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Research Article

Vertex energy invariance in double graphs and bipartite double covers

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Abstract: Vertex energy is a local spectral invariant that measures the contribution of individual vertices to the overall energy of a graph. Understanding how vertex energy behaves under graph transformations is essential for both theoretical insights and practical applications in spectral graph theory and network analysis. In this paper, we investigate the preservation of vertex energy under two fundamental graph constructions: the double graph and the bipartite double cover. We prove that for any connected graph G, the vertex energies of the duplicated vertices in both $\mathrm{D}(G)$ and $\mathrm{DC}(G)$ remain identical to those in G. These results demonstrate the robustness of vertex energy as a spectral measure invariant under these duplication operations. To illustrate the theorems, we provide explicit examples and computational verifications using SAGEMATH, with code publicly available for reproducibility. Our findings contribute to the deeper understanding of spectral properties in graph operations and open avenues for further research in spectral invariants of graph transformations.

Keywords: graph energy, vertex energy, eigenvalues, bipartite double cover, double graph.

AMS Subject classification: 05C50, 05C76

1. Introduction

The study of graph energy, introduced by Gutman [6] in 1978, has provided profound insights into the interplay between spectral graph theory and mathematical chemistry.

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The energy of a graph, defined as the sum of the absolute values of the eigenvalues of its adjacency matrix, was initially motivated by applications in Hückel Molecular Orbital theory to model conjugated hydrocarbons. Over time, this invariant has been extensively explored, leading to numerous generalizations and applications across disciplines. For further details, refer to [4, 8].
A recent and notable extension of this concept is the energy associated with individual vertices, referred to as the *energy of a vertex*, introduced by Arizmendi *et al.* [1]. This refinement offers a localized perspective on graph energy, quantifying the contribution of each vertex to the overall graph energy. The vertex energy is defined in terms of the diagonal entries of the matrix $|A| = (AA^*)^{1/2}$, where A is the adjacency matrix of the graph. Specifically, for a graph $G = (V, E)$ with vertex set $\{v_1, v_2, \dots, v_n\}$, the energy of the vertex v_i in G, denoted as $\mathscr{E}_G(v_i)$, is defined as:

$$\mathscr{E}_G(v_i) = |A|_{ii}, \quad \text{for } i = 1, \dots, n.$$

As noted by Nikiforov [9], the energy of a graph is defined as the $\mathscr{E}(G) = \mathrm{Tr}(|A(G)|)$. Consequently, the energy of a graph can be calculated as the sum of the individual energies of its vertices:

$$\mathscr{E}(G) = \mathscr{E}_G(v_1) + \cdots + \mathscr{E}_G(v_n).$$

In recent years, the concept of vertex energy in graphs has gained significant attention. Arizmendi et al. [2] introduced the Coulson integral formula for the vertex energy of a graph, providing a deeper understanding of its properties. Furthermore, Arizmendi and Sigarreta [3] explored the change in vertex energy when joining trees, offering valuable information on how graph transformations affect energy measures. Ramane *et al.* [10] extended the study of vertex energy by calculating the energies of vertices for subdivision graphs. An insightful tutorial by Gutman and Furtula [7] provides a detailed methodology for calculating vertex energies, emphasizing the importance of both eigenvalues and eigenvectors of the adjacency matrix. According to them, the vertex energy of a vertex v_i is defined as:

$$\mathscr{E}_G(v_i) = \sum_{k=1}^{n} |\lambda_k| C_{k,i}^2,$$

where λ_k are the eigenvalues of the adjacency matrix $A(G)$, and $C_{k,i}$ are the components of the orthonormal eigenvectors of $A(G)$. Specifically, if $\mathbf{v}_k = (C_{k,1}, C_{k,2}, \dots, C_{k,n})^T$ is the k-th eigenvector of $A(G)$, then $C_{k,i}$ represents the i-th component of $\mathbf{v}_k$. The orthonormality condition ensures:

$$\sum_{j=1}^{n} C_{k,j}^2 = 1 \quad \text{and} \quad \sum_{j=1}^{n} C_{k,j} C_{\ell,j} = 0 \quad \text{for } k \neq \ell.$$

This guarantees that the eigenvectors form a basis for the spectral decomposition of $A(G)$, enabling the calculation of vertex energy using the above formula.

