associated with algebraic structures. The benefit of studying graphs associated with algebraic structures is that one may find some results about algebraic structures using graphs and vice versa. A lot of research has been done on graphs defined on an algebraic structure such as groups, rings, vector spaces, modules, etc. In 2019, Muneshwar and Bondar [4] introduced the open subset inclusion graph $\gamma(\tau)$ on a finite topological space (X, τ), where the vertex set is the collection of proper open subsets of a topological space (X, τ) and two distinct vertices A and B are adjacent if either $A \subseteq B$ or $B \subseteq A$. The completely separated topological graph $G(\tau)$ [2] is defined on an arbitrary non-indiscrete topological space (X, τ), where the vertex set is the collection of proper open subsets of X and two vertices are adjacent if and only if there is a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(A) \leq r < s \leq f(B)$, $r,\ s \in \mathbb{R}$, where $f(A) \leq r$ means $f(a) \leq r$ for every $a \in A$ and $f(B) \geq s$ means $f(b) \geq s$ for every $b \in B$. The main objective of studying graphs associated with a topological space is to find interrelationships between graph theoretical results and topological results.

In this paper, we study the complement graph $\overline{G(\tau)}$ of the completely separated topological graph $G(\tau)$. We show that for any T_1 topological space X, $\overline{G(\tau_X)}$ is connected if and only if $|X| \geq 3$, and $\overline{G(\tau_X)}$ is triangulated and hypertriangulated if and only if $|X| \geq 3$. Also, for a discrete topological space such that $|X| \geq 3$, $\overline{G(\tau_X)}$

is Hamiltonian and Eulerian. We also show that a vertex A of $\overline{G(\tau)}$ is simplicial if and only if A is a singleton and $\{A, X\backslash A\}$ dominate $\overline{G(\tau)}$ if and only if A is an open and closed subset of X. We determine the clique number of $\overline{G(\tau)}$ for a discrete space, and for any T_1 space, we observe that $dt(\overline{G(\tau)}) = 2$ if and only if X has at least one isolated point. In the last section, we study some properties of $L(\overline{G(\tau)})$ and show that $L(\overline{G(\tau)})$ is isomorphic to $\overline{G(\tau)}$ if and only if the topology on X contains exactly three non-trivial open sets and every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.

2. The complement graph of the completely separated topological graph

Definition 1. [3] Two subsets A and B of a topological space X are said to be completely separated if there exists a continuous function $g : X \longrightarrow [0, 1]$ such that $g(A) = 0$ and $g(B) = 1$.

Remark 1. The next theorem provides an alternative expression of the notion of complete separation of subsets of a topological space. This well-known result in topology will be particularly useful in our study. It is brought to our notice by the learned referee regarding the existence of alternative proofs of this result in literature, as can be seen in [5, Page 96].

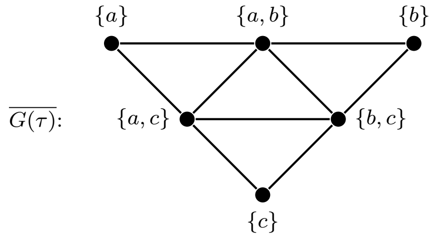
Theorem 1. *Let A and B be two subsets of a topological space X. Then, A, B are completely separated if and only if there is a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(A) \leq r < s \leq f(B)$, $r,\ s \in \mathbb{R}$, where $f(A) \leq r$ means $f(a) \leq r$ for every $a \in A$ and $f(B) \geq s$ means $f(b) \geq s$ for every $b \in B$.*

Proof. Suppose $f : X \longrightarrow \mathbb{R}_u$ is a continuous function such that $f(A) \leq r < s \leq f(B)$, $r,\ s \in \mathbb{R}$. Then, $f_1 : X \longrightarrow \mathbb{R}_u$ is continuous, where $f_1(x) = \frac{f(x)-r}{s-r}$. It is clear that $f_1(A) \leq 0 < 1 \leq f_1(B)$ and $g_1 : X \longrightarrow \mathbb{R}_u$ defined by $g_1(x) = \frac{f_1(x)+|f_1(x)|}{2}$, is continuous. Also, $g : X \longrightarrow \mathbb{R}_u$ defined by $g(x) = \frac{g_1(x)+1-|g_1(x)-1|}{2}$, is continuous. If $x \in A$, then $f_1(x) \leq 0$ and $g_1(x) = 0$. This implies that $g(x) = 0$ for all $x \in A$. If $x \in B$, then $f_1(x) \geq 1$ and $g_1(x) \geq 1$. This implies that $g(x) = 1$ for all $x \in B$. This shows that $g(A) = 0$ and $g(B) = 1$ and hence A, B are completely separated. □

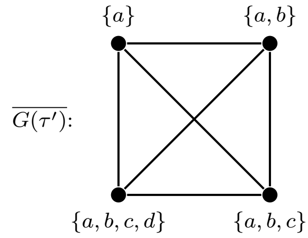
Definition 2. Let (X, τ) be any non-indiscrete topological space. The complement graph of the completely separated topological graph denoted by $\overline{G(\tau)}$, is a graph whose vertex set is the collection of all proper open subsets of X with two vertices $A,\ B$ adjacent if and only if A and B are not completely separated.

Example 1. For the discrete space (X, τ), where $X = \{a, b, c\}$, $\overline{G(\tau)}$ is shown in $Fig(i)$. For the topological space (X, τ'), where $X = \{a, b, c, d, e\}$ with $\tau' = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$, $\overline{G(\tau')}$ is shown in $Fig(ii)$.

Remark 2. Since any two intersecting subsets A, B of a topological space (X, τ) cannot be completely separated, therefore such subsets A, B are adjacent as vertices in $\overline{G(\tau)}$. However, the converse of this is false. To see this, we consider $X = \{1, 2, 3\}$ with topology $\tau =$



Fig(i)



Fig(ii)

$\{\phi, \{1\}, \{2\}, \{1, 2\}, X\}$. Then, as every real-valued continuous function on X is constant, $A = \{1\}$ and $B = \{2\}$ cannot be completely separated and as such are adjacent in $\overline{G(\tau)}$.

