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Research Article

Weak signed total Italian domination in digraphs

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Dedicated to Odile Favaron

Abstract: A *weak signed total Italian dominating function* (WSTIDF) of a digraph D with vertex set $V(D)$ is defined as a function $f : V(D) \to \{-1, 1, 2\}$ having the property that $\sum_{x \in N^-(v)} f(x) \geq 1$ for each $v \in V(D)$, where $N^-(v)$ consists of all vertices of D from which arcs go into v. The weight of a WSTIDF is the sum of its function values over all vertices. The *weak signed total Italian domination number* of D, denoted by $\gamma_{wstI}(D)$, is the minimum weight of a WSTIDF on D. We initiate the study of the weak signed total Italian domination number in digraphs, and we present different sharp bounds on $\gamma_{wstI}(D)$. In addition, we determine the weak signed total Italian domination number of some classes of digraphs.

Keywords: digraphs, weak signed total Italian domination, signed total Italian domination, signed total Roman domination, total domination.

AMS Subject Classification: 05C20, 05C69

1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [8]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G)|uv \in E(G)\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. A graph G is *regular* or *r-regular* if $\delta(G) = \Delta(G) = r$. Let K_n be the complete graph of order n, C_n the cycle of order n and $K_{p,q}$ the complete bipartite graph with partite sets X and Y, where $|X| = p$ and $|Y| = q$.
A *signed total Roman dominating function* (STRDF) on a graph G is defined in [12] as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $f(N(v)) = \sum_{x \in N(v)} f(x) \geq$

1 for each $v \in V(G)$ and if $f(u) = -1$, then the vertex u must have a neighbor w with $f(w) = 2$. The weight of a signed total Roman dominating function is the value $\sum_{u\in V(G)} f(u)$. The *signed total Roman domination number* $\gamma_{stR}(G)$ is the minimum weight of a signed total Roman dominating function on G.

A *signed total Italian dominating function* (STIDF) of a graph G is defined in [15] as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $f(N(v)) \geq 1$ for each $v \in V(G)$ and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$ or adjacent to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIDF f is the value $\sum_{v\in V(G)} f(v)$. The *signed total Italian domination number* of G, denoted by $\gamma_{stI}(G)$, is the minimum weight of an STIDF in G.

The signed total Roman and signed total Italian domination numbers are well-defined for graphs G without isolated vertices, since the function $f : V(G) \to \{-1, 1, 2\}$ with $f(x) = 1$ for each $x \in V(G)$ is an STRDF as well as an STIDF. The definitions lead to $\gamma_{stI}(G) \leq \gamma_{stR}(G) \leq n(G)$.

Now let D be a finite and simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. The sets $N^+_D(v) = N^+(v) = \{x|(v, x) \in A(D)\}$ and $N^-_D(v) = N^-(v) = \{x|(x, v) \in A(D)\}$ are called the *out-neighborhood* and *in-neighborhood* of the vertex v. Likewise $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$. The integers $d^+_D(v) = d^+(v) = |N^+(v)|$ and $d^-_D(v) = d^-(v) = |N^-(v)|$ are the *out-degree* and *in-degree* of the vertex v, respectively. The *minimum* and *maximum out-degree* are $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$ and the *minimum* and *maximum in-degree* are $\delta^-(D) = \delta^-$ and $\Delta^-(D) = \Delta^-$. A digraph D is *out-regular* or *r-out-regular* if $\delta^+(D) = \Delta^+(D) = r$. For a subset $X \subseteq V(D)$, we use $D[X]$ to denote the subdigraph of D induced by X. For an arc $(x, y) \in A(D)$, the vertex y is an *out-neighbor* of x and x is an *in-neighbor* of y, we also say that x *dominates* y and y is dominated by x. For a real-valued function $f : V(D) \longrightarrow \mathbf{R}$, the weight of f is $\omega(f) = \sum_{x\in V(G)} f(x)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{x\in S} f(x)$. We define a set $S \subseteq V(D)$ to be a *total dominating set* of D if for all $v \in V(D)$, there exists a vertex $u \in S$ such that v is dominated by u. The *total domination number* $\gamma_t(D)$ equals the minimum cardinality of a total dominating set in D.