Definition 1. The *Double Graph* $\mathrm{D}(G)$ of a connected graph G is constructed by taking two copies of G, denoted as G' and G''. Each vertex u' in G' is joined to the neighbors of the corresponding vertex u'' in G''.

It was shown in [12] that the eigenvalues of the graph $\mathrm{D}(G)$ are given by $2\lambda_i$ for $i = 1, 2, \ldots, n$, along with 0 occurring n times, where λ_i are the eigenvalues of the graph G.

Definition 2. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *Bipartite Double Cover* $\mathrm{DC}(G)$ of a graph G is the bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, where a vertex x_i in X is adjacent to a vertex y_j in Y if and only if v_i and v_j are adjacent in G.

In [5], it was demonstrated that if λ is an eigenvalue of G, then both λ and $-\lambda$ are eigenvalues of bipartite double cover $\mathrm{DC}(G)$.
The aim of this paper is to investigate the behavior of the vertex energy of a graph under specific transformations, namely the double graph and the bipartite double cover. Through detailed analysis, the paper aims to demonstrate that the vertex energy of the original graph remains invariant under these graph operations, advancing the study of energy-related properties in spectral graph theory.

1.1. Contributions

One of the key contributions of this study is the analysis of how vertex energy behaves under specific graph transformations that duplicate the vertex set. In particular, we focus on the double graph and the bipartite double cover of a given graph G, both of which are well-studied constructions in algebraic graph theory.

- We prove that in the double graph $\mathrm{D}(G)$, the vertex energy of each duplicated vertex (in both copies of G) remains equal to the vertex energy in the original graph. This result demonstrates that the operation of forming a double graph preserves local spectral properties at the vertex level.
- Similarly, we establish that in the bipartite double cover $\mathrm{DC}(G)$, each of the corresponding vertex copies retains the same vertex energy as in the original graph. This further confirms the robustness of vertex energy under common covering and lifting operations.

These findings contribute to the understanding of vertex energy as a stable spectral invariant under graph duplication processes, and they may serve as a basis for further studies involving graph coverings, symmetry, and spectral-based network analysis.

1.2. Outline

The structure of the paper is organized as follows. In Section 2, we present the main results of the paper concerning the behavior of vertex energy under two important graph constructions: the double graph $\mathrm{D}(G)$ and the bipartite double cover $\mathrm{DC}(G)$. We formally state and prove that the vertex energy of each vertex remains unchanged in both constructions, thereby establishing the invariance of this spectral quantity under vertex duplication.

To support the theoretical results, we include an illustrative example that visually demonstrates the preservation of vertex energy in $\mathrm{D}(G)$ and $\mathrm{DC}(G)$, with detailed vertex-by-vertex energy comparisons. Computational verification is carried out using SAGEMATH, and the implementation is shared via a public GitHub repository (https://github.com/cahitdede/vertex_energy) for reproducibility.

Finally, Section 3 concludes the paper with a summary of the findings and a discussion of possible directions for future research, including the study of vertex energy under other graph operations and extensions to broader classes of graphs.

2. Main Results

In spectral graph theory, it is often insightful to examine how various graph operations influence structural and spectral properties. Two notable constructions in this context are the double graph and the bipartite double cover, both of which involve duplicating the vertex set of a given graph and modifying the edge set according to specific rules. These constructions have applications in chemistry, network theory, and the study of graph symmetries.

This section focuses on the behavior of vertex energy under such constructions. Vertex energy, which quantifies the spectral contribution of a single vertex to the total energy of a graph, is a finer invariant than classical graph energy. We show that, in both the double graph and the bipartite double cover of a graph G, the vertex energies of the duplicated vertices are preserved from the original graph. This invariance highlights the structural regularity of these transformations and supports the robustness of vertex energy as a local spectral measure. The following theorems formalize these observations.