Theorem 2. *Let (X, τ) be any topological space. Then, $\overline{G(\tau)}$ is a complete graph if and only if every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.*

Proof. Suppose $\overline{G(\tau)}$ is a complete graph. If possible, let $g : X \longrightarrow \mathbb{R}_u$ be a non-constant continuous function, say $g(a) = r < s = g(b)$, $a, b \in X$. Let $z = \frac{r+s}{2}$. Then, $g^{-1}((-\infty, z))$ and $g^{-1}((z, \infty))$ are distinct, proper, open subsets of X, which are completely separated by g. This shows that $g^{-1}((-\infty, z))$ and $g^{-1}((z, \infty))$ are not adjacent in $\overline{G(\tau)}$, which contradicts the fact that $\overline{G(\tau)}$ is a complete graph. The converse part is straightforward. □

Corollary 1. *Let (X, τ) be a non-indiscrete topological space with $|\tau| > 2$, say $|\tau| = n+2$ for some $n \in \mathbb{N}$. Then, $\overline{G(\tau)} = K_n$ if and only if every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.*

Theorem 3. *Let (X, τ) be any topological space. Then, $\overline{G(\tau)}$ is a cycle if and only if τ has exactly three distinct proper open subsets and every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.*

Proof. Suppose $\overline{G(\tau)}$ is a cycle C_n, $n \geq 3$.
Case (i). $n = 3$. In this case, there exists three distinct vertices A, B, C such that $A - B - C - A$ is a cycle. This shows that $\tau = \{\phi, A, B, C, X\}$. If possible, let $f : X \longrightarrow \mathbb{R}_u$ be a non-constant continuous function, with $f(p) = r < s = f(q)$ for some p, $q \in X$. Put $z = \frac{r+s}{2}$. Then, $f^{-1}((-\infty, z))$ and $f^{-1}((z, \infty))$ are proper open subsets in X which are completely separated by f, hence they are not adjacent in $\overline{G(\tau)}$, which is a contradiction as $\overline{G(\tau)} = C_3$. Therefore, every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.
Case (ii). $n = 4$. In this case, there exist four distinct vertices A, B, C, D such that $A - B - C - D - A$ is a cycle. Since A and C are not adjacent, this implies that $A \bigcap C = \phi$. Similarly, $B \bigcap D = \phi$. We can now consider two subcases: $A \bigcap B \neq \phi$ and $A \bigcap B = \phi$.
If $A \bigcap B \neq \phi$, then $A \bigcap B$ can assume one of these values - $A, B, C,$ or D. It is clear

that $A \bigcap B = C$ and $A \bigcap B = D$ is not possible since $A \bigcap C = \phi$ and $B \bigcap D = \phi$. If $A \bigcap B = A$, then $A \subseteq B$. As B and D are not adjacent, there exists a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(B) \leq r < s \leq f(D)$, $r,\ s \in \mathbb{R}$. Since $A \subseteq B$, so $f(A) \leq r < s \leq f(D)$, $r,\ s \in \mathbb{R}$, which contradicts the fact that $A,\ D$ are adjacent. Similarly, if $A \bigcap B = B$, then $B \subseteq A$. Since A and C are not adjacent, so there exists a continuous function f such that $f(A) \leq r < s \leq f(C)$, $r,\ s \in \mathbb{R}$. Since $B \subseteq A$, so $f(B) \leq r < s \leq f(C)$, $r,\ s \in \mathbb{R}$, which contradicts the fact that $B,\ C$ are adjacent.
If $A \bigcap B = \phi$, then $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C) = \phi$. This implies that $B \bigcup C \neq A$, so that $B \bigcup C = B$ *or* C *or* D. If $B \bigcup C = B$, then $C \subseteq B$. Again, since B and D are not adjacent, there exists a continuous function f such that $f(B) \leq r < s \leq f(D)$, $r,\ s \in \mathbb{R}$. This implies that $f(C) \leq r < s \leq f(D)$, which contradicts the fact that C and D are adjacent. Similarly, the equality $B \bigcup C = C$ results in a contradiction. Finally, if $B \bigcup C = D$, then $B \bigcap D \neq \phi$, so that B and D will be adjacent, which is not possible. This shows that $\overline{G(\tau)}$ is never C_4.
Case(iii). $n \geq 5$. In this case, there exists four distinct vertices $A,\ B,\ C,\ D$ such that $-A - B - C - D-$ is part of the cycle C_n, $n \geq 5$. Then, $A \bigcap C = \phi$, $A \bigcap D = \phi$, and $B \bigcap D = \phi$. We have two subcases: $A \bigcap B = \phi$ or $A \bigcap B \neq \phi$. If $A \bigcap B = \phi$, then $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C) = \phi$. This implies that $B \bigcup C \neq A$, so that $B \bigcup C = B$ *or* C *or* D *or* $B \bigcup C$ is a new vertex distinct from $B,\ C,\ D$. If $B \bigcup C$ is a new vertex distinct from $B,\ C,\ D$, then $B - B \bigcup C - C - B$ is a triangle, which is not possible. If $B \bigcup C = B$ *or* C *or* D, we get a contradiction as in case (ii). Next, if $A \bigcap B \neq \phi$, then either $A \bigcap B = A$ *or* B *or* C *or* D or $A \bigcap B$ is a new vertex distinct from $A,\ B,\ C,\ D$. If $A \bigcap B$ is a new vertex distinct from $A,\ B,\ C,\ D$, then $A - B - A \bigcap B - A$ forms a triangle, which is not possible. As in Case (ii), if $A \bigcap B = A$ or $A \bigcap B = B$ or $A \bigcap B = C$ or $A \bigcap B = D$, then we will get a contradiction. This shows that $\overline{G(\tau)}$ is never C_n, for $n \geq 5$.
Therefore, if $\overline{G(\tau)}$ is a cycle, then $\overline{G(\tau)} = C_3$ and, every continuous functions $f : X \longrightarrow \mathbb{R}_u$ is constant. The proof of the converse part is straightforward. □

Theorem 4. *Let (X, τ) be any T_1 topological space. Then, $\overline{G(\tau_X)}$ is connected if and only if $|X| \geq 3$.*

Proof. Suppose $\overline{G(\tau_X)}$ is connected. If possible, let $|X| = 2$, say, $X = \{a, b\}$. Since the topology is T_1, it follows that X must be a discrete space in this case. However, the open sets $\{a\}, \{b\}$ are completely separated, hence they are not adjacent in $\overline{G(\tau_X)}$, which is a contradiction. Conversely, let A and B be any two proper open subsets of X. If $A \bigcap B \neq \phi$, then by Remark 2, A and B are adjacent. If $A \bigcap B = \phi$, then, as $|X| \geq 3$, we see that either both A, B are singleton sets, or $A($ *or* $B)$ is not singleton sets. In case both A and B are singleton sets, then $A - A \bigcup B - B$ is a path joining A and B. If A is not a singleton set, say $\exists\ a \in A$ with $a \notin B$, then $A - X \backslash \{a\} - B$ is a path connecting A and B. If B is not a singleton set, say $\exists\ b \in B$ with $b \notin A$, then $B - X \backslash \{b\} - A$ is a path connecting A and B. Similarly, if both A and B are not singleton then it is easy to get a path connecting A and B. Hence, $\overline{G(\tau_X)}$ is connected. $\square$

Corollary 2. *Let (X, τ) be any T_1 topological space such that $|X| \geq 3$. Then, $diam(\overline{G(\tau_X)}) \leq 2$.*

Corollary 3. *Let (X, τ) be a discrete topological space and A, B be any two proper open subsets of X such that $|X| \geq 3$. Then,*
(i) $d(A, B) = 1$ if and only if $A \bigcap B \neq \phi$.
(ii) $d(A, B) = 2$ if and only if $A \bigcap B = \phi$.
(iii) $\rho(\overline{G(\tau_X)}) = 2$.

Corollary 4. *Let (X, τ) be a discrete topological space and A, B be any two proper open subsets of X such that $|X| \geq 3$. Then,*
(i) $c(A, B) = 3$ if and only if $A \bigcap B \neq \phi$.