In this paper we continue the study of signed (total) Roman (Italian) domination in graphs and digraphs (see, for example, the survey articles [2–5] and [1, 6, 9–14]).

A *signed total Roman dominating function* (STRDF) on a digraph D is defined in [13] as a function $f : V(D) \to \{-1, 1, 2\}$ having the property that $f(N^-(v)) = \sum_{x\in N^-(v)} f(x) \geq 1$ for each $v \in V(D)$ and every vertex u for which $f(u) = -1$, has an in-neighbor v for which $f(v) = 2$. The weight of an STRDF f is the value $\sum_{u\in V(D)} f(u)$. The *signed total Roman domination number* $\gamma_{stR}(D)$ is the minimum weight of an STRDF on D.

A *signed total Italian dominating function* (STIDF) of a digraph D is defined in [16] as a function $f : V(D) \to \{-1, 1, 2\}$ having the property that $f(N^-(v)) \geq 1$ for each $v \in V(D)$ and every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$ or two in-neighbors w and z with $f(w) = f(z) = 1$. The weight of an

STIDF f is the value $\sum_{v\in V(D)} f(v)$. The *signed total Italian domination number* of D, denoted by $\gamma_{stI}(D)$, is the minimum weight of an STIDF on D. A $\gamma_{stI}(D)$-function is an STIDF of weight $\gamma_{stI}(D)$.

A *weak signed total Italian dominating function* (WSTIDF) of a digraph D is defined as a function $f : V(D) \to \{-1, 1, 2\}$ having the property that $f(N^-(v)) \geq 1$ for each $v \in V(D)$. The weight of a WSTIDF f is $\omega(f) = \sum_{v\in V(D)} f(v)$. The *weak signed total Italian domination number* of G, denoted by $\gamma_{wstI}(D)$, is the minimum weight of a WSTIDF on D. A $\gamma_{wstI}(D)$-function is a WSTIDF of weight $\gamma_{wstI}(D)$. For a WSTIDF f on D, let $V_i = \{v \in V(D) : f(v) = i\}$ for $i = -1, 1, 2$. A WSTIDF f can be represented by the ordered partition $f = (V_{-1}, V_1, V_2)$.

The weak signed total Italian domination number, the signed total Italian domination number and the signed total Roman domination exist when $\delta^-(D) \geq 1$, because the function $f : V(D) \to \{-1, 1, 2\}$ with $f(x) = 1$ for each vertex $x \in V(D)$ is a WSTIDF, an STIDF as well as an STRDF on D of weight $n(D)$ and thus the definitions lead to $\gamma_{wstI}(D) \leq \gamma_{stI}(D) \leq \gamma_{stR}(D) \leq n(D)$.

Our purpose in this work is to initiate the study of the weak signed total Italian domination number in digrphs. We present basic properties and sharp bounds for the weak signed total Italian domination number of a digraph. In particular, we show that many lower bounds on $\gamma_{stI}(D)$ and on $\gamma_{stR}(D)$ are also valid for $\gamma_{wstI}(D)$. In addition, we show that the difference $\gamma_{stI}(D) - \gamma_{wstI}(D)$ can be arbitrarily large, and we determine the weak signed total Italian domination number of some classes of digraphs.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1. *If $D(G)$ is the associated digraph of a graph G, then $\gamma_{stR}(D(G)) = \gamma_{stR}(G)$ and $\gamma_{stI}(D(G)) = \gamma_{stI}(G)$.*

Let K^*_n, $K^*_{p,q}$ and C^*_n be the associated digraphs of K_n, $K_{p,q}$ and C_n, respectivly. We make use of the following known results.

Proposition 1. *[16]* If $n \geq 2$, then $\gamma_{stI}(K^*_n) = 2$ when n is even and $\gamma_{stI}(K^*_n) = 3$ when n is odd.

Proposition 2. *[16]* If $p, q \geq 2$ are integers, then $\gamma_{stI}(K^*_{p,q}) = 2$.

Proposition 3. *[16]* If $n \geq 3$, then $\gamma_{stI}(C^*_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stI}(C^*_n) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stI}(C^*_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.

