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Research Article

Join standard graph of a lattice

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Abstract: In this paper, we introduce and investigate the join standard graph $G_S(L)$ of a finite lattice L. We explore structural properties of the graph such as connectedness, girth, and provide necessary and sufficient conditions for the existence of universal and isolated vertices. We show that a lattice homomorphism φ from L_1 to L_2 induces a graph homomorphism between $G_S(L_1)$ and $G_S(L_2)$. We further analyze the relationship between the graph of a lattice product and the product of graphs of its constituent lattices. Subsequently, we establish a condition under which the graph becomes hypertriangulated. We prove that the graph $G_S(L)$ is complemented if and only if the underlying lattice has cardinality at most two. Finally, we provide a criterion under which the subgraph $G_S(L) - 1$ becomes disconnected.

Keywords: standard element, lattice, cartesian product, totally disconnected.

AMS Subject classification: 06B10, 05C25, 06D05, 06D50, 05C76

1. Introduction and Preliminaries

Distributive lattices are a fundamental part of lattice theory, which evolved from Boolean algebra. Grätzer and Schmidt [5] introduced the concept of standard elements in lattices, which form distributive sublattices, behaving like elements of a

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distributive lattice. A major area of research involves exploring graphs that are based on algebraic structures via groups, rings, lattices etc. These graphs provide valuable insights into the behaviour and the properties of the algebraic structures they represent. The zero divisor graph was first introduced by Beck [3] in the context of commutative rings. Later, authors like Alizadeh et al. [1], Joshi et al. [11], Halaš and Länger [8] studied it in ordered structures. Nimbhokar et al. [15] extended the concept to meet-semilattices with 0 and proved a version of Beck's conjecture. Halaš and Jukl [7] extended these results to posets (qosets) with 0. Khiste and Joshi [12] studied the basic properties such as connectivity, diameter and girth of the zero-divisor graph $\Gamma(M_n(L))$ of $n \times n$ matrices over lattices with 0. They established that Beck's Conjecture is true for $\Gamma(M_n(L))$ and showed that $\Gamma(M_2(C_n))$ is a hyper-triangulated graph. Ülker [17] defined a graph for bounded lattices using essentiality of elements and studied graphs of those lattices whose zero divisor graphs and incomparability graphs coincide with essential element graph. Sahoo et al. [16] introduced and investigated the notion of superfluous element graph $S(L)$ and showed that $S(L)$ is complete if and only if every proper non-superfluous element is a dual atom.

In this paper, we introduce the join standard graph $G_S(L)$ of a lattice L and explore the structural properties like connectivity, diameter etc. of $G_S(L)$. We give a necessary and sufficient condition for a vertex to be an isolated vertex in $G_S(L) - 1$. We explore the conditions under which $G_S(L)$ is hyper-triangulated and prove that $G_S(L)$ is complemented if and only if $|L| \leq 2$.

For a lattice $(L, \leq)$ and $a, b \in L$, we say a is covered by b if $a \leq b$ and there is no $c \in L$ such that $a < c < b$ and denote it by $a \prec b$. The principal ideal generated by $b \in L$ is denoted by $(b]$ and is given by $(b] = \{x \in L : x \leq b\}$. Also $[b)$ denotes the principal filter generated by b and it is given by $[b) = \{x \in L : b \leq x\}$. By $a \parallel b$, we mean a is incomparable with b. A chain with n elements is denoted by C_n [6].

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. A simple graph is a graph having no loops or multiple edges. We denote an edge in G by uv for $u, v \in V(G)$. A path of a graph G is an alternating sequence of distinct vertices and edges $v_0, e_1, v_1, \cdots, v_{n-1}, e_n, v_n$, beginning and ending with vertices in which each edge is incident with the two vertices immediately preceding and following it. The distance $d(u, v)$ between vertices u and v of a graph G is the length of the shortest path between u and v. A graph G is connected if there is a path between each pair of vertices. The diameter $diam(G)$ of a connected graph G is the maximum distance between two vertices of G. A vertex v of a graph G is called universal if it is adjacent to all the other vertices of G. By a clique, we mean a complete subgraph of the graph G [9, 10, 18].

Definition 1 ([6]). A lattice L is said to be distributive, if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in L$.

Definition 2 ([5]). An element s of the lattice $(L, \wedge, \vee)$ is called standard if

$$x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)$$

for all $x, y \in L$.

In any lattice L, the minimum element 0 and the maximum element 1 (if they exist) are standard elements.

Proposition 1 ([6]). *Let L and L' be two lattices, let φ be a homomorphism of L onto L'. If $a \in L$ is standard in L then $\varphi(a)$ is standard in L'.*

2. Join standard graph of a Lattice

In the sequel, L denotes a lattice and $0, 1$ denote the minimum and maximum element of L respectively.

Let L be a finite lattice. Let $S \subseteq L$ be the set of all standard elements of L. The join standard graph of L, denoted by $G_S(L)$, is the simple graph whose vertex set is $V(G_S(L)) = L$ and edge set is $E(G_S(L)) = \{xy : x \neq y,\ x \vee y \in S\}$.

If the condition $x = y$ is allowed in the edge definition, the resulting graph may contain loops. In this case, the join standard graph is denoted by $G_S^o(L)$.

For any subset $K \subseteq L$, the subgraph of $G_S^o(L)$ induced by K (allowing loops) is denoted by $G_S^o[K]$ while the subgraph of $G_S(L)$ induced by K is denoted by $G_S[K]$.

Example 1. Consider the lattice $L = N_5$ shown in Figure 1. The set of standard elements of L is $\{0, p_2, 1\}$. The corresponding join standard graph $G_S(N_5)$ is given in Figure 2.

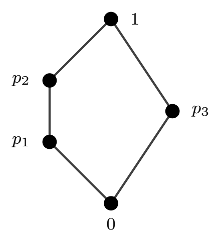


Figure 1. **Lattice N_5**

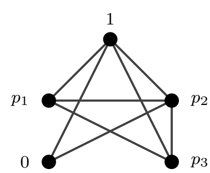


Figure 2. **Join standard graph $G_S(N_5)$**

Example 2. Consider the lattices L_1 and L_2 shown in Figures 4 and 5 respectively. Their corresponding join standard graphs are given in Figures 6 and 7 respectively.

Note that the lattices L_1 and L_2 are not isomorphic whereas their join standard graphs are isomorphic.