(ii) For $|X| = 3$, $\begin{cases} c(A,B) = 4 \text{ if and only if } A \bigcap B = \phi \text{ and } A \bigcup B = X. \\ c(A,B) = 5 \text{ if and only if } A \bigcap B = \phi \text{ and } A \bigcup B \neq X. \end{cases}$

(iii)For $|X| \geq 4$, $c(A, B) = 4$ if and only if $A \bigcap B = \phi$.

Theorem 5. *Let (X, τ) be a T_1 topological space and $p \in X$ such that $|X| \geq 3$. Then, $e(X \backslash \{p\}) = 2$ if and only if p is an isolated point of X.*

Proof. If p is not an isolated point of X, then $\{p\}$ is not a vertex of $\overline{G(\tau_X)}$. This shows that for any vertex A of $\overline{G(\tau_X)}$, $A \bigcap (X \backslash \{p\}) \neq \phi$. This shows that $e(X \backslash \{p\}) = 1$, which is a contradiction. Hence, p is an isolated point of X. Conversely, suppose p is an isolated point of X. As (X, τ) is T_1, $\{p\}$ is both open and closed. Notice that the characteristic function of $\{p\}$ completely separates $\{p\}$ and $X \backslash \{p\}$. This shows that $\{p\}$ is not adjacent to $X \backslash \{p\}$. Hence, it follows that $e(X \backslash \{p\}) = 2$. $\square$

Corollary 5. *Let (X, τ) be any topological space such that $|X| \geq 3$. Then, $e(X \backslash \{p\}) = 2$ for all $p \in X$ if and only if (X, τ) is a discrete space.*

Theorem 6. *Let (X, τ) be any T_1 topological space. Then, $\overline{G(\tau)}$ is triangulated if and only if $|X| \geq 3$.*

Proof. Suppose $\overline{G(\tau)}$ is triangulated. Then, it is obvious that $|X| \geq 3$. Conversely, let A be a proper open subset of X. As $|X| \geq 3$, there exist three distinct points $a,\ b,\ c \in X$ such that $a \in A$. As X is T_1, $X \backslash \{b\}$, $X \backslash \{c\}$ are proper open subsets of X. If A is distinct from $X \backslash \{b\}$ and $X \backslash \{c\}$, then as $A \bigcap (X \backslash \{b\}) \neq \phi$, $A \bigcap (X \backslash \{c\}) \neq \phi$, and $(X \backslash \{b\}) \bigcap (X \backslash \{c\}) \neq \phi$, it follows that $A - (X \backslash \{b\}) - (X \backslash \{c\}) - A$ is a triangle. If $A = X \backslash \{b\}$, then $a,\ c \in A$ and $A - (X \backslash \{a\}) - (X \backslash \{c\}) - A$ is a triangle. Similarly, if $A = X \backslash \{c\}$, then $a,\ b \in A$, and $A - (X \backslash \{a\}) - (X \backslash \{b\}) - A$ is a triangle. Hence, $\overline{G(\tau)}$ is triangulated. $\square$

Theorem 7. *Let (X, τ) be any T_1 topological space. Then, $\overline{G(\tau)}$ is hypertriangulated if and only if $|X| \geq 3$.*

Proof. Suppose $\overline{G(\tau)}$ is hypertriangulated. Then, it is obvious that $|X| \geq 3$. Conversely, let A and B be two proper open subsets of X such that AB is an edge of $\overline{G(\tau)}$. If $A \bigcap B \neq \phi$, then there exists $x \in A \bigcap B$. Let $y \in X$ such that $y \neq x$. Then, it is easy to see that $A - (X \backslash \{y\}) - B - A$ is a triangle. If $A \bigcap B = \phi$, let $a \in A$ and $b \in B$ such that $a \neq b$. As $|X| \geq 3$, there exists $x \in X \backslash \{a, b\}$ and $A - (X \backslash \{x\}) - B - A$ is a triangle since $A \bigcap (X \backslash \{x\}) \neq \phi$ and $B \bigcap (X \backslash \{x\}) \neq \phi$. This shows that $\overline{G(\tau)}$ is hypertriangulated.

□

Theorem 8. *Let (X, τ) be any T_1 topological space. Then, $\overline{G(\tau)}$ is planar if and only if $|X| \leq 3$.*

Proof. Suppose $\overline{G(\tau)}$ is planar. Let A be any proper open subset of X with $a \in A$ for some $a \in X$. If $|X| \geq 4$, then there exists three distinct points $b,\ c,\ d \in X$ different from a. If A is singleton, then A is adjacent to $X \backslash \{b\}$, $X \backslash \{c\}$, $X \backslash \{d\}$, $X \backslash \{b, c\}$, $X \backslash \{b, d\}$, and $X \backslash \{c, d\}$. This shows that $deg(A) \geq 6$. Similarly, it is easy to see that if $|A| > 1$, then $deg(A) \geq 6$. But, A is an arbitrary proper open subset of X, so that the degree of every vertex in $\overline{G(\tau)}$ is ≥ 6, which is a contradiction as $\overline{G(\tau)}$ is assumed to be planar. The converse part is straightforward.

□

Theorem 9. *Let (X, τ) be any discrete topological space such that $|X| = n$. Then, $\overline{G(\tau)}$ is a Hamiltonian graph if and only if $|X| \geq 3$.*

Proof. Suppose $\overline{G(\tau)}$ is a Hamiltonian graph. Then, it is obvious that $|X| \geq 3$. Conversely, suppose $|X| \geq 3$, say $X = \{x_1, x_2, x_3, ..., x_n\}$, $n \geq 3$. For each $i \in \{1, 2, 3, ..., n\}$ let $\beta_i = \{A \in \tau \backslash \{X\} : x_i \in A\}$. It is easy to see that $V(\overline{G(\tau)}) = \bigcup_{i=1}^{n} \beta_i$. Clearly, each β_i induces a complete subgraph. Consider $V_1 = \beta_1 \backslash \{\{x_1, x_2\}\}$, $V_2 = \beta_2 \backslash \{\{x_2, x_3\}\}$,..., $V_n = \beta_n \backslash \{\{x_n, x_1\}\}$. Then, each V_i induces a complete subgraph. Now, consider a path beginning at $\{x_1\}$ and passing through all vertices in V_1 except $\{x_1, x_n\}$, then continue this path through $\{x_1, x_2\}$ to all the vertices of V_2 except those vertices that have already passed through before, then through $\{x_2, x_3\}$ to all the vertices of V_3 except those vertices that have already passed through before. Continuing in this way until this path passes through all those vertices of V_n that have not been passed before. Finally, we join this path to $\{x_1\}$ through $\{x_n, x_1\}$, which is a cycle that contains all vertices of $\overline{G(\tau)}$. This shows that $\overline{G(\tau)}$ is a Hamiltonian graph. □

Theorem 10. *Let (X, τ) be a discrete topological space such that $|X| = n$. Then, $\overline{G(\tau)}$ is an Eulerian graph if and only if $|X| \geq 3$.*