Let $n = 2r + 1$ with an integer $r \geq 1$. We define the *circulant tournament* $CT(n)$ of order n with vertex set $\{u_0, u_1, \ldots, u_{n-1}\}$ as follows. For each i, the arcs are going from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n.

Proposition 4. *[16]* Let $n = 2r+1$ with an integer $r \geq 1$. Then $\gamma_{stI}(CT(n)) = 3$ when $r = 2p+1$ is odd and $\gamma_{stI}(CT(n)) = 4$ when $r = 2p$ is even.

2. Preliminary results and first bounds

In this section we present basic properties and some first bounds on the weak signed total Italian domination number.

Observation 2. *If $f = (V_{-1}, V_1, V_2)$ is a WSTIDF of a digraph D of order n with $\delta^-(D) \geq 1$, then the following holds.*

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) $V_1 \cup V_2$ *is a total dominating set of* D.

Proof. Since (a) and (b) are immediate, we only prove (c). By the definition, each vertex of V_{-1} has an in-neighbor in $V_1 \cup V_2$. Thus $V_1 \cup V_2$ dominates V_{-1}. Suppose that $V_1 \cup V_2$ contains a vertex v without an in-neighbor in $V_1 \cup V_2$. As $\delta^-(D) \geq 1$, the vertex v must have an in-neighbor in V_{-1} and all its in-neighbors are in V_{-1}. This leads to the contradiction $f(N^-(v)) \leq -1$. Therefore each vertex of $V_1 \cup V_2$ has an in-neighbor in $V_1 \cup V_2$ and thus $V_1 \cup V_2$ is a total dominating set of D. □

If $\delta^- \geq 3$, then we can improve the bounds $\gamma_{wstI}(D) \leq \gamma_{stI}(D) \leq n(D)$.

Theorem 1. If D is a digraph of order n with $\delta^-(D) = \delta^- \geq 3$, then

$$\gamma_{wstI}(D) \leq \gamma_{stI}(D) \leq n - 2\left\lfloor \frac{\delta^- - 1}{2} \right\rfloor.$$

Proof. Let $t = \lfloor \frac{\delta^- - 1}{2} \rfloor$, and let $A = \{u_1, u_2, \ldots, u_t\}$ be an arbitrayr set of t vertices of D. Define the function $f : V(D) \to \{-1, 1, 2\}$ by $f(u_i) = -1$ for $1 \leq i \leq t$ and $f(x) = 1$ for $x \in V(D) \setminus A$. Then

$$f(N^-(v)) \geq -t + (\delta^- - t) = \delta^- - 2t = \delta^- - 2\left\lfloor \frac{\delta^- - 1}{2} \right\rfloor \geq 1$$

for each vertex $v \in V(D)$. If $f(u) = -1$, then it follows from $\delta^- \geq 3$ that u has at least two in-neighbors w and z with $f(w) = f(z) = 1$. Therefore f is an STIDF on D of weight $-t + (n-t) = n - 2t$ and the proof is complete. □

Proposition 1 shows that Theorem 1 is sharp.
The proof of the next lower bounds are identically with the proofs of Propositions 6, 7 and 8 in [16] and are therefore omitted.

Proposition 5. If D is a digraph of order n with $\delta^-(D) \geq 1$, then $\gamma_{wstI}(D) \geq \Delta^-(D) + 1 - n$.

Proposition 6. If D is a digraph of order n with $\delta^-(D) \geq 1$, then $\gamma_{wstI}(D) \geq \delta^-(D) + 3 - n$.

Proposition 7. If D is a digraph of order n with $\delta^-(D) \geq 1$, then $\gamma_{wstI}(D) \geq 2\gamma_t(D) - n$.