Remark 1. Let L be a lattice. Then,

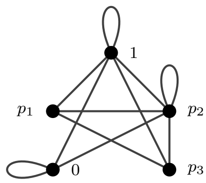


Figure 3. **Join standard graph with loops** $G_S^o(N_5)$

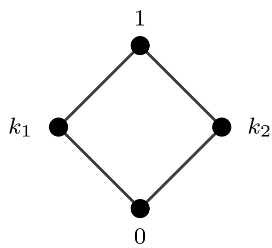


Figure 4. **Lattice** L_1

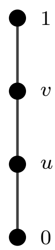


Figure 5. **Lattice** L_2

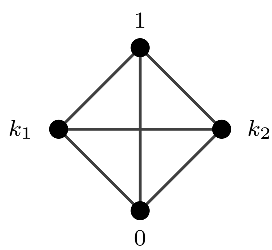


Figure 6. **Join standard graph** $G_S(L_1)$

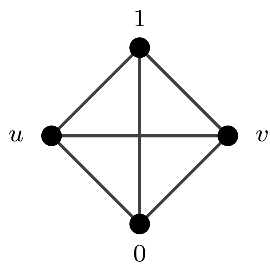


Figure 7. **Join standard graph** $G_S(L_2)$

(i) 0 is adjacent to only standard elements in $G_S(L)$.

(ii) Only standard elements will have loop in $G_S^o(L)$.

(iii) $\deg(0) - 1$ in $G_S^o(L)$ gives the number of standard elements in L.

(iv) It can be observed that 1 is always a universal vertex in $G_S(L)$, since $1 \vee x = 1$ for any $x \in L$.

Proposition 2. *If 0 is a universal vertex, then L is distributive and $G_S(L)$ is a complete graph.*

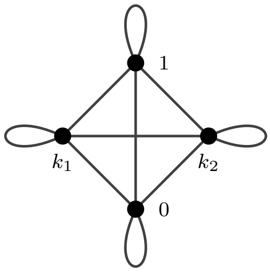


Figure 8. Join standard graph with loops $G_S^o(L_1)$

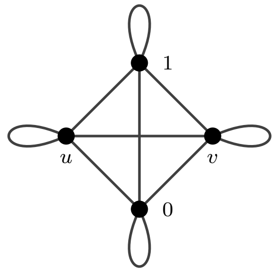


Figure 9. Join standard graph with loops $G_S^o(L_2)$

Proof. In any lattice, 0 is always standard. Note that every standard element of L is a distributive element of L. We have $0x \in E(G_S(L))$ for any $0 \neq x \in L$ which implies x is standard for any $0 \neq x \in L$. Thus, every element of L is distributive which implies L is distributive. Since every element of L is standard, for any $x, y \in V(G_S(L))$, xy is an edge in $G_S(L)$. Hence $G_S(L)$ is a complete graph. □

Proposition 3. *$G_S(L)$ is connected and the* $\operatorname{diam} G_S(L) \leq 2$.

Proof. Follows from the fact that 1 is a universal vertex in $G_S(L)$. □

Proposition 4. *The subgraph induced by dual atoms of a lattice L is complete in $G_S(L)$.*

Proof. Let $D = \{x \in L : x \prec 1\}$ be the set of dual atoms of L. For $x, y \in D$, since $x < x \vee y \leq 1$ and x is a dual atom, it follows that $x \vee y = 1$, which is a standard element in L. Therefore, $xy \in E(G_S(L))$. Since x, y are arbitrary in D, it follows that $G_S[D]$ is a complete graph. □

From the above proposition, we can observe that $G_S[D]$ is a clique and the clique number of $G_S(L)$ is at least the number of dual atoms of L.

Remark 2. For a complete subgraph of $G_S(L)$, the vertices need not be dual atoms in L. Consider the lattice L_3 shown in Figure 10. Note that the induced subgraph obtained from $\{0, h_1, 1\}$ is complete, even though none of the vertices are dual atoms in L.

Proposition 5 ([5]). *The set S of all standard elements of a lattice L forms a sublattice of L. In fact, it forms a distributive sublattice of L.*

From Proposition 5, it follows that $G_S[S]$ is a complete subgraph of $G_S(L)$.

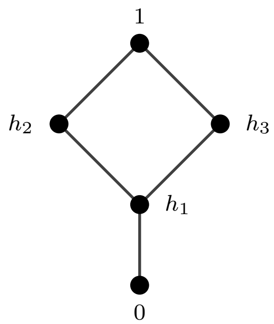


Figure 10. **Lattice L_3**

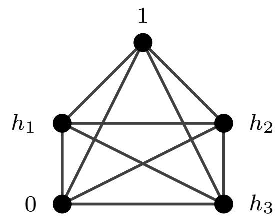


Figure 11. Join standard graph $G_S(L_3)$

Proposition 6. *If L is a lattice having a unique dual atom p, then p is a standard element of L.*

Proof. Let $x, y \in L$. Then there exist two cases $y = 1$ and $y \neq 1$.
Case 1: $y = 1$
In this case, we have $x \wedge (p \vee 1) = x \wedge 1 = x$ and $(x \wedge p) \vee (x \wedge 1) = (x \wedge p) \vee x = x$.
Case 2: $y \neq 1$
We have $p \leq p \vee y \leq 1$. Since p is the unique dual atom and $y \neq 1$, we must have $y \leq p$. Therefore, $x \wedge y \leq x \wedge p$. This leads to $x \wedge (p \vee y) = x \wedge p$ and $(x \wedge p) \vee (x \wedge y) = (x \wedge p)$. Thus, p is a standard element. □

Remark 3. Let L be a lattice with $|L| = n \geq 3$. Since 1 is a universal vertex, $G_S(L)$ has at least $(n-1)$ edges. Suppose L has a unique dual atom d, then $0d \in E(G_S(L))$. If L has two distinct dual atoms c and f, then $cf \in E(G_S(L))$. Therefore $|E(G_S(L))| \geq (n - 1 + 1) = n$. This implies that $G_S(L)$ can never be a tree. Consequently, the subgraph $G_S(L) - 1$ can never be totally disconnected.

Proposition 7. *Let K be a sublattice of L. Then, there exists a one-to-one homomorphism from $G_S[K]$ to $G_S(K)$.*

Proof. Consider the identity map $\mathcal{I} : G_S[K] \to G_S(K)$. It is obvious that $\mathcal{I}$ is a one-to-one map. Let $x, y \in K$ such that $xy \in E(G_S[K])$. Then $x \vee y$ is a standard element in L. Note that $x \vee y$ is standard in K implies that $xy \in E(G_S(K))$. Therefore $\mathcal{I}(x)\mathcal{I}(y) \in E(G_S(K))$, showing that $\mathcal{I}$ is a graph homomorphism. □

We illustrate the above proposition with the following example.

Example 3. Consider the lattice $L_4 = \{0, a, b, c, d, 1\}$ shown in Figure 12. Consider the sublattice $K = \{0, b, d\}$. Then $G_S(K)$ and $G_S[K]$ are shown in Figures 14 and 15 respectively. Here, identity map $\mathcal{I}$ is a one-to-one graph homomorphism from $G_S[K]$ to $G_S(K)$.