Proof. If $\overline{G(\tau)}$ is an Eulerian graph, then it is obvious that $|X| \geq 3$. Conversely, let $|X| = n \geq 3$, say, $X = \{x_1, x_2, \ldots, x_n\}$. Let S be any proper subset of X. It is enough to show that the degree of S is even. Without loss of generality, we can assume that $S = \{x_1, x_2, ..., x_m\}$, $1 \leq m < n$. Notice that the degree of S equals $|\alpha| + |\beta| + |\gamma|$, where $\alpha = \{A \in V(\overline{G(\tau)}) \mid \phi \neq A \subsetneq S\}$, $\beta = \{B \in V(\overline{G(\tau)}) \mid B \supsetneq S\}$ and $\gamma = \{C \in V(\overline{G(\tau)}) \mid C \notin \alpha \bigcup \beta, C \bigcap S \neq \phi\}$. It is clear that S is adjacent to every proper subset of S and $|\alpha| = 2^m - 2$. Also, there are $2^{n-m} - 2$ proper supersets of S and S is adjacent to each one of them, thus it follows that $|\beta| = 2^{n-m} - 2$. Let T be a subset of $S^c = \{x_{m+1}, x_{m+2}, ..., x_n\}$ such that $T \neq \phi$. The number of such subsets T of X is $2^{n-m} - 1$. If $D = \{x_i\} \bigcup T$ where $i \in \{1, 2, ..., m\}$, then $D \bigcap S \neq \phi$ and the number of such proper subsets is $m(2^{n-m} - 1)$. If $D = \{x_{i_1}, x_{i_2}\} \bigcup T$ where $i_1,\ i_2$ are distinct members of $\{1, 2, ..., m\}$, then the number of such proper subsets is $(m(m-1)/2!)(2^{n-m} - 1)$. Continuing in this manner, if $D = \{x_{i_1}, x_{i_2}, ..., x_{i_j}\} \bigcup T$ where $i_1,\ i_2,\ \ldots,\ i_j$ are distinct members of $\{1, 2, ..., m\}$, then the number of such proper subsets is $(m!/((m-j)!j!)(2^{n-m} - 1)$. Therefore, the degree of a vertex S is $deg(S) = (2^m - 2) + (2^{(n-m)} - 2) + (2^{(n-m)} - 1)(\{\sum_{j=1}^{m-1}(m!/((m-j)!j!)\})$. This implies that $deg(S) = (2^m + 2^{n-m} - 4) + (2^{n-m} - 1)(2^m - 2)$, which is even since $\sum_{j=0}^{m}(m!/((m-j)!j!) = 2^m$. This shows that $\overline{G(\tau)}$ is an Eulerian graph. □

Theorem 11. *Suppose (X, τ_X) and (Y, τ_Y) are homeomorphic. Then, $\overline{G(\tau_X)}$ is isomorphic to $\overline{G(\tau_Y)}$.*

Proof. Let $h : X \longrightarrow Y$ be a homeomorphic map. Consider the map $\overline{h} : V(\overline{G(\tau_X)}) \longrightarrow V(\overline{G(\tau_Y)})$ defined by $\overline{h}(A) = h(A),\ \forall\ A \in V(\overline{G(\tau_X)})$. Clearly, $\overline{h}$ is bijective as h is bijective. Next, we show that $\overline{h}$ preserves adjacency. Suppose A and B are vertices of $\overline{G(\tau_X)}$ such that A and B are adjacent in $\overline{G(\tau_X)}$. Then, there doesn't exist any continuous map $f \colon X \longrightarrow \mathbb{R}_u$ such that $f(A) \leq r < s \leq f(B)$, for some $r,\ s \in \mathbb{R}$. Since h is a homeomorphism, $\overline{h}(A)$ and $\overline{h}(B)$ are open in Y. If $\overline{h}(A)$ and $\overline{h}(B)$ are not adjacent in $\overline{G(\tau_Y)}$, then there exist a continuous function $g : Y \longrightarrow \mathbb{R}_u$ such that $g(\overline{h}(A)) \leq r < s \leq g(\overline{h}(B))$, which is a contradiction as $g \circ h : X \longrightarrow \mathbb{R}_u$ is continuous and $A,\ B$ are adjacent in $\overline{G(\tau_X)}$. So we must have $\overline{h}(A)$ and $\overline{h}(B)$ adjacent in $\overline{G(\tau_Y)}$. Conversely, let $\overline{A}$ and $\overline{B}$ be two vertices of $\overline{G(\tau_Y)}$ such that $\overline{A}$ and $\overline{B}$ are adjacent in $\overline{G(\tau_Y)}$. As h is a homeomorphic map, we have $\overline{A} = h(A)$ and $\overline{B} = h(B)$ for some $A,\ B \in V(\overline{G(\tau_X)})$. We need to show that A and B are adjacent in $\overline{G(\tau_X)}$. As $\overline{A} = h(A)$ and $\overline{B} = h(B)$ are adjacent, there doesn't exists any continuous map $g : Y \longrightarrow \mathbb{R}_u$ such that $g(\overline{A} = h(A)) \leq r < s \leq g(\overline{B} = h(B))$, $r,\ s \in \mathbb{R}$. If A and B are not adjacent, then there exists a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(A) \leq r < s \leq f(B)$. This shows that $f \circ h^{-1} : Y \longrightarrow \mathbb{R}_u$ is continuous and $f \circ h^{-1}(\overline{A}) \leq r < s \leq f \circ h^{-1}(\overline{B})$, which contradicts the fact that $\overline{A}$ and $\overline{B}$ are adjacent in $\overline{G(\tau_Y)}$. This shows that A and B are adjacent in $\overline{G(\tau_X)}$. □

Corollary 6. *Let (X, τ_X) and (Y, τ_Y) be two discrete topological spaces. Then, the following conditions are equivalent:*
(i) $\overline{G(\tau_X)}$ is isomorphic to $\overline{G(\tau_Y)}$.
(ii) X and Y are homeomorphic.

Theorem 12. *Let (X, τ) be any discrete topological space. Then, a vertex A of $\overline{G(\tau)}$ is simplicial if and only if A is a singleton.*

Proof. Suppose A is simplicial, i.e., $N[A]$ induces a complete subgraph. If possible, suppose A is not a singleton set, say, there exists $p,\ q \in A,\ p \neq q$. It follows that $B_1 = \{p\}$ and $B_2 = \{q\}$ are elements of $N[A]$. But B_1 and B_2 are not adjacent, which contradicts the fact that $N[A]$ induces a complete graph. So, A must be a singleton set. Conversely, suppose $A = \{p\}$. Let $B_1,\ B_2 \in N[A]$. This shows that $p \in B_1$ and $p \in B_2$ and so B_1 is adjacent to B_2. Since B_1 and B_2 are arbitrary vertices, therefore, $N[A]$ induces a complete subgraph. □

Theorem 13. *Let (X, τ) be a discrete topological space and $A,\ B$ be any two vertices of $\overline{G(\tau)}$. Then, $N[A] \subseteq N[B]$ if and only if $A \subseteq B$.*