Theorem 1. *Let $\mathrm{D}(G)$ be the double graph of a connected graph G, where G' and G'' are the two copies of G. For each vertex $v \in V(G)$, let $v' \in V(G')$ and $v'' \in V(G'')$ be the corresponding vertices in G' and G'', respectively. Then, we have:*

$$\mathscr{E}_{\mathrm{D}(G)}(v') = \mathscr{E}_G(v) \quad \text{and} \quad \mathscr{E}_{\mathrm{D}(G)}(v'') = \mathscr{E}_G(v).$$

Proof. Let A be the adjacency matrix of the graph G. The adjacency matrix of $D(G)$ is given by:

$$M = \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Since A is a real symmetric matrix, it admits an orthonormal basis of eigenvectors. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be the orthonormal eigenvectors of A, with the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then, the eigenvalue equation for M is:

$$M\mathbf{x} = \lambda \mathbf{x},$$

which expands as:

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}.$$

Expanding the block matrix equation yields:

$$\begin{cases} A\mathbf{u} + A\mathbf{v} = \lambda \mathbf{u}, \\ A\mathbf{u} + A\mathbf{v} = \lambda \mathbf{v}. \end{cases}$$

Subtract the second equation from the first:

$$A\mathbf{u} + A\mathbf{v} - (A\mathbf{u} + A\mathbf{v}) = \lambda \mathbf{u} - \lambda \mathbf{v},$$

which simplifies to:

$$0 = \lambda(\mathbf{u} - \mathbf{v}).$$

Therefore, either $\lambda = 0$ or $\mathbf{u} = \mathbf{v}$.

Case 1: Non-Zero Eigenvalues $(\lambda \neq 0)$

Consider $\mathbf{x} = \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}$. Then:

$$M\mathbf{x} = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_i + A\mathbf{v}_i \\ A\mathbf{v}_i + A\mathbf{v}_i \end{bmatrix}.$$

Using $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$, this becomes:

$$M\mathbf{x} = \begin{bmatrix} 2\lambda_i \mathbf{v}_i \\ 2\lambda_i \mathbf{v}_i \end{bmatrix} = 2\lambda_i \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}.$$

Thus, $\mathbf{x} = \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}$ is an eigenvector of M with eigenvalue $2\lambda_i$.

Case 2: Zero Eigenvalues $(\lambda = 0)$

Consider $\mathbf{x} = \begin{bmatrix} \mathbf{v}_i \\ -\mathbf{v}_i \end{bmatrix}$. Then:

$$M\mathbf{x} = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ -\mathbf{v}_i \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_i - A\mathbf{v}_i \\ A\mathbf{v}_i - A\mathbf{v}_i \end{bmatrix}.$$

Using $A\mathbf{v}_i = 0$, this becomes:

$$M\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{x} = \begin{bmatrix} \mathbf{v}_i \\ -\mathbf{v}_i \end{bmatrix}$ is an eigenvector of M with eigenvalue 0. Let:

$$\mathbf{x}_i^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}, \quad \mathbf{x}_i^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{v}_i \\ -\mathbf{v}_i \end{bmatrix}.$$

The set

$$\{\mathbf{x}_i^{(1)}, \mathbf{x}_i^{(2)} \mid i = 1, 2, \dots, n\}$$

forms an orthonormal basis for $\mathbb{R}^{2n}$, and hence, the orthonormal eigenvectors of M. Here,

$$\mathbf{v}_i = (C_{i,1}, C_{i,2}, \dots, C_{i,n})^T$$

is the i-th eigenvector of A, where $C_{i,j}$ represents the j-th component of $\mathbf{v}_i$.
The vertex energy $\mathscr{E}(v)$ of a vertex v in $D(G)$ is defined as:

$$\mathscr{E}_{D(G)}(v) = \sum_{k=1}^{2n} |\mu_k| C_{k,v}^2,$$

where μ_k are the eigenvalues of M, and $C_{k,v}$ represents the component of the k-th orthonormal eigenvector of M corresponding to vertex v.
For corresponding vertices $v_i' \in G'$ and $v_i'' \in G''$, let their eigenvector components be denoted as $C_{k,i'}$ and $C_{k,i''}$, respectively. Using the definition of vertex energy for v_i':