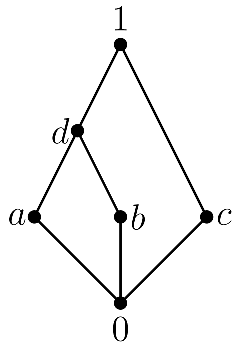


Figure 12. **Lattice** L_4

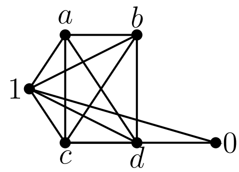


Figure 13. Join standard graph $G_S(L_4)$

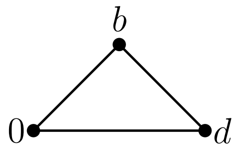


Figure 14. Join standard graph $G_S(K)$

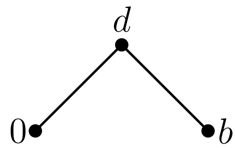


Figure 15. Subgraph $G_S[K]$ induced by K

Proposition 8. *For a lattice L,* girth($G_S(L)$)=*3 where* $|L| \geq 3$.

Proof. Suppose L has a unique dual atom a, then a is a standard element. Hence the set $\{0, a, 1\}$ forms a triangle in $G_S(L)$. Now, suppose L has at least two dual atoms, x and y. Then $x \vee y = 1$, which is a standard element. This implies that $\{1, x, y\}$ forms a triangle in $G_S(L)$. □

Remark 4. Proposition 8 shows that $G_S(L)$ can never be a tree whenever $|L| \geq 3$.

Proposition 9. *Let $k \in L$. Then $k \in V(G_S(L))$ is universal vertex in $G_S(L)$ if and only if every element of $[k)$ is standard in L.*

Proof. Let $k \in V(G_S(L))$ be a universal vertex in $G_S(L)$. Then $0k \in E(G_S(L))$, which implies k is a standard element in L. Now, let $x \in [k), x \neq k$. Then $xk \in E(G_S(L))$. So $x \vee k = x$ is standard. Since x was arbitrary, every element of $[k)$ is standard. Conversely, assume that every element of $[k)$ is standard. Since $l \vee k \in [k)$ for any $l \in L$, we get $l \vee k$ is standard for any $l \in L$. Hence k is universal vertex in $G_S(L)$. □

Proposition 10. *Let p and q be two vertices of $G_S(L)$. Then there exists a path between p and q in $G_S(L)$ if and only if there exists a standard element $s \in L$ such that $p \vee q \leq s$.*

Proof. Let p, q be two vertices of $G_S(L)$. Suppose there exists a path $p - u_1 - u_2 - \cdots - u_k - q$ (assumed to be the shortest path). Then the elements $p \vee u_1, u_1 \vee u_2, \ldots, u_k \vee q$ are standard in L. Since the set of standard elements forms a sublattice, the join $s = p \vee u_1 \vee u_2 \vee \cdots \vee u_k \vee q$ is also a standard element and $p, q \leq s$. Hence $p \vee q \leq s$. Conversely, suppose there exists a standard element s in L such that $p \vee q \leq s$. Then since $p \vee s = s$ and $q \vee s = s$, we obtain the path $p - s - q$ in $G_S(L)$ from p to q. □

Proposition 11. *$G_S(L)$ is regular if and only if L is distributive.*

Proof. Suppose $G_S(L)$ is regular. Then $\deg(0) = \deg(1)$. Since 1 is a universal vertex, we get that 0 is a universal vertex. Then, by Proposition 2, L is distributive. Conversely, suppose L is distributive. Then $G_S(L) = K_{|L|}$. Hence, $G_S(L)$ is regular. □

Definition 3 ([9]). For a graph G, $G - e$ is the graph obtained by removing the edge e and, $G - v$ is the graph obtained by removing the vertex v and all the edges incident on it.

Definition 4 ([4]). In a lattice L with 1, an element $a \in L$ is called superfluous if $a \vee b \neq 1$ for every $b \neq 1$.

Proposition 12. *Let b be a non-zero proper element of L. Then b is an isolated vertex in $G_S(L) - 1$ if and only if any element of $[b) \setminus \{1\}$ is not standard in L and b is superfluous in L.*

Proof. Let $b \in L$ be an isolated vertex in $G_S(L) - 1$. By hypothesis, it is clear that $b \neq 1$ and $b \neq 0$. Since b is an isolated vertex, $0b \notin E(G_S(L) - 1)$, which implies that $0 \vee b = b$ is not standard in L. Now, let $c > b$. Suppose $c \neq 1$. Since $cb \notin E(G_S(L) - 1)$, it follows that $c \vee b = c$ is not standard. Thus, any element of $[b) \setminus \{1\}$ is not standard. Next, suppose there exists $d \neq 1$ in L such that $b \vee d = 1$. Then, $bd \in E(G_S(L) - 1)$, which contradicts the assumption that b is an isolated vertex. Hence b is superfluous. Conversely, since $b \in [b) \setminus \{1\}$, b is not standard. For any $a < b$, we have $a \vee b = b$ is not a standard element in L. Hence $ab \notin E(G_S(L) - 1)$ for any $a < b$. By hypothesis, $bd \notin E(G_S(L) - 1)$ for any $1 \neq d > b$. Let $1 \neq c \in L$. Since b is superfluous in L, $b \vee c \neq 1$. So $b \vee c \in [b) \setminus \{1\}$ which implies $bc \notin E(G_S(L) - 1)$. Therefore, b is an isolated vertex in $G_S(L) - 1$. □

Definition 5 ([13]). Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. A mapping $f : V \to V'$ is called a

(i) graph homomorphism if $xy \in E$ implies $f(x)f(y) \in E'$.

(ii) graph weak homomorphism if $xy \in E$ and $f(x) \neq f(y)$ implies $f(x)f(y) \in E'$.

Definition 6 ([6]). Let L be a lattice and α be a congruence relation on L. Let $L/\alpha = \{a/\alpha : a \in L\}$. Define the operations on L/α as follows:

$$\begin{aligned} a/\alpha \wedge b/\alpha &= (a \wedge b)/\alpha \\ a/\alpha \vee b/\alpha &= (a \vee b)/\alpha. \end{aligned}$$

With these operations, L/α forms a lattice, called the quotient lattice of L modulo α.

Proposition 13. *Let $f : L_1 \to L_2$ be a lattice homomorphism from L_1 into L_2. Then*

(i) If f is onto, then f is a graph homomorphism (graph weak homomorphism) from $G_S^o(L_1)$ to $G_S^o(L_2)$ ($G_S(L_1)$ to $G_S(L_2)$).

(ii) If f is one-to-one, then there exists a graph homomorphism (graph weak homomorphism) from $G_S^o(L_2)$ to $G_S^o(L_1)$ ($G_S(L_2)$ to $G_S(L_1)$).

Proof. (i) Consider the map $f : V(G_S^o(L_1)) \to V(G_S^o(L_2))$. We have $f(L_1) = L_2$. Let $xy \in E(G_S^o(L_1))$ where $x, y \in V(G_S^o(L_1))$. Then $x \vee y = s$ is a standard element in L_1. Since f is a lattice homomorphism, we get $f(x) \vee f(y) = f(s)$. By Proposition 1, $f(s)$ is standard in L_2. Therefore $f(x)f(y) \in E(G_S^o(L_2))$. Thus f is a graph homomorphism from $G_S^o(L_1)$ to $G_S^o(L_2)$. A similar argument applies in the case of graph without loops.