Proof. Suppose $N[A] \subseteq N[B]$. If possible $A \nsubseteq B$, then there exists $p \in A$ such that $p \notin B$. This shows that $P = \{p\} \in N[A]$ but $P \notin N[B]$, which contradicts the fact that $N[A] \subseteq N[B]$. Hence, $A \subseteq B$. The proof of the converse part is straightforward. □

Corollary 7. *Let (X, τ_X) and (Y, τ_Y) be two discrete topological spaces such that $\Phi : \overline{G(\tau_X)} \longrightarrow \overline{G(\tau_Y)}$ is a graph isomorphism, then for any two vertices $A,\ B$ of $\overline{G(\tau_X)}$, $A \subseteq B$ if and only if $\Phi(A) \subseteq \Phi(B)$.*

Proof. Suppose $A \subseteq B$. Then, by Theorem 13, $N[A] \subseteq N[B]$. Since Φ is a graph isomorphism, $N[\Phi(A)] \subseteq N[\Phi(B)]$. This implies that $\Phi(A) \subseteq \Phi(B)$. Conversely, suppose $\Phi(A) \subseteq \Phi(B)$, then $N[\Phi(A)] \subseteq N[\Phi(B)]$. Since Φ preserves adjacency, $N[A] \subseteq N[B]$. This shows that $A \subseteq B$. □

3. Dominating number and clique number

In this section, we study the dominating number and clique number of $\overline{G(\tau)}$. For a graph G, a dominating set is a set $\mathcal{D}$ of vertices such that any vertex in G either belongs to $\mathcal{D}$ or is adjacent to at least one member of $\mathcal{D}$. The dominating number of a graph G is defined as $dt(G) = inf\{|\mathcal{D}| : \mathcal{D}\ is\ a\ dominating\ set\ of\ G\}$.

Theorem 14. *Let (X, τ) be any topological space. Then, the set $\mathcal{C}_p = \{A \in \tau \setminus \{X\} : p \in A\}$ induces a complete subgraph of $\overline{G(\tau)}$.*

Proof. The proof follows from the fact that every members of $\mathcal{C}_p$ contain p. $\square$

Corollary 8. *Let (X, τ) be a discrete topological space such that $|X| = n$. Then, the clique number of $\overline{G(\tau)}$ is $2^{n-1} - 1$.*

Proof. The proof follows from the fact that the number of vertices of $\overline{G(\tau)}$ containing p is $2^{n-1} - 1$ and $\mathcal{C}_p$ is a largest collection of pairwise non-disjoint subsets of X. $\square$

Corollary 9. *Let (X, τ) be a T_1 topological space. Then, the set $\mathcal{C} = \{X \backslash \{p\} : p \in X\}$ induces a complete subgraph of $\overline{G(\tau)}$ if and only if $|X| \geq 3$.*

Theorem 15. *Let (X, τ) be a T_1 topological space and $p \in X$. Then, $\{\{p\}, X \backslash \{p\}\}$ dominate $\overline{G(\tau)}$ if and only if p is an isolated point of X.*

Proof. If $\{\{p\}, X \backslash \{p\}\}$ dominate $\overline{G(\tau)}$, then it is obvious that p is an isolated point of X. Conversely, suppose p is an isolated point of X. Let A be any proper open subset of X such that $A \neq \{p\}$ and $A \neq X \backslash \{p\}$. Then, it clear that either $p \in A$ or $A \bigcap (X \backslash \{p\}) \neq \phi$ as $X \backslash \{p\} \bigcup \{p\} = X$. This shows that A is adjacent to either $\{p\}$ or $X \backslash \{p\}$. Hence, $\{\{p\}, X \backslash \{p\}\}$ dominate $\overline{G(\tau)}$. $\square$

Corollary 10. *Let (X, τ) be any topological space. Then, $\mathcal{D} = \{\{p\}, X \backslash \{p\}\}$ dominates $\overline{G(\tau)}$ for each $p \in X$ if and only if (X, τ) is a discrete space.*

Corollary 11. *Let (X, τ) be a T_1 topological space. Then, $dt(\overline{G(\tau)}) = 2$ if and only if X has at least one isolated point.*

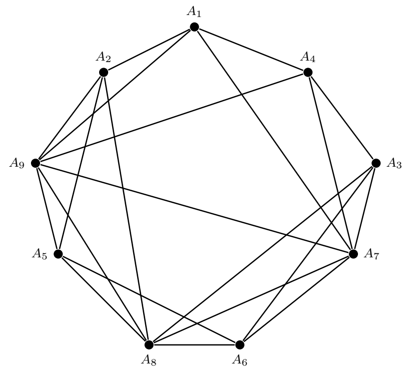
Corollary 12. *Let (X, τ) be any topological space and A be any vertex of $\overline{G(\tau)}$. Then, $\{A, X \backslash A\}$ dominate $\overline{G(\tau)}$ if and only if A is both open and closed as a subset of X.*

Remark 3. The Corollary 11 is not true if X is not a T_1 topological space. As an example, consider $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a, b\}, \{c, d\}, X\}$. Then, $\mathcal{D} = \{\{a, b\}, \{c, d\}\}$ is the smallest dominating set of $\overline{G(\tau)}$, but X has no isolated point.

4. Line graph of the complement graph of the completely separated topological graph

In this section, we study some properties of $L(\overline{G(\tau)})$, which is the line graph of $\overline{G(\tau)}$. Let A, B be any two vertices of $\overline{G(\tau)}$. Then, $[A, B]$ is a vertex of $L(\overline{G(\tau)})$ if and only if A, B are adjacent in $\overline{G(\tau)}$. Since $\overline{G(\tau)}$ is an undirected graph, we have, $[A, B] = [B, A]$. It is clear that for distinct vertices $[A_1, A_2]$ and $[B_1, B_2]$ in $L(\overline{G(\tau)})$, $[A_1, A_2]$ is adjacent to $[B_1, B_2]$ if and only if $A_i = B_j$, for some $i, j \in \{1, 2\}$.

$L(\overline{G(\tau)})$:



$Fig(iii)$

Example 2. $L(\overline{G(\tau)})$ for the discrete space $X = \{a, b, c\}$ is shown in $Fig(iii)$ where $A_1 = [\{a\}, \{a, b\}]$, $A_2 = [\{a\}, \{a, c\}]$, $A_3 = [\{b\}, \{b, c\}]$, $A_4 = [\{b\}, \{a, b\}]$, $A_5 = [\{c\}, \{a, c\}]$, $A_6 = [\{c\}, \{b, c\}]$, $A_7 = [\{a, b\}, \{b, c\}]$, $A_8 = [\{a, c\}, \{b, c\}]$, and $A_9 = [\{a, b\}, \{a, c\}]$.

Theorem 16. *Let (X, τ) be any topological space such that $\overline{G(\tau)}$ is connected. Then,*
(i) $L(\overline{G(\tau)})$ is isomorphic to K_1 if and only if τ has exactly two distinct proper open subsets and every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.
(ii) $L(\overline{G(\tau)})$ is isomorphic to K_3 if and only if τ has exactly three distinct proper open subsets and every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.