Example 1. *Let $p \geq 3$ be an integer, and let $v_1, v_2, \ldots, v_p$ be the vertex set of the complete graph K_p. Let the graph H consisting of K_p and the $p(p-2)$ new vertices $w_i^1, w_i^2, \ldots, w_i^{p-2}$ for $1 \leq i \leq p$ such that v_i is adjacent to the vertices $w_i^1, w_i^2, \ldots, w_i^{p-2}$ for $1 \leq i \leq p$. Now let $D(H)$ be the associated digraph of H. Define $f : V(D(H)) \to \{-1, 1, 2\}$ by $f(v_i) = 1$ for $1 \leq i \leq p$ and $f(x) = -1$ otherwise. Then f is a WSTIDF on $D(H)$ of weight $p - p(p-2) = 3p - p^2$ and thus $\gamma_{wstI}(D(H)) \leq 3p - p^2$. In addition, we note that $\gamma_t(D(H)) = p$. Combining this with Proposition 7, we obtain*

$$3p - p^2 = 2\gamma_t(D(H)) - n(D(H)) \leq \gamma_{wstI}(D(H)) \leq 3p - p^2$$

and thus $\gamma_{wstI}(D(H)) = 3p - p^2$ and $\gamma_{wstI}(D(H)) = 2\gamma_t(D(H)) - n(D(H))$.

Example 1 shows that Proposition 7 is sharp. Example 1 will also demonstrate that the difference $\gamma_{stI}(D) - \gamma_{wstI}(D)$ can be arbitrarily large.

Example 2. *If f is an STIDF on the digraph $D(H)$ of Example 1, then we show that $f(v_j) + \sum_{i=1}^{p-2} f(w_j^i) \geq 4 - p$ for $1 \leq j \leq p$. If $f(w_j^i) = -1$ for an index $1 \leq i \leq p-2$, then $f(v_j) = 2$ and therefore $f(v_j) + \sum_{i=1}^{p-2} f(w_j^i) \geq 2 - (p-2) = 4 - p$. If $f(w_j^i) \geq 1$ for each $i \in \{1, 2, \ldots, p-2\}$, then $f(v_j) + \sum_{i=1}^{p-2} f(w_j^i) \geq 1 + (p-2) = p - 1 \geq 4 - p$. This leads to $\sum_{x \in V(D(H))} f(x) \geq p(4-p) = 4p - p^2$ and thus $\gamma_{stI}(D(H)) \geq 4p - p^2$. Using the fact that $\gamma_{wstI}(D(H)) = 3p - p^2$, we deduce that $\gamma_{stI}(D(H)) - \gamma_{wstI}(D(H)) \geq 4p - p^2 - (3p - p^2) = p$.*

We present a further example which will show that the difference $\gamma_{stI}(D) - \gamma_{wstI}(D)$ can be arbitrarily large.

Example 3. *Let $p \geq 2$ be an integer, and let SP_{2p+1} be the spider with the central vertex w, the neighbors $u_1, u_2, \ldots, u_p$ of w and the leaves v_i adjacent to u_i for $1 \leq i \leq p$. Now let $D(SP_{2p+1})$ be the associated digraph of SP_{2p+1}. If f is a $\gamma_{stI}(D(SP_{2p+1}))$-function, then we observe that $f(u_i) + f(v_i) \geq 1$ for $1 \leq i \leq p$ and $f(u_i) + f(v_i) = 1$ if and only if $f(w) = 2$. Therefore $\gamma_{stI}(D(SP_{2p+1})) \geq p + 2$, and in fact we observe $\gamma_{stI}(D(SP_{2p+1})) = p + 2$.*
On the other hand, the function g defined by $g(v_i) = -1$, $g(u_i) = 1$ for $1 \leq i \leq p$ and $g(w) = 2$ is a WSTIDF on $D(SP_{2p+1})$ of weight 2 and thus $\gamma_{wstI}(D(SP_{2p+1})) \leq 2$. In fact we have $\gamma_{wstI}(D(SP_{2p+1})) = 2$.
Consequently, we deduce that $\gamma_{stI}(D(SP_{2p+1})) - \gamma_{wstI}(D(SP_{2p+1})) \geq p + 2 - 2 = p$.

The next propostion is a supplement to Proposition 6.

Proposition 8. Let D be a digraph of order n. If $\delta^-(D) \geq 2$ is even, then $\gamma_{wstI}(D) \geq 4 + \delta^-(D) - n$.