(ii) Define $\phi : V(G_S^o(L_2)) \to V(G_S^o(L_1))$ by

$$\phi(x) = \begin{cases} f^{-1}(x), & x \in f(L_1) \\ 1_{L_1}, & x \in L_2 \setminus f(L_1) \end{cases}$$

Note that $f(L_1)$ is a sublattice of L_2. Let $xy \in E(G_S^o(L_2))$.
Case 1. Suppose $x, y \in f(L_1)$. Then $x = f(a), y = f(b)$ for some $a, b \in L_1$. We have $\phi(x) = a,\ \phi(y) = b$ and $x \vee y = z$ is a standard element in L_2. Since $f(L_1)$ is a sublattice of L_2, $z \in f(L_1)$. So $z = f(c)$ for some $c \in L_1$. We have $x \vee y = f(a) \vee f(b) = z = f(c)$. Since f is a one-to-one homomorphism, it follows that $a \vee b = c$. Note that, $f^{-1} : f(L_1) \to L_1$ is an onto map and $f(c)$ is standard in $f(L_1)$. By Proposition 1, we get $c = f^{-1}(f(c))$ is standard in L_1. Hence $ab \in E(G_S^o(L_1))$, which implies $\phi(x)\phi(y) \in E(G_S^o(L_1))$.
Case 2. Suppose $x \in f(L_1), y \notin f(L_1)$. Then $\phi(y) = 1_{L_1}$. So, $\phi(x) \vee \phi(y) = \phi(x) \vee 1_{L_1} = 1_{L_1}$, which is standard in L_1. Hence $\phi(x)\phi(y) \in E(G_S^o(L_1))$.
Thus in both cases, ϕ defines a graph homomorphism.

□

Remark 5. In Proposition 13 (i), the surjectivity of the map f is necessary. If f is not onto, then it need not be a graph homomorphism. For example, consider the lattices shown in Figures 16 and 17. The join standard graphs of C_3 and N_5 are shown in Figures 18 and 19 respectively. Define a map $f : C_3 \to N_5$ by $f(0) = 0,\ f(x) = a,\ f(1) = b$. Clearly, f is not surjective. It can be observed that f is a lattice homomorphism. However it is not a graph homomorphism because $0x \in E(G_S^o(C_3))$ but $f(0)f(x) = 0a \notin E(G_S^o(N_5))$.

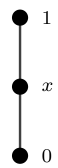


Figure 16. **Lattice** $C_3 = \{0, x, 1\}$

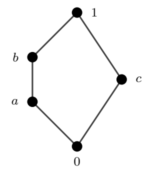


Figure 17. **Lattice** N_5

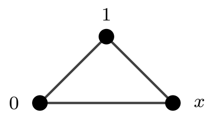


Figure 18. Join standard graph $G_S(C_3)$

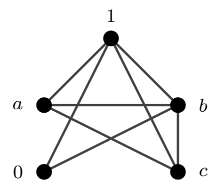


Figure 19. Join standard graph $G_S(N_5)$

Corollary 1. *For a lattice L, there exists a graph homomorphism from $G^o_S(L)$ to $G^o_S(L/\theta)$. More generally, if θ_1, θ_2 are two congruence relations on L with $\theta_1 \subseteq \theta_2$ then there exists a surjective map φ from L/θ_1 to L/θ_2. Moreover, φ is a graph homomorphism from $G^o_S(L/\theta_1)$ to $G^o_S(L/\theta_2)$.*

Proof. The proof follows from the second isomorphism theorem for lattices and Proposition 13. □

Remark 6. Let $\mathrm{Aut}(L)$ be the group of automorphisms of a lattice L and $\mathrm{Aut}(G_S(L))$ be the group of automorphisms of $G_S(L)$. Note that for any lattice L, $\mathrm{Aut}(L)$ is a subgroup of $\mathrm{Aut}(G_S(L))$. For the lattice M_5 shown in Figure 32, we have $\mathrm{Aut}(M_5) = \mathrm{Aut}(G_S(M_5))$. The following example illustrates a case where $\mathrm{Aut}(L)$ is a proper subgroup of $\mathrm{Aut}(G_S(L))$. Consider the map $f : C_3 \to C_3$ on the lattice C_3 shown in Figure 16, defined by $f(0) = a, f(x) = 0$ and $f(1) = 1$. This map is a graph isomorphism from $G_S(C_3)$ to $G_S(C_3)$ but not a lattice isomorphism on C_3 since $f(0 \wedge x) = a \neq 0 = f(x) \wedge f(0)$. For a distributive lattice L, the automorphism group of its associated graph $G_S(L)$ is the symmetric group S_n where n is the number of elements in L. Consider the map $f : L \to L$ defined by $f(0) = 1, f(a) = a$ for $a \notin \{0, 1\}$ and $f(1) = 0$. Clearly $f \in \mathrm{Aut}(G_S(L))$ but $f \notin \mathrm{Aut}(L)$. This shows that in the case of distributive lattices, the lattice automorphism group $\mathrm{Aut}(L)$ is a proper subgroup of the graph automorphism group $\mathrm{Aut}(G_S(L))$. This leads to an interesting question: For which non-distributive lattices does the equality $\mathrm{Aut}(L) = \mathrm{Aut}(G_S(L))$ hold? Identifying such lattices may help us to understand how the elements are positioned or related within the lattice by examining the structure of the graph.

Remark 7 shows that the definition of the join standard graph has catergory theoretic significance. For the basic definitions of category theory, we refer to Tom Leinster [14].

Remark 7. From Proposition 13 (i), recall that if f is a surjective lattice homomorphism, then $f = f^*$ is a graph homomorphism between the corresponding join standard graphs. Let $(\mathfrak{L}, surj)$ denote the category of lattices with surjective lattice homomorphisms and let $(\mathfrak{G}, gh)$ denote the category of graphs (with loops) with graph homomorphisms. The mapping $G_S^o : \mathfrak{L} \to \mathfrak{G}$ defined earlier becomes a functor if we define $G_S^o(f) = f^*$. If $g : L_1 \to L_2$ and $f : L_2 \to L_3$, then $G_S^o(f \circ g) = (f \circ g)^* = G_S^o(f) \circ G_S^o(g)$. Also $G_S^o(\mathrm{id}_L) = \mathrm{id}_{G_S^o(L)}$ where id_L is identity map on L. Thus, G_S^o defines a covariant functor from $(\mathfrak{L}, surj)$ to $(\mathfrak{G}, gh)$.

Definition 7 ([13]). Let $\varrho \subseteq V \times V$ be an equivalence relation on the vertex set V of a graph $G = (V, E)$, and denote by x_ϱ the equivalence class of $x \in V$ with respect to ϱ. Then $G_\varrho = (V_\varrho, E_\varrho)$ is called the factor graph of G with respect to ϱ where $V_\varrho = V/\varrho$ and $x_\varrho y_\varrho \in E_\varrho$ if there exist $x' \in x_\varrho$ and $y' \in y_\varrho$ with $x'y' \in E$.