Proof. (i) The proof is straightforward.
(ii) Suppose $L(\overline{G(\tau)})$ is isomorphic to K_3. Then, $\overline{G(\tau)}$ is isomorphic to the complete graph K_3 or the complete bipartite graph $K_{1,3}$. If $\overline{G(\tau)}$ is isomorphic to $K_{1,3}$, then τ has 4 non-trivial open sets, say A, B, C, D. Without loss of generality, we can assume the degree of A to be 3. It is easy to see that $B \bigcap C = \phi$, $B \bigcap D = \phi$, $C \bigcap D = \phi$. This implies that $B \bigcap (C \bigcup D) = (B \bigcap C) \bigcup (B \bigcap D) = \phi$. This shows that $C \bigcup D = A$ as $C \bigcup D \neq B$, $C \bigcup D \neq C$, $C \bigcup D \neq D$, $C \bigcup B \neq X$, and $C \bigcup D \neq \phi$. Similarly, $C \bigcap (B \bigcup D) = (C \bigcap B) \bigcup (C \bigcap D) = \phi$ and $B \bigcup D = A$. This shows that $C \bigcup D = A = B \bigcup D$, which is not possible as $B \bigcap C = \phi$, $B \bigcap D = \phi$. This shows that $\overline{G(\tau)}$ is isomorphic to K_3 and hence the proof follows from Theorem 3. The proof of the converse part is straightforward. □

Theorem 17. *Let (X, τ) be any topological space such that $\overline{G(\tau)}$ is connected. Then, $L(\overline{G(\tau)})$ is isomorphic to $\overline{G(\tau)}$ if and only if τ has exactly three distinct proper open subsets and every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.*

Proof. The proof follows from Theorem 3 and from the fact that a simple connected graph is isomorphic to its line graph if and only if it is a cycle. □

Theorem 18. *Let (X, τ) be any T_1 topological space such that $|X| \geq 3$. Then, $L(\overline{G(\tau)})$ is connected with $diam(L(\overline{G(\tau)})) \leq 3$.*

Proof. Let $[A_1, A_2]$, $[B_1, B_2]$ be any two vertices of $L(\overline{G(\tau)})$. If $A_i = B_j$ for some $i,\ j \in \{1,2\}$, then $[A_1, A_2]$ and $[B_1, B_2]$ are adjacent. If $A_i \neq B_j$, $\forall\ i,\ j \in \{1,2\}$, then as $A_1 \subsetneq X$, $\exists\ p \in X$ such that $p \notin A_1$ say. As X is T_1 and $|X| \geq 3$, we have $(X\backslash\{p\}) \bigcap A_1 \neq \phi$ and $(X\backslash\{p\}) \bigcap B_j \neq \phi$ for some $j \in \{1,2\}$. Then, $[A_1, A_2] - [A_1, X\backslash\{p\}] - [X\backslash\{p\}, B_j] - [B_1, B_2]$ is a path of length 3. Hence, $L(\overline{G(\tau)})$ is connected with $diam(L(\overline{G(\tau)})) \leq 3$. □

Corollary 13. *Let (X, τ) be any discrete topological space. Then,*

$$diam(L(\overline{G(\tau)})) = \begin{cases} 2, & \textit{if} \quad |X| = 3. \\ 3, & \textit{if} \quad |X| > 3. \end{cases} \tag{4.1}$$

Theorem 19. *Let (X, τ) be any T_1 topological space such that $|X| \geq 3$. Then, $L(\overline{G(\tau)})$ is both triangulated and hypertriangulated.*

Proof. Let $[A_1, A_2]$ be any vertex of $L(\overline{G(\tau)})$. Then, $[A_1, A_2]$ is an edge of $\overline{G(\tau)}$. As $\overline{G(\tau)}$ is hypertriangulated, there exists a vertex B of $\overline{G(\tau)}$ such that $A_1 - B - A_2 - A_1$ forms a triangle. This shows that $[A_1, A_2] - [A_1, B] - [B, A_2] - [A_1, A_2]$ is triangle. Hence, $L(\overline{G(\tau)})$ is triangulated. Let $[A_1, A_2] - [B_1, B_2]$ be an edge of $L(\overline{G(\tau)})$. Then, $A_i = B_j$ for some $i,\ j \in \{1,2\}$. As $[A_1, A_2]$ is an edge of $\overline{G(\tau)}$ and $\overline{G(\tau)}$ is hypertriangulated, there exists a vertex U of $\overline{G(\tau)}$ such that $A_1 - U - A_2 - A_1$ forms a triangle. As $A_i = B_j$ for some $i,\ j \in \{1,2\}$, $[U, A_i] - [A_1, A_2] - [B_1, B_2] - [U, A_i]$ is a triangle. Hence, $L(\overline{G(\tau)})$ is hypertriangulated. □

Theorem 20. *Let (X, τ) be any T_1 topological space such that $|X| \geq 3$ and $[A_1, A_2]$ be a vertex in $L(\overline{G(\tau)})$. Then,*

$$e([A_1, A_2]) = \begin{cases} 2, & \textit{if} \quad A_1 \bigcup A_2 = X. \\ 3, & \textit{if} \quad A_1 \bigcup A_2 \neq X. \end{cases} \tag{4.2}$$

Proof. Let $[B_1, B_2]$ be any vertex distinct from $[A_1, A_2]$.
Case (i): If $A_1 \bigcup A_2 = X$, then $(A_1 \bigcup A_2) \bigcap B_i \neq \phi$, $\forall\, i \in \{1,2\}$. This shows that $\forall\, i \in \{1,2\}$, $A_1 \bigcap B_i \neq \phi$ or $A_2 \bigcap B_i \neq \phi$. If $A_1 \bigcap B_i \neq \phi$, then $[A_1, A_2] - [A_1, B_i] - [B_1, B_2]$ is a path of length 2. If $A_2 \bigcap B_i \neq \phi$, then $[A_1, A_2] - [A_2, B_i] - [B_1, B_2]$ is a path of length 2.
Case (ii): If $A_1 \bigcup A_2 \neq X$, then there exists $p \in X\backslash(A_1 \bigcup A_2)$. As B_1, B_2 are distinct, we have either $(X\backslash\{p\}) \bigcap B_1 \neq \phi$ or $(X\backslash\{p\}) \bigcap B_2 \neq \phi$. If $(X\backslash\{p\}) \bigcap B_1 \neq \phi$, then $[A_1, A_2] - [A_i, X\backslash\{p\}] - [X\backslash\{p\}, B_1] - [B_1, B_2]$ is a path of length 3, where $i \in \{1,2\}$. Similarly, if $(X\backslash\{p\}) \bigcap B_2 \neq \phi$, then $[A_1, A_2] - [A_i, X\backslash\{p\}] - [X\backslash\{p\}, B_2] - [B_1, B_2]$ is a path of length 3, where $i \in \{1,2\}$. Hence, $e([A_1, A_2]) = \begin{cases} 2, & \text{if } A_1 \bigcup A_2 = X. \\ 3, & \text{if } A_1 \bigcup A_2 \neq X. \end{cases}$ □

Corollary 14. *For any T_1 topological space (X, τ) such that $|X| \geq 3$, radius of $L(\overline{G(\tau)})$ is 2.*