Proof. Let f be a $\gamma_{wstI}(D)$-function. Then there exists a vertex w with $f(w) \geq 1$. If $f(w) = 2$, then it follows from the definitions that

$$\begin{aligned}\gamma_{wstI}(D) &= \sum_{x\in V(D)} f(x) = f(w) + \sum_{x\in N^-(w)} f(x) + \sum_{x\in V(D)-N^-[w]} f(x)\\ &\geq 2 + 1 - (n - d^-(w) - 1) = 4 + d^-(w) - n \geq 4 + \delta^-(D) - n.\end{aligned}$$

Next assume that $f(x) \in \{-1, 1\}$ for each vertex $x \in V(D)$. If there exists a vertex v with $f(v) = 1$ and $d^-(v) \geq \delta^-(D) + 1$, then we obtain as above

$$\begin{aligned}\gamma_{wstI}(D) &= \sum_{x\in V(D)} f(x) = f(v) + \sum_{x\in N^-(v)} f(x) + \sum_{x\in V(D)-N^-[v]} f(x)\\ &\geq 1 + 1 - (n - d^-(v) - 1) = 3 + d^-(v) - n \geq 4 + \delta^-(D) - n.\end{aligned}$$

Finally assume that $d^-(u) = \delta^-(D)$ for each vertex u with $f(u) = 1$. Since $\delta^-(D) = d^-(u)$ is even, we observe $\sum_{x\in N^-(u)} f(x) \geq 2$ for each vertex u with $f(u) = 1$ and thus for each such vertex u

$$\begin{aligned}\gamma_{wstI}(D) &= \sum_{x\in V(D)} f(x) = f(u) + \sum_{x\in N^-(u)} f(x) + \sum_{x\in V(D)-N^-[u]} f(x)\\ &\geq 1 + 2 - (n - d^-(u) - 1) = 4 + d^-(u) - n = 4 + \delta^-(D) - n.\end{aligned}$$

□

Since $\gamma_{wstI}(D) \leq \gamma_{stI}(D)$, Proposition 8 leads to the next result immediately.

Corollary 1. Let D be a digraph of order n. If $\delta^-(D) \geq 2$ is even, then $\gamma_{stI}(D) \geq 4 + \delta^-(D) - n$.

The proof of the next proposition is identically with the proof of Proposition 8 in [13] and is therefore omitted.

Proposition 9. Let $f = (V_{-1}, V_1, V_2)$ be a WSTIDF of a digraph D of order n with $\delta^-(D) \geq 1$. If $\Delta^+ = \Delta^+(D)$ and $\delta^+ = \delta^+(D)$, then the following holds.

(a) $(2\Delta^+ - 1)|V_2| + (\Delta^+ - 1)|V_1| \geq (\delta^+ + 1)|V_{-1}|$.

(b) $(2\Delta^+ + \delta^+)|V_2| + (\Delta^+ + \delta^+)|V_1| \geq (\delta^+ + 1)n$.

(c) $(\Delta^+ + \delta^+)\omega(f) \geq (\delta^+ - \Delta^+ + 2)n + (\delta^+ - \Delta^+)|V_2|$.

(d) $\omega(f) \geq (\delta^+ - 2\Delta^+ + 2)n/(2\Delta^+ + \delta^+) + |V_2|$.

As an immediate consequence of Proposition 9-(c), we obtain a lower bound on the weak signed total Italian domination number for r-out-regular digraphs.

Corollary 2. If D is an r-out-regular digraph of order n with $r \geq 1$, then $\gamma_{wstI}(D) \geq \lceil n/r \rceil$.

Therefore $\gamma_{stR}(D) \geq \gamma_{stI}(D) \geq \gamma_{wstI}(D) \geq \lceil n/r \rceil$ for each r-out-regular digraph of order n with $r \geq 1$ (see [13, 16]). Using Corollary 2 and Observation 1, we obtain the next known bounds immediately.

Corollary 3. *[12, 15]* If G is an r-regular graph of order n with $r \geq 1$, then $\gamma_{stR}(G) \geq \gamma_{stI}(G) \geq \lceil n/r \rceil$.

If D is not out-regular, then the next lower bound on the weak signed total Italian domination number is valid.