$$\begin{aligned}
\mathscr{E}_{D(G)}(v_i') &= \sum_{k=1}^{2n} |\mu_k| C_{k,i'}^2 \\
&= \sum_{k=1}^{n} |2\lambda_k| \left(\frac{1}{\sqrt{2}} C_{k,i}\right)^2 + \sum_{k=1}^{n} |0| \left(\frac{1}{\sqrt{2}} C_{k,i}\right)^2 \\
&= \sum_{k=1}^{n} |2\lambda_k| \frac{1}{2} C_{k,i}^2 \\
&= \sum_{k=1}^{n} |\lambda_k| C_{k,i}^2 \\
&= \mathscr{E}_G(v_i).
\end{aligned}$$

Similarly:

$$\mathscr{E}_{D(G)}(v_i'') = \sum_{k=1}^{2n} |\mu_k| C_{k,i''}^2 = \sum_{k=1}^{n} |2\lambda_k| \left(\frac{1}{\sqrt{2}} C_{k,i}\right)^2 + \sum_{k=1}^{n} |0| \left(-\frac{1}{\sqrt{2}} C_{k,i}\right)^2 = \mathscr{E}_G(v_i).$$

This completes the proof. □

Theorem 1 establishes that the process of forming the double graph $\mathrm{D}(G)$ does not alter the vertex energy of any vertex, as each copy of a vertex in the resulting structure preserves the energy of its original counterpart. This result highlights the spectral consistency of the double graph construction. We now turn our attention to a related transformation, the bipartite double cover, which similarly involves duplicating the vertex set of G, but with a distinct edge configuration that ensures bipartiteness. The next theorem shows that vertex energy remains invariant under this construction as well.

Theorem 2. *Let $\mathrm{DC}(G)$ be the bipartite double cover of a graph G. Then, for each vertex $v \in V(G)$, the vertex energy in $\mathrm{DC}(G)$ is given by:*

$$\mathscr{E}_{\mathrm{DC}(G)}(x_i) = \mathscr{E}_G(v_i) \quad \text{and} \quad \mathscr{E}_{\mathrm{DC}(G)}(y_i) = \mathscr{E}_G(v_i).$$

Proof. Let A be the adjacency matrix of graph G. Then the adjacency matrix of $D(G)$ is given by:

$$M = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}.$$

Since A is a real symmetric matrix, it admits an orthonormal basis of eigenvectors. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be the eigenvectors of A, with the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then the eigenvalue equation for M is given by:

$$M\mathbf{x} = \lambda \mathbf{x},$$

which expands as:

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}.$$

Expanding the block matrix equation yields:

$$\begin{cases} A\mathbf{v} = \lambda \mathbf{u}, \\ A\mathbf{u} = \lambda \mathbf{v}. \end{cases}$$

Case 1: $\lambda = 0$

If $\lambda = 0$, the equations reduce to:

$$A\mathbf{v} = 0 \quad \text{and} \quad A\mathbf{u} = 0.$$

Thus, $\mathbf{v}$ and $\mathbf{u}$ are in the null space of A. Let $\mathbf{v}_i$ be an eigenvector of A corresponding to $\lambda = 0$. Then the eigenvectors of M are:

$$\begin{bmatrix} \mathbf{v}_i \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \mathbf{v}_i \end{bmatrix}.$$

These correspond to the eigenvalue $\lambda = 0$ for M.
Case 2: $\lambda \neq 0$

For $\lambda \neq 0$, substituting $\mathbf{u} = \frac{1}{\lambda} A\mathbf{v}$ into the second equation gives:

$$A\left(\frac{1}{\lambda} A\mathbf{v}\right) = \lambda \mathbf{v},$$

which simplifies to:

$$\frac{1}{\lambda} A^2\mathbf{v} = \lambda \mathbf{v} \implies A^2\mathbf{v} = \lambda^2 \mathbf{v}.$$