Corollary 15. *Let (X, τ) be a T_1 topological space such that $|X| \geq 3$ and $[A_1, A_2]$ be a vertex in $L(\overline{G(\tau)})$. Then, $[A_1, A_2]$ is a center vertex of $L(\overline{G(\tau)})$ if and only if $A_1 \bigcup A_2 = X$.*

Theorem 21. *Let (X, τ) be a discrete topological space such that $|X| \geq 3$ and $[A_1, A_2]$, $[B_1, B_2]$ be two distinct vertices of $L(\overline{G(\tau)})$. Then,*
(i) $d([A_1, A_2], [B_1, B_2]) = 1$ if and only if $A_i = B_j$ for some $i,\ j \in \{1, 2\}$.
(ii) $d([A_1, A_2], [B_1, B_2]) = 2$ if and only if $A_i \neq B_j$, $\forall\ i,\ j \in \{1, 2\}$ and for some $i,\ j$, $A_i \bigcap B_j \neq \phi$.
(iii) If $|X| \geq 4$, $d([A_1, A_2], [B_1, B_2]) = 3$ if and only if $A_i \bigcap B_j = \phi$, $\forall\ i,\ j \in \{1, 2\}$.

Proof. (i) The proof is straightforward.
(ii) Suppose $d([A_1, A_2], [B_1, B_2]) = 2$. By (i), $A_i \neq B_j$, $\forall\ i,\ j \in \{1, 2\}$. Also, there exists a vertex $[P_1, P_2]$ such that $[A_1, A_2] - [P_1, P_2] - [B_1, B_2]$ is a path of length 2. This implies that $P_1 = A_i$ and $P_2 = B_j$ for some $i,\ j \in \{1, 2\}$. But P_1 and P_2 are adjacent in $\overline{G(\tau)}$. This shows that $A_i \bigcap B_j \neq \phi$ for some $i,\ j \in \{1, 2\}$. Conversely, if $A_i \bigcap B_j \neq \phi$ for some $i,\ j \in \{1, 2\}$, then $[A_1, A_2] - [A_i, B_j] - [B_1, B_2]$ is a path of length 2. So, $d([A_1, A_2], [B_1, B_2]) = 2$.
(iii) Suppose $d([A_1, A_2], [B_1, B_2]) = 3$. Then, from (i) and (ii), we have $A_i \neq B_j$, $\forall$ $i,\ j \in \{1, 2\}$ and $A_i \bigcap B_j = \phi$, $\forall\ i,\ j \in \{1, 2\}$. Conversely, suppose $A_i \bigcap B_j = \phi$, $\forall$ $i,\ j \in \{1, 2\}$. As X is discrete and A_1, A_2 are distinct, we have $|A_1| > 1$ or $|A_2| > 1$. Suppose $|A_1| > 1$ and $a \in A_1$. Then, $[A_1, A_2] - [A_1, X \backslash \{a\}] - [X \backslash \{a\}, B_1] - [B_1, B_2]$ is a path of length 3. As $diam(L(\overline{G(\tau)})) \leq 3$, therefore $d([A_1, A_2], [B_1, B_2]) = 3$. □

Theorem 22. *Let (X, τ) be a discrete topological space such that $|X| \geq 3$ and $[A_1, A_2]$, $[B_1, B_2]$ be two distinct vertices of $L(\overline{G(\tau)})$. Then,*
(i) $c([A_1, A_2], [B_1, B_2]) = 3$ if and only if $A_i = B_j$ for some $i,\ j \in \{1, 2\}$.
(ii) $c([A_1, A_2], [B_1, B_2]) = 4$ if and only if $A_i \neq B_j$, $\forall\ i,\ j \in \{1, 2\}$ and one of the following conditions is satisfied:
(a) for some i, $A_i \bigcap B_j \neq \phi$, $\forall\ j \in \{1, 2\}$
(b) $A_1 \bigcap B_i \neq \phi$ and $A_2 \bigcap B_j \neq \phi$, for some $i,\ j \in \{1, 2\}$.
(iii) $c([A_1, A_2], [B_1, B_2]) = 5$ if and only if $A_i \neq B_j$, $\forall\ i,\ j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$, $A_i \bigcap B_j \neq \phi$ for only one $j \in \{1, 2\}$.
(iv) For $|X| \geq 4$, $c([A_1, A_2], [B_1, B_2]) = 6$ if and only if $A_i \bigcap B_j = \phi$, $\forall\ i,\ j \in \{1, 2\}$.

Proof. (i) Suppose $c([A_1, A_2], [B_1, B_2]) = 3$. Then, $[A_1, A_2]$ and $[B_1, B_2]$ are adjacent. This shows that $A_i = B_j$ for some $i,\ j \in \{1, 2\}$. Conversely, suppose $A_i = B_j$ for some $i,\ j \in \{1, 2\}$. Then, $[A_1, A_2]$ and $[B_1, B_2]$ are adjacent. As $L(\overline{G(\tau)})$ is hypertriangulated, there exists another vertex $[P_1, P_2]$ such that $[A_1, A_2] - [B_1, B_2] - [P_1, P_2] - [A_1, A_2]$ is a cycle of length 3.

(ii) Suppose $c([A_1, A_2], [B_1, B_2]) = 4$. By (i), $A_i \neq B_j$, $\forall\ i,\ j \in \{1, 2\}$. So, we have a cycle $[A_1, A_2] - [P_1, P_2] - [B_1, B_2] - [Q_1, Q_2] - [A_1, A_2]$, where $P_1, Q_2 \in \{A_1, A_2\}$ and