Corollary 4. Let D be a digraph of order n with $\delta^-(D) \geq 1$, maximum out-degree Δ^+ and minimum out-degree δ^+. If $\delta^+ < \Delta^+$, then

$$\gamma_{wstI}(D) \geq \left\lceil \frac{-2\Delta^+ + 2\delta^+ + 3}{2\Delta^+ + \delta^+} n \right\rceil.$$

Proof. Multiplying both sides of the inequality in Proposition 9-(d) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Proposition 9-(c), we obtain the desired lower bound. □

Because of $\gamma_{stR}(D) \geq \gamma_{stI}(D) \geq \gamma_{wstI}(D)$ the bound of Corollary 4 is also valid for $\gamma_{stR}(D)$ and $\gamma_{stI}(D)$ (see [13, 16]).
Since $\Delta^+(D(G)) = \Delta(G)$ and $\delta^+(D(G)) = \delta(G)$, Corollary 4 and Observation 1 lead to the next known corollary.

Corollary 5. *[12, 15]* Let G be a graph of order n, maximum degree Δ and minimum degree $\delta \geq 1$. If $\delta < \Delta$, then

$$\gamma_{stR}(G) \geq \gamma_{stI}(G) \geq \left\lceil \frac{-2\Delta + 2\delta + 3}{2\Delta + \delta} n \right\rceil.$$

Example 11 in [12] demonstrates that Corollary 5 is sharp. This example together with Observation 1 show that Corollary 4 is sharp too.
Using Example 2 or 3 we see that the difference $\gamma_{stI}(D) - \gamma_{wstI}(D)$ can be arbitrarily large. However, if $\delta^-(D) \geq 2$, then we will show that $\gamma_{wstI}(D) = \gamma_{stI}(D)$.

Theorem 2. If D is a digraph with $\delta^-(D) \geq 2$, then $\gamma_{wstI}(D) = \gamma_{stI}(D)$.

Proof. Clearly, $\gamma_{wstI}(D) \leq \gamma_{stI}(D)$. Let now f be a $\gamma_{wstI}(D)$-function. By the definition, we have $\sum_{x\in N^-(v)} f(x) \geq 1$ for each vertex $v \in V(D)$. If $f(u) = -1$, then it follows from $d^-(u) \geq 2$ and $\sum_{x\in N^-(u)} f(x) \geq 1$ that u has an in-neighbor v with $f(v) = 2$ or two in-neighbors w and z with $f(w) = f(z) = 1$. Hence f is also an STIDF on D and thus $\gamma_{stI}(D) \leq \gamma_{wstI}(D)$. This leads to $\gamma_{wstI}(D) = \gamma_{stI}(D)$. □

3. Special classes of digraphs

In this section, we determine the weak signed total italian domination number for special classes of digraphs. Since $\gamma_{wstI}(K_2^*) = 2$, Theorem 2 and Proposition 1 lead to the first result in this section immediately.

Proposition 10. If $n \geq 2$, then $\gamma_{wstI}(K_n^*) = 2$ when n is even and $\gamma_{wstI}(K_n^*) = 3$ when n is odd.

For even n, Proposition 10 shows that Proposition 6 is sharp, and for odd n, Proposition 10 shows that Proposition 8 is sharp.
Theorem 2, Proposition 2, Proposition 3 and Proposition 4 yield to the next results immediately.

Proposition 11. If $p, q \geq 2$ are integers, then $\gamma_{wstI}(K_{p,q}^*) = 2$.

Proposition 12. If $n \geq 3$, then $\gamma_{wstI}(C_n^*) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{wstI}(C_n^*) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{wstI}(C_n^*) = (n+6)/2$ when $n \equiv 2 \pmod 4$.

Proposition 13. Let $n = 2r + 1$ with an integer $r \geq 1$. Then $\gamma_{wstI}(CT(n)) = 3$ when $r = 2p + 1$ is odd and $\gamma_{wstI}(CT(n)) = 4$ when $r = 2p$ is even.

A *rooted tree* is a connected digraph with a vertex of in-degree 0, called the *root*, such that every vertex different from the root has in-degree 1. A digraph D is *contrafunctional* if each vertex of D has in-degree 1. Harary, Norman and Cartwright [7] have shown that every connected contrafunctional digraph has a unique directed cycle, and the removal of any arc of the directed cycle results in a rooted tree. If D is a connected contrafunctional digraph, then let $\ell(D)$ be the vertex set of D with $d^+(v) = 0$.