Remark 8. Let θ be a congruence relation on a lattice L. Then if an element s is a standard element in L, then s/θ is a standard element in L/θ.

Proposition 14. *$G_S^o(L)/\theta$ is subgraph of $G_S^o\,(L/\theta)$.*

Proof. Let $(a/\theta)(b/\theta)$ be an edge in $G_S^o(L)/\theta$. Then there exist $u \in a/\theta$, $v \in b/\theta$ such that $u \vee v$ is standard in L. So $u \vee v = s$ is a standard element in L. This implies that $(u/\theta) \vee (v/\theta)$ is standard element in L/θ. Hence, $(u/\theta)(v/\theta)$ that is, $(a/\theta)(b/\theta) \in E(G_S^o(L/\theta))$. □

From the construction of the quotient graph of $G_S^o(L)$ and the join standard graph of the quotient lattice L/θ, a natural question arises: are $G_S^o(L)/\theta$ and $G_S^o\,(L/\theta)$ always isomorphic?
The following example shows that they need not be isomorphic always.

Example 4. Consider the lattice shown in Figure 20. Let Θ be the congruence relation on L_5 such that $L_5/\Theta = \{\{0\}, \{1, d\}, \{a, c\}, \{b, e\}\}$. Note that $G_S^o(L_5)/\Theta$ is subgraph of $G_S^o\,(L_5/\Theta)$ but they are not isomorphic.

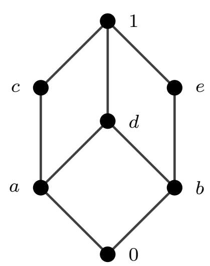


Figure 20. Lattice $L_5 = \{0, a, b, c, d, e, 1\}$

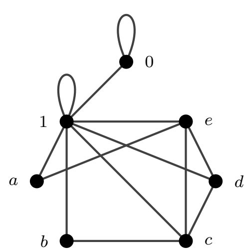
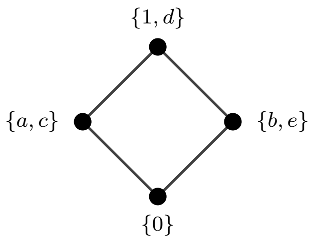


Figure 21. Join standard graph with loops $G_S^o(L_5)$



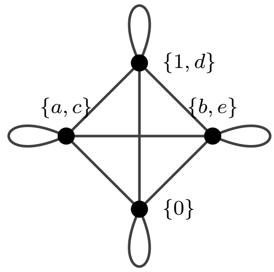


Figure 22. **The quotient lattice L_5/Θ** Figure 23. Join standard graph $G_S^o(L_5/\Theta)$

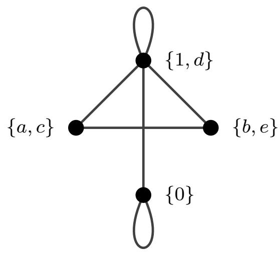


Figure 24. **The quotient graph $G_S^o(L_5)/\Theta$**

For the definitions of the Cartesian product ($\Box$), the Direct product ($\times$), and the Strong product ($\boxtimes$) of graphs, we refer to Hammack et al.[9].

Definition 8 ([6]). The direct product of lattices L and K is denoted by $L \times K$ and $\wedge, \vee$ on $L \times K$ are defined by

$$\begin{aligned}(a_0, b_0) \vee (a_1, b_1) &= (a_0 \vee a_1, b_0 \vee b_1) \\ (a_0, b_0) \wedge (a_1, b_1) &= (a_0 \wedge a_1, b_0 \wedge b_1)\end{aligned}$$

where $a_0, a_1 \in L$ and $b_0, b_1 \in K$.

Remark 9. Let L and K be two lattices. Then an element (p_1, p_2) is standard in $L \times K$ if and only if p_1 and p_2 are standard in L and K respectively.

Proposition 15. *For two lattices L_1 and L_2, $G_S^o(L_1 \times L_2) = G_S^o(L_1) \times G_S^o(L_2)$.*

Proof. Define a map $\varphi : V(G_S^o(L_1 \times L_2)) \to V(G_S^o(L_1) \times G_S^o(L_2))$ by $\varphi((p,q)) = (p,q)$. Clearly φ is one-to-one and onto. We have $V(G_S^o(L_1 \times L_2)) = V(G_S^o(L_1) \times G_S^o(L_2))$. Let $(p,q)(r,s) \in E(G_S^o(L_1 \times L_2))$. Then $(p \vee r, q \vee s)$ is standard in $L_1 \times L_2$.

This implies $p \vee r$ is standard in L_1 and $q \vee s$ is standard in L_2. Thus, $pr \in E(G_S^o(L_1))$ and $qs \in E(G_S^o(L_2))$. Hence, $\varphi(p,q)\varphi(r,s) = (p,q)(r,s) \in E(G_S^o(L_1) \times G_S^o(L_2))$. Now, let $(g,h)(u,v) \in E(G_S^o(L_1) \times G_S^o(L_2))$. Then $gu \in E(G_S^o(L_1))$ and $hv \in E(G_S^o(L_2))$. Thus $(g \vee u, h \vee v)$ is standard in $L_1 \times L_2$. Hence, $\varphi(g,h)\varphi(u,v) = (g,h)(u,v) \in E(G_S^o(L_1 \times L_2))$. Thus, φ and φ^{-1} are graph homomorphisms and hence, φ is an isomorphism. □

Proposition 16. *Let L_1 and L_2 be two lattices. Then $G_S^o(L_1 \times L_2) = G_S^o(L_1) \boxtimes G_S^o(L_2)$ if and only if both L_1, L_2 are distributive.*