$P_2, Q_1 \in \{B_1, B_2\}$. If $P_1 = A_1$, $Q_2 = A_2$, $P_2 = B_2$, and $Q_1 = B_1$, then $A_1 \bigcap B_2 \neq \phi$ and $A_2 \bigcap B_1 \neq \phi$. If $P_1 = A_1$, $Q_2 = A_2$, $P_2 = B_1$, and $Q_1 = B_2$, then $A_1 \bigcap B_1 \neq \phi$ and $A_2 \bigcap B_2 \neq \phi$. If $P_1 = A_2$, $Q_2 = A_1$, $P_2 = B_2$, and $Q_1 = B_1$, then $A_1 \bigcap B_1 \neq \phi$ and $A_2 \bigcap B_2 \neq \phi$. If $P_1 = A_2$, $Q_2 = A_1$, $P_2 = B_1$, and $Q_1 = B_2$, then $A_1 \bigcap B_2 \neq \phi$ and $A_2 \bigcap B_1 \neq \phi$. If $P_1 = Q_2 = A_1$, $P_2 = B_1$, and $Q_1 = B_2$, then $A_1 \bigcap B_1 \neq \phi$ and $A_1 \bigcap B_2 \neq \phi$. If $P_1 = Q_2 = A_1$, $P_2 = B_2$, and $Q_1 = B_1$, then $A_1 \bigcap B_2 \neq \phi$ and $A_1 \bigcap B_1 \neq \phi$. If $P_1 = Q_2 = A_2$, $P_2 = B_1$, and $Q_1 = B_2$, then $A_2 \bigcap B_1 \neq \phi$ and $A_2 \bigcap B_2 \neq \phi$. If $P_1 = Q_2 = A_2$, $P_2 = B_2$, and $Q_1 = B_1$, then $A_2 \bigcap B_2 \neq \phi$ and $A_2 \bigcap B_1 \neq \phi$. If $P_1 = A_1$, $Q_2 = A_2$ and $P_2 = Q_1 = B_1$, then $A_1 \bigcap B_1 \neq \phi$ and $A_2 \bigcap B_1 \neq \phi$. If $P_1 = A_2$, $Q_2 = A_1$ and $P_2 = Q_1 = B_1$, then $A_2 \bigcap B_1 \neq \phi$ and $A_1 \bigcap B_1 \neq \phi$. If $P_1 = A_2$, $Q_2 = A_1$ and $P_2 = Q_1 = B_2$, then $A_2 \bigcap B_2 \neq \phi$ and $A_1 \bigcap B_2 \neq \phi$. If $P_1 = A_1$, $Q_2 = A_2$ and $P_2 = Q_1 = B_2$, then $A_1 \bigcap B_2 \neq \phi$ and $A_2 \bigcap B_2 \neq \phi$. If $P_1 = Q_2 = A_i$, $P_2 = Q_1 = B_j$ for some $i, j \in \{1, 2\}$, then $[P_1, P_2] = [Q_1, Q_2]$ which is not possible as $[P_1, P_2]$ and $[Q_1, Q_2]$ are distinct.
Conversely, if $A_i \neq B_j$, $\forall\, i, j \in \{1, 2\}$ and for some i, $A_i \bigcap B_j \neq \phi$, $\forall\, j \in \{1, 2\}$, then $[A_1, A_2] - [A_i, B_1] - [B_1, B_2] - [B_2, A_i] - [A_1, A_2]$ is a cycle of length 4. Again, if $A_1 \bigcap B_i \neq \phi$ and $A_2 \bigcap B_j \neq \phi$, for some $i, j \in \{1, 2\}$, then $[A_1, A_2] - [A_1, B_i] - [B_1, B_2] - [B_j, A_2] - [A_1, A_2]$ is a cycle of length 4.
(iii) Suppose $c([A_1, A_2], [B_1, B_2]) = 5$. By (i), we have $A_i \neq B_j$, $\forall\, i, j \in \{1, 2\}$. There exists three vertices $[P_1, P_2]$, $[Q_1, Q_2]$, and $[U_1, U_2]$ such that $[A_1, A_2] - [P_1, P_2] - [B_1, B_2] - [Q_1, Q_2] - [U_1, U_2] - [A_1, A_2]$ is a cycle of length 5 or $[A_1, A_2] - [P_1, P_2] - [Q_1, Q_2] - [B_1, B_2] - [U_1, U_2] - [A_1, A_2]$ is a cycle of length 5. If $[A_1, A_2] - [P_1, P_2] - [B_1, B_2] - [Q_1, Q_2] - [U_1, U_2] - [A_1, A_2]$ is a cycle, then $P_1 \in \{A_1, A_2\}$ and $P_2 \in \{B_1, B_2\}$. This shows that for only one $i \in \{1, 2\}$, $A_i \bigcap B_j \neq \phi$ for only one $j \in \{1, 2\}$ as P_1 is adjacent to P_2. Similarly, if $[A_1, A_2] - [P_1, P_2] - [Q_1, Q_2] - [B_1, B_2] - [U_1, U_2] - [A_1, A_2]$ is a cycle, then $U_2 \in \{A_1, A_2\}$ and $U_1 \in \{B_1, B_2\}$ and so we have for only one $i \in \{1, 2\}$, $A_i \bigcap B_j \neq \phi$ for only one $j \in \{1, 2\}$. Conversely, by (i) and (ii), no cycle of length 3 or 4 contains both $[A_1, A_2]$ and $[B_1, B_2]$. It is given that for only one $i \in \{1, 2\}$, $A_i \bigcap B_j \neq \phi$ for only one $j \in \{1, 2\}$. Suppose $i = 1$ and $j = 1$ then $A_1 \bigcap B_1 \neq \phi$. As A_1 and A_2 are distinct, we have either $|A_1| > 1$ or $|A_2| > 1$. If possible $|A_1| > 1$, then there exists $p \in A_1$. It is easy to show that $[A_1, A_2] - [A_1, B_1] - [B_1, B_2] - [B_1, X\backslash\{p\}] - [X\backslash\{p\}, A_1] - [A_1, A_2]$ is a cycle of length 5. So, $c([A_1, A_2], [B_1, B_2]) = 5$.
(iv) Suppose $c([A_1, A_2], [B_1, B_2]) = 6$. Then, by (i), (ii), (iii), we have $A_i \neq B_j$, $\forall\, i, j \in \{1, 2\}$ and $A_i \bigcap B_j = \phi$, $\forall\, i, j \in \{1, 2\}$. Conversely, Suppose $A_i \bigcap B_j = \phi$ for all $i, j \in \{1, 2\}$. By (i), (ii), (iii) no cycle of length 3 or 4 or 5 contains both $[A_1, A_2]$ and $[B_1, B_2]$. Since A_1, A_2, B_1, B_2 are distinct and $A_1 \bigcap A_2 \neq \phi$, $B_1 \bigcap B_2 \neq \phi$, we have either $|A_1| > 1$ or $|A_2| > 1$ and $|B_1| > 1$ or $|B_2| > 1$. If possible $|A_1| > 1$ and $|B_1| > 1$, then there exists $p \in A_1$ and $q \in B_1$, where $p \neq q$ as $A_1 \bigcap B_1 = \phi$. It is easy to see that $[A_1, A_2] - [X\backslash\{p\}, A_1] - [X\backslash\{p\}, B_1] - [B_1, B_2] - [B_1, X\backslash\{q\}] - [X\backslash\{q\}, A_1] - [A_1, A_2]$ is a cycle of length 6. □

Theorem 23. *Let (X, τ_X) and (Y, τ_Y) be any two discrete topological spaces with $|X| \geq 3$ and $|Y| \geq 3$. Then, the following conditions are equivalent:*

(i) $\overline{G(\tau_X)}$ *is isomorphic to* $\overline{G(\tau_Y)}$.
(ii) X *and* Y *are homeomorphic.*
(iii) $L(\overline{G(\tau_X)})$ *is isomorphic to* $L(\overline{G(\tau_Y)})$.

Proof. The equivalence of (i) and (ii) follows from Corollary 6. Since $|X| \geq 3$ and $|Y| \geq 3$, therefore by Theorem 4, $\overline{G(\tau_X)}$ and $\overline{G(\tau_Y)}$ are connected graphs. It is clear that $\overline{G(\tau_X)}$ and $\overline{G(\tau_Y)}$ are neither K_3 nor $K_{1,3}$ for discrete spaces. Thus, from Theorem 3.10.1 [6] and Theorem 3.10.3 [6], it follows that (i) and (iii) are equivalent. □

Open Problem 24. To investigate whether Theorem 23 still holds or not if (X, τ_X) and (Y, τ_Y) are T_1 or T_2 topological spaces.

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