Theorem 3. If D is a connected contrafunctional digraph of order n, then $\gamma_{wstI}(D) = n - 2|\ell(D)|$.

Proof. Let f be a $\gamma_{wstI}(D)$-function. If u is a vertex of D with $d^+(u) \geq 1$, then assume that u dominates v. Since $d^-(v) = 1$, we deduce that $f(u) \geq 1$. Therefore $\gamma_{wstI}(D) \geq (n - |\ell(D)|) - |\ell(D)| = n - 2|\ell(D)|$.
On the other hand if we define the function g by $g(x) = 1$ for $x \in V(D) \setminus \ell(D)$ and $g(x) = -1$ for $x \in \ell(D)$, then we observe that g is a WSTIDF on D of weight $n - 2|\ell(D)|$ and thus $\gamma_{wstI}(D) \leq n - 2|\ell(D)|$. This leads to the desired result. □

Corollary 6. If C_n^o is an oriented cycle of order $n \geq 2$, then $\gamma_{wstI}(C_n^o) = n$.

4. Further lower bounds

We call a set $S \subseteq V(D)$ a *2-packing* of the digraph D if $N^-[u] \cap N^-[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The maximum cardinality of a 2-packing is the *2-packing number* of D, denoted by $\rho(D)$.

Theorem 4. If D is a digraph with $\delta^-(D) \geq 1$, then

$$\gamma_{wstI}(D) \geq \rho(D)(\delta^-(D) + 1) - n.$$

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(D)}\}$ be a 2-packing of D, and let f be a $\gamma_{wstI}(D)$-function. If we define $A = \bigcup_{i=1}^{\rho(D)} N^-(v_i)$, then, since $\{v_1, v_2, \ldots, v_{\rho(D)}\}$ is a 2-packing of D, we note that $|A| = \sum_{i=1}^{\rho(D)} d^-(v_i) \geq \rho(D) \cdot \delta^-(D)$. This leads to

$$\begin{aligned}
\gamma_{wstI}(D) &= \sum_{x \in V(D)} f(x) = \sum_{i=1}^{\rho(D)} f(N^-(v_i)) + \sum_{x \in V(D) \setminus A} f(x) \\
&\geq \rho(D) + \sum_{x \in V(D) \setminus A} f(x) \geq \rho(D) - n + |A| \\
&\geq \rho(D) - n + \rho(D) \cdot \delta^-(D) = \rho(D)(\delta^-(D) + 1) - n.
\end{aligned}$$

□

Theorem 4 leads to $\gamma_{stR}(D) \geq \gamma_{stI}(D) \geq \gamma_{wstI}(D) \geq \rho(D)(\delta^-(D)+1)-n$ for digraphs with $\delta^-(D) \geq 1$. The bound $\gamma_{stR}(D) \geq \rho(D)(\delta^-(D) + 1) - n$ can be found in [13]. The next family of examples will demonstrate that Theorem 4 is sharp.

Example 4. *Let $p \geq 2$ be an integer, and let L be a strongly connected digraph with vertex set $\{v_1, v_2, \ldots, v_p\}$. Let $\{u_i^1, u_i^2, \ldots, u_i^{t_i}\}$ be further pairwise disjoint vertex sets with integers $t_i \geq 1$ for $1 \leq i \leq p$. Now let H be the digraph consisting of L, the vertex sets $\{u_i^1, u_i^2, \ldots, u_i^{t_i}\}$ for $1 \leq i \leq p$ such that v_i dominates $u_i^1, u_i^2, \ldots, u_i^{t_i}$ for $1 \leq i \leq p$. We observe that $\rho(H) = p$, $\delta^-(H) = 1$ and $n(H) = p + t_1 + t_2 + \ldots + t_p$. Therefore Theorem 4 leads to*

$$\gamma_{wstI}(H) \geq p(1 + 1) - (p + t_1 + t_2 + \ldots + t_p) = p - (t_1 + t_2 + \ldots + t_p).$$

If we define the function $f : V(H) \to \{-1, 1, 2\}$ by $f(v_i) = 1$ for $1 \le i \le p$ and $f(x) = -1$ for the remaining vertices x, then f is a WSTIDF on H and thus $\gamma_{wstI}(H) \le p-(t_1+t_2+\ldots+t_p)$. Consequently, $\gamma_{wstI}(H) = \rho(H)(\delta^-(H) + 1) - n(H)$.