Proof. Let $G_S^o(L_1 \times L_2) = G_S^o(L_1) \boxtimes G_S^o(L_2)$. By Proposition 15, we get $G_S^o(L_1) \times G_S^o(L_2) = G_S^o(L_1) \boxtimes G_S^o(L_2)$. Suppose one of L_1 and L_2 is non-distributive, say L_1 is non-distributive. Then there exists a non-standard element $n' \in L_1$. Clearly $(n',0)(n',0) \in E(G_S^o(L_1) \boxtimes G_S^o(L_2))$ for $0 \in L_2$. However, since n' is non-standard, we have $(n',0)(n',0) \notin E(G_S^o(L_1) \times G_S^o(L_2))$, which leads to a contradiction. Therefore, both L_1 and L_2 are distributive. Now we show that, if L_1 and L_2 are distributive, then $G_S^o(L_1) \boxtimes G_S^o(L_2) = G_S^o(L_1) \times G_S^o(L_2)$. Let $G_S^o(L_1) \boxtimes G_S^o(L_2) = H$ and $G_S^o(L_1) \times G_S^o(L_2) = K$. We show that $K = H$. Clearly, $V(K) = V(H)$ and $E(K) \subseteq E(H)$. It remains to show $E(H) \subseteq E(K)$. Let $(u,v)(w,x)$ be an edge in H. Then, either $u = w$ and $vx \in E(G_S^o(L_2))$ or $uw \in E(G_S^o(L_1))$ and $v = x$ or $uw \in E(G_S^o(L_1))$ and $vx \in E(G_S^o(L_2))$.
Case 1: Suppose $u = w$ and $vx \in E(G_S^o(L_2))$. Since L_1 is distributive and vertex u has loop on it, we get $(u,v)(w,x) \in E(K)$.
Case 2: Suppose $uw \in E(G_S^o(L_1))$ and $v = x$. Since L_2 is distributive and vertex v has loop on it, we get $(u,v)(w,x) \in E(K)$.
Case 3: Suppose $uw \in E(G_S^o(L_1))$ and $vx \in E(G_S^o(L_2))$, it follows that $(u,v)(w,x) \in E(K)$. Thus in any case we get $(u,v)(w,x) \in E(K)$ which implies $E(H) \subseteq E(K)$ and hence, $E(H) = E(K)$. Thus, $K = H$. Converse part follows from Proposition 15 with the argument $K = H$. □

As shown in Example 5, the converse of Proposition 16 may not be true if one of the lattices is non-distributive. In the same example, it can be seen that Proposition 15 need not hold for a join standard graph without loops. However, $G_S(L_1) \times G_S(L_2)$ is always a subgraph of $G_S(L_1 \times L_2)$.

Example 5. Consider the lattice N_5 shown in Figure 1 and the lattice C_2 shown in Figure 25. Then the product $N_5 \times C_2$ is shown in Figure 26.

Definition 9 ([12]). A graph G is said to be

(i) triangulated if every vertex of G is contained in a triangle.

(ii) hypertriangulated if every edge of G is contained in a triangle.

Figure 25. **Lattice** $C_2 = \{0, 1\}$

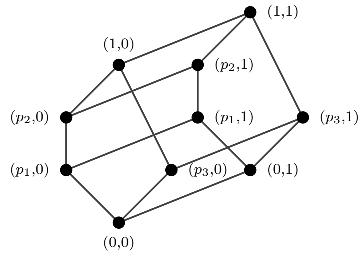


Figure 26. **Lattice** $N_5 \times C_2$

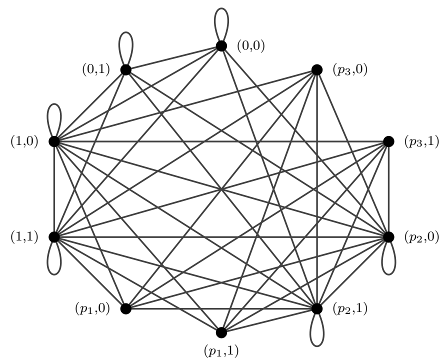


Figure 27. **Join standard graph with loops** $G_S^o(N_5 \times C_2)$

Proposition 17. *If a lattice L with $|L| \geq 3$ has a dual atom which is standard, then $G_S(L)$ is hypertriangulated.*

Proof. Let d be a dual atom which is standard in L and, let uv be an edge in $G_S(L)$. If $u, v \notin \{0, 1\}$, then clearly the set $\{u, v, 1\}$ forms a triangle. Suppose $u = 0, v = 1$. Then, we get a triangle uvd. Suppose $u = 0$ and $v \neq 1$. Then, uv is contained in triangle $0v1$, that is, the triangle $uv1$. Assume that $u = 1$. If v is a non-zero standard element, then uv is contained in triangle $0uv$. If v is non-standard, then clearly $v \neq 1$. We have either $v \leq d$ or $v \parallel d$. If $v \parallel d$, then since d is dual atom, we get $v \vee d = 1$, and hence, uv is contained in triangle udv. When $v \leq d$, we get $v \vee d = d$ which is a standard element and hence, we get triangle uvd. Thus, in all cases the edge uv is contained in a triangle. Therefore, $G_S(L)$ is hypertriangulated. □

Definition 10 ([2]). For distinct vertices a and b in a graph, we say a and b are orthogonal, if a and b are adjacent and there is no vertex c of G which is adjacent to both a and b. We denote it by $a \perp b$ and we call a as orthogonal complement of b. If a and b are not orthogonal, then we denote it by $a \not\perp b$.

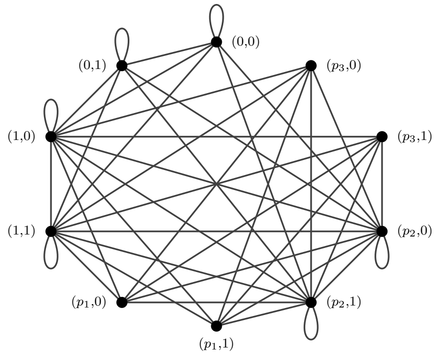


Figure 28. The direct product of the join standard graphs-$G_S^o(N_5) \times G_S^o(C_2)$

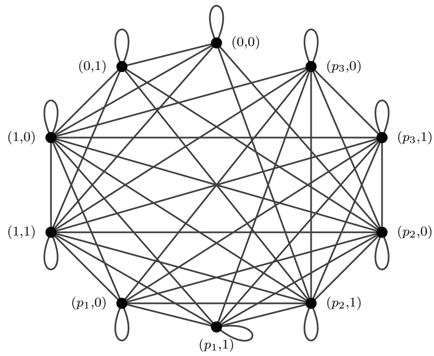


Figure 29. The strong product of the join standard graphs-$G_S^o(N_5) \boxtimes G_S^o(C_2)$

Definition 11 ([2]). A graph G is called complemented if for each vertex a of G, there is a vertex b such that $a \perp b$.

Definition 12 ([2]). A graph G is uniquely complemented if G is complemented and whenever $a \perp b, a \perp c$, b and c are adjacent to exactly same vertices.

Proposition 18. *In $G_S(L)$, if $u \perp v$ for some $u, v \in V(G_S(L))$, then either $u = 1$ or $v = 1$. Furthermore, if $u = 1$, then either $v = 0$ or v is a non-standard element.*

Proof. Let $u \perp v$ in $G_S(L)$. Then, $uv \in E(G_S(L))$ and u and v do not have a common neighbourhood. Suppose $u, v \neq 1$. Then, 1 will be in the common neighbourhood of u and v, a contradiction. Therefore, either $u = 1$ or $v = 1$. Without loss of generality, let $u = 1$. Then, $v \neq 1$. If v is standard and $v \neq 0$, then 0 will be in the common neighbourhood of u and v which is a contradiction. Thus, either $v = 0$ or v is a non-standard element. □

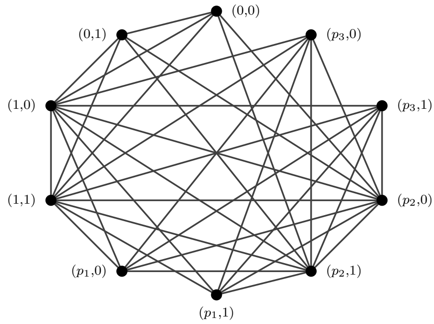


Figure 30. Join standard graph $G_S(N_5 \times C_2)$

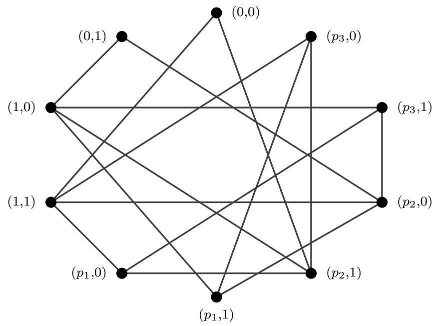


Figure 31. Direct product of join standard graphs-$G_S(N_5) \times G_S(C_2)$

Proposition 19. *A lattice L has no non-zero proper standard element if and only if $0 \perp u$ in $G_S(L)$ for some $u \in V(G_S(L))$. Furthermore, if 0 has an orthogonal complement in $G_S(L)$, then 1 is the unique orthogonal complement of 0 in $G_S(L)$.*

Proof. Assume that L has no non-zero proper standard element. Then, $01 \in E(G_S(L))$ and $0a$ is not an edge in $G_S(L)$ for any $a \neq 1$. This implies that, there is no element in the common neighbourhood of 0 and 1. Therefore, $0 \perp 1$. Conversely, let $0 \perp u$ in $G_S(L)$ for some $u \in V(G_S(L))$. Then, $0u \in E(G_S(L))$, and u is a standard element. By Proposition 18, we get $u = 1$. Suppose there exists a proper non-zero standard element s in L, then s will be in the common neighbourhood of 0 and 1 which is a contradiction to $0 \perp 1$. Now, assume that 0 has an orthogonal complement a in $G_S(L)$. By Proposition 18, we conclude that $a = 1$, which gives the uniqueness of orthogonal complement of 0. □

Proposition 20. *$G_S(L)$ is complemented if and only if $|L| \leq 2$.*

Proof. Assume that $G_S(L)$ is complemented. Suppose that $|L| = 3$. Then the only possibility for L is C_3 (see Fig 16). Clearly $G_S(L)$ is not complemented which is a contradiction. Now let $|L| > 3$. Suppose L has at least two dual atoms d_1 and d_2. Since $d_1 \vee d_2 = 1$, the elements $1, d_1, d_2$ form a triangle. Clearly, 1 and d_2 are not orthogonal to d_1 in $G_S(L)$. Suppose that $d_1 \perp c$ for some $c \in V(G_S(L))$, then $d_1 c \in E(G_S(L))$ and $c \neq 1, d_2$. But then 1 belongs to the common neighbourhood of c and d_1 which is a contradiction to $d_1 \perp c$. So $d_1 \not\perp c$ for any $c \in V(G_S(L))$. Suppose L has a unique dual atom d_3, it is standard by Proposition 6 and the set $\{1, 0, d_3\}$ forms a triangle. This implies that $0, 1$ are not orthogonal to d_3. If $d_3 \perp p$ for some $p \in V(G_S(L))$, then since d_3 is unique dual atom, we get $p \leq d_3$. Moreover, 1 will be in common neighbourhood of d_3 and p which is a contradiction to $d_3 \perp p$. In both the cases we get an element which doesn't have an orthogonal complement in $G_S(L)$ which is a contradiction. Thus $|L| \leq 2$. Converse follows directly. □

Proposition 21. *If $b \in L$ is non-standard and has an orthogonal complement in $G_S(L)$, then b is superfluous in L.*

Proof. Let $b \in L$ be a non-standard element in L, and suppose $b \perp c$ for some $c \in V(G_S(L))$. Then, $bc \in E(G_S(L))$. Since b is non-standard, we note that $c \neq 0$. By Proposition 18 and $b \neq 1$, we get that $c = 1$. Now, suppose $bd \in E(G_S(L))$ for some $d \neq 1$. Then d will be in common neighbourhood of 1 and b which is a contradiction to $b \perp c$ that is, $b \perp 1$. Thus, for any $d \neq 1$, we must have $bd \notin E(G_S(L))$. This implies that $b \vee d$ is not a standard element for any $d \neq 1$. In particular, $b \vee d \neq 1$ for $d \neq 1$. Therefore, b is a superfluous element in L. □

Remark 10.

1. If an element is non-standard in L, then it need not be superfluous. Consider the lattice N_5 shown in Figure 1. The element $p_3 \in N_5$ is non-standard but it is not superfluous. Furthermore, observe that p_3 doesn't have orthogonal complement in $G_S(N_5)$, which shows that if an element is non-standard in L, then it need not have an orthogonal complement in $G_S(L)$.

2. If an element has an orthogonal complement in $G_S(L)$, then it need not be superfluous in L. Consider the lattice M_5 shown in Figure 32. Note that 0 is an orthogonal complement of 1 in $G_S(M_5)$. However 1 is not superfluous in M_5.

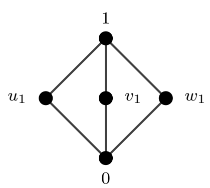


Figure 32. **Lattice** M_5

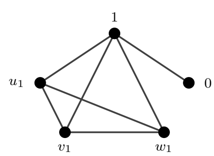


Figure 33. Join standard graph $G_S(M_5)$

3. If an element is non-standard and superfluous in a lattice L, it need not have an orthogonal complement in $G_S(L)$. For example, the element p_3 of the lattice L_6 shown in Figure 34 is non-standard and superfluous. But, it doesn't have an orthogonal complement in $G_S(L_6)$.

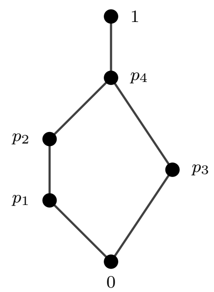


Figure 34. **Lattice L_6**

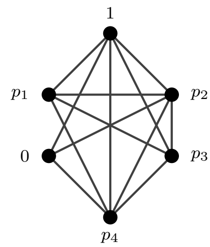


Figure 35. Join standard graph $G_S(L_6)$

Proposition 22. *Let $b \in L$ and S be the set of all standard elements of L. Then $1 \perp b$ in $G_S(L)$ if and only if b is superfluous and there exists no standard element of L in $[b) \setminus \{0, 1\}$.*