Let F_n be the digraph of order $n \ge 2$ with vertex sets $\{u, v\}$, $X = \{x_1, x_2, \ldots, x_s\}$, $Y = \{y_1, y_2, \ldots, y_t\}$ and $Z = \{z_1, z_2, \ldots, z_p\}$ such that u dominates v, v dominates u, u dominates x_i for $1 \le i \le s$ and z_i for $1 \le i \le p$ and v dominates y_i for $1 \le i \le t$ and z_i for $1 \le i \le p$ with $n = 2 + p + s + t$. We define the family $\mathcal{F}_n$ as follows. The digraph F_n belongs to $\mathcal{F}_n$. There is no arc from $X \cup Y \cup Z$ to $\{u, v\}$. In addition, $d^-(x_i) = -1$ for $1 \le i \le s$, $d^-(y_i) = -1$ for $1 \le i \le t$ and there are admissible arcs between vertices of $X \cup Y \cup Z$ such that $d^-(z_i) \le 3$ for $1 \le i \le p$.

Theorem 5. Let D be a digraph of order $n \ge 2$ with $\delta^-(D) \ge 1$. Then $\gamma_{wstI}(D) \ge 4-n$, with equality if and only if D is a member of $\mathcal{F}_n$.

Proof. Proposition 6 leads to the desired bound $\gamma_{wstI}(D) \ge \delta^-(D) + 3 - n \ge 4 - n$. Let now $\gamma_{wstI}(D) = 4 - n$, and let f be a $\gamma_{wstI}(D)$-function. Then there exist at least two vertices u and v with $f(u), f(v) \ge 1$. Since $\gamma_{wstI}(D) = 4 - n$, we note that $f(u) = f(v) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u, v\}$. Clearly, u dominates v, v dominates u and there is no arc from $V(D) \setminus \{u, v\}$ to $\{u, v\}$. Since $f(N^-(x)) \ge 1$, for each vertex x, there is an arc (x, u) or (x, v) for each vertex $x \in V(D) \setminus \{u, v\}$. Let now $V(D) \setminus \{u, v\} = X \cup Y \cup Z$ such that $X = \{x_1, x_2, \ldots, x_s\}$, $Y = \{y_1, y_2, \ldots, y_t\}$ and $Z = \{z_1, z_2, \ldots, z_p\}$ with the properties that u dominates x_i for $1 \le i \le s$ and z_i for $1 \le i \le p$ and v dominates y_i for $1 \le i \le t$ and z_i for $1 \le i \le p$ with $n = 2+p+s+t$. In addition, there are no arcs between X and v and no arcs between Y and u. If $d^-(x_i) \ge 2$ for an integer $1 \le i \le s$, then we obtain the contradiction $f(N^-(x_i)) \le 0$. Therefore $d^-(x_i) = 1$ for an integer $1 \le i \le s$. Analogously, we observe that $d^-(y_i) = 1$ for $1 \le i \le t$. If $d^-(z_i) \ge 4$ for an integer $1 \le i \le p$, then we obtain again the contradiction $f(N^-(z_i)) \le 0$. We deduce that $d^-(z_i) \le 3$ for $1 \le i \le p$, and thus D is a member of $\mathcal{F}_n$.
Conversely, assume that D is a member of $\mathcal{F}_n$. Define the function g by $g(u) = g(v) = 1$ and $g(x) = -1$ for $x \in V(D) \setminus \{u, v\}$. Then it is straightforward to verify that f is a WSTIDF on D of weight $4 - n$ and therefore $\gamma_{wstI}(D) = 4 - n$. □

Let F be a member of $\mathcal{F}_n$ such that $\Delta^-(F) = 3$. Then it follws from Theorem 5 that $\gamma_{wstI}(F) = 4 - n = \Delta^-(F) + 1 - n$. Therefore Proposition 5 is sharp too.

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