Proof. Let $b \in L$ such that $1 \perp b$ in $G_S(L)$. Clearly $b \neq 1$. If $b = 0$, the result holds. Now suppose $b \neq 0$. Then b is non-standard (because if b is standard, 0 will be in common neighbourhood of 1 and b). Suppose that b is not superfluous. Then there exists an element $g \neq 1$ such that $b \vee g = 1$. Then g belongs to the common neighbourhood of 1 and b, which is a contradiction. Hence, b must be superfluous. Next, suppose that s is a standard element of L in $[b) \setminus \{0, 1\}$. Then $b < s$ and s will be in common neighbourhood of 1 and b, which is a contradiction to $1 \perp b$. Conversely, assume b is superfluous and there is no standard element of L in $[b) \setminus \{0, 1\}$. Since b is superfluous, it follows that $b \neq 1$. Note that $1b \in E(G_S(L))$. Suppose $b = 0$. If c is in common neighbourhood of 1 and 0 for some $c \in V(G_S(L))$, then c is standard and $c \in [0) \setminus \{0, 1\}$, which contradicts the hypothesis. Thus, no such c exists and $1 \perp b$ holds. Suppose $b \neq 0$. If the set $\{b, d, 1\}$ forms a triangle for some $d \in V(G_S(L))$, then $b \vee d = s$ or 1 where s is a proper standard element of L. Since b is not standard, $b \neq s$, and so $0 < b < s < 1$, which again contradicts the assumption that there is no standard element in $[b) \setminus \{0, 1\}$. Therefore no such triangle $bd1$ exists in $G_S(L)$, and thus $1 \perp b$. □

Remark 11. In the converse part of Proposition 22, both conditions are necessary.

(i) If b is superfluous, then it does not necessarily follow that $1 \perp b$. For example, in the lattice C_3 shown in Figure 16, the element x is superfluous but $1 \not\perp x$ in $G_S(C_3)$.

(ii) Similarly, if there is no standard element of L in $[b) \setminus \{0, 1\}$, then it need not imply that $1 \perp b$. Consider the lattice N_5 shown in Figure 1. Note that there is no standard element of N_5 in $[p_3) \setminus \{0, 1\} = \{p_3\}$. However, $p_3 \not\perp 1$.

Remark 12. Suppose a dual atom d of L is standard. For any $b \in L$, $d \vee b \in \{d, 1\}$, where both d and 1 are standard elements. Hence, $db \in E(G_S(L))$ for all $b \in L$. Thus, d is a universal vertex in $G_S(L)$. Therefore, the subgraph $G_S(L) - 1$ is connected.

Remark 3 shows that $G_S(L) - 1$ is never totally disconnected, whereas, Remark 12 gives us a case where the subgraph $G_S(L) - 1$ is connected. On the other hand, if 0 and 1 are the only standard elements of lattice L, then $G_S(L) - 1$ is always disconnected. This observation motivates us to investigate the conditions under which the subgraph $G_S(L) - 1$ is connected or disconnected.

Proposition 23. *Let L be a lattice with at least two dual atoms and suppose every standard element of L is comparable with each dual atom, then the subgraph $G_S(L) - 1$ is disconnected.*

Proof. Let L have $k \geq 2$ dual atoms $d_1, d_2, d_3, \ldots, d_k$. From the hypothesis, it is clear that, d_i is non-standard for any $1 \leq i \leq k$. Assume that $G_S(L) - 1$ is connected. Then there must exist at least one proper non-zero standard element because otherwise the vertex 0 would be isolated. By hypothesis, $0d_1 \notin E(G_S(L)-1)$. Suppose there is a path $0 - u - d_1$ from 0 to d_1. Then we observe that u is standard, $u \neq d_i$ for any i and $u \leq d_1$. This implies $u \vee d_1 = d_1$. Because of the presence of the edge ud_1, we get that $u \vee d_1 = d_1$ is standard in L which is a contradiction to the hypothesis. Hence, there is no path of the form $0 - u - d_i$ for any proper standard element u and any $1 \leq i \leq k$. This implies that the length of the shortest path from 0 to d_1 is at least 3. Let $0 - s - u - d_1$ be the shortest path from 0 to d_1. Note that u is non-standard, $u \parallel d_1$ and $u \neq d_i$ for any $1 \leq i \leq k$. Without loss of generality, let d_2 be the dual atom containing u. Suppose $s \vee u$ is a proper standard element, then $u \leq s \vee u \leq d_1$, contradicting the assumption that $u \parallel d_1$. Therefore, $s \vee u = 1$, which implies $d_2 \vee s \vee u = d_2 \vee u = 1$, contradicting the fact that $u \vee d_2 = d_2$. Hence, such a path $0 - s - u - d_1$ does not exist for any proper standard element s and non-standard element u. This implies that the shortest path from 0 to d_1 must have length at least 4. Let $0 - u_1 - u_2 - \cdots - u_m - d_1$ be the shortest path from 0 to d_1 where $m \geq 3$. Then, the elements $u_1 \vee u_2, u_2 \vee u_3, \ldots, u_m \vee d_1$ are standard. Suppose $u_1 \vee u_2 = x$ is a proper standard element. Then $0 - x - u_3 - \cdots - u_m - d_1$ is a path from 0 to d_1 shorter than the original, contradicting its minimality. Therefore, $u_1 \vee u_2 = 1$. Since u_1 is standard, we must have $u_1 \leq d_1$. Now, suppose $u_2 \parallel d_1$. Then, $u_2 \vee d_1 = 1$, implies that the path $0 - u_1 - u_2 - d_1$ of length 3 from 0 to d_1 exists, which is a contradiction. Thus, $u_2 \leq d_1$. Hence, we have $1 = u_1 \vee u_2 \leq d_1$, contradicting the fact that d_1 is a dual atom. Therefore, no path exists from 0 to d_1 in $G_S(L) - 1$, contradicting the assumption that $G_S(L) - 1$ is connected. Hence, $G_S(L) - 1$ is disconnected. $\square$

Conclusion

In this work, we have introduced and explored the join standard graph of a finite lattice, focusing on its structural properties such as connectedness and girth. We have identified the conditions under which the graph has universal or isolated vertices and have showed how a homomorphism between lattices leads to a corresponding homomorphism between their graphs. We have described the conditions when the graph forms a hypertriangulated structure. Moreover, we found that the graph is complemented only when the lattice has at most two elements. We have discussed several results that hold only for a join standard graph with loops, but they fail for the graphs without loops. This emphasizes the need to analyze graphs with loops, as loops can hold key structural information lost in their absence. We have established that if a lattice L contains at least two dual atoms and every standard element of L is comparable with each dual atom, then the subgraph $G_S(L) - 1$ is disconnected. In the complementary case, where not every standard element is comparable with each dual atom, the disconnection of $G_S(L) - 1$ remains an open question. However, based on preliminary analysis and structural observations, we conjecture that $G_S(L) - 1$ is still disconnected in this more general setting.
The study of the standard elements leads naturally to the concept of standard ideals, which serve as lattice-theoretic analogues of normal subgroups in group theory. The present work enhances the interaction between graph theory and lattice theory by associating lattices with graphs constructed from their standard elements. This graphical approach offers a strong visual and combinatorial framework for analyzing key lattice properties like congruences and homomorphisms. By studying these properties through the lens of graph structures, we gain deeper insights into the behaviour of lattices.

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