Thus, λ^2 is an eigenvalue of A^2, and the eigenvalues of M are λ_i and $-\lambda_i$ for each nonzero eigenvalue λ_i of A. Next, we show that $\begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}$ and $\begin{bmatrix} -\mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}$ are eigenvectors of M corresponding to λ_i and $-\lambda_i$, respectively.
Let $\mathbf{x} = \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}$. Then:

$$M\mathbf{x} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_i \\ A\mathbf{v}_i \end{bmatrix}.$$

Using $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$, this becomes:

$$M\mathbf{x} = \begin{bmatrix} \lambda_i \mathbf{v}_i \\ \lambda_i \mathbf{v}_i \end{bmatrix} = \lambda_i \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix} = \lambda_i \mathbf{x}.$$

Thus, $\mathbf{x}$ is an eigenvector of M corresponding to the eigenvalue λ_i.
Let $\mathbf{y} = \begin{bmatrix} -\mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}$. Then:

$$M\mathbf{y} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \begin{bmatrix} -\mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_i \\ -A\mathbf{v}_i \end{bmatrix}.$$

Using $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$, this becomes:

$$M\mathbf{y} = \begin{bmatrix} \lambda_i \mathbf{v}_i \\ -\lambda_i \mathbf{v}_i \end{bmatrix} = -\lambda_i \begin{bmatrix} -\mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix} = -\lambda_i \mathbf{y}.$$

Thus, $\mathbf{y}$ is an eigenvector of M corresponding to eigenvalue $-\lambda_i$.
The corresponding orthonormal eigenvectors are:

$$\text{For } \lambda_i \neq 0: \quad \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix}.$$

$$\text{For } \lambda_i = 0: \quad \begin{bmatrix} \mathbf{v}_i \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \mathbf{v}_i \end{bmatrix}.$$

The vertex energy of $x_i \in X$ for all $i = 1, 2, \ldots, n$ in $\mathrm{DC}(G)$ is computed as:

$$\mathscr{E}_{\mathrm{DC}(G)}(x_i) = \sum_{\lambda_k \neq 0} |\lambda_k| C_{k,i}^2 + \sum_{\lambda_k = 0} |\lambda_k| C_{k,i}^2.$$

This simplifies to:

$$\mathscr{E}_{\mathrm{DC}(G)}(x_i) = \sum_k |\lambda_k| \left(\frac{C_{k,i}}{\sqrt{2}} \right)^2 + \sum_k |-\lambda_k| \left(\frac{-C_{k,i}}{\sqrt{2}} \right)^2 + \sum_k |0| C_{k,i}^2.$$

Since $|\lambda_k| = |-\lambda_k|$, this becomes:

$$\mathscr{E}_{\mathrm{DC}(G)}(x_i) = \sum_{k=1}^{n} |\lambda_k| C_{k,i}^2 = \mathscr{E}_G(v_i).$$

Similarly, the vertex energy of $y_i \in Y$ for all $i = 1, 2, \ldots, n$ in $\mathrm{DC}(G)$ is computed as:

$$\mathscr{E}_{\mathrm{DC}(G)}(y_i) = \sum_{\lambda_k \neq 0} |\lambda_k| C_{k,i}^2 + \sum_{\lambda_k = 0} |\lambda_k| C_{k,i}^2.$$

This simplifies to:

$$\mathscr{E}_{\mathrm{DC}(G)}(y_i) = \sum_k |\lambda_k| \left(\frac{C_{k,i}}{\sqrt{2}} \right)^2 + \sum_k |-\lambda_k| \left(\frac{C_{k,i}}{\sqrt{2}} \right)^2 + \sum_k |0| C_{k,i}^2.$$

Thus, we get:

$$\mathscr{E}_{\mathrm{DC}(G)}(y_i) = \sum_{k=1}^{n} |\lambda_k| C_{k,i}^2 = \mathscr{E}_G(v_i).$$

This completes the proof. □

To complement and clarify the theoretical findings presented in the previous theorems, we now provide a detailed illustrative example that visually demonstrates the behavior of vertex energy under two fundamental graph constructions: the double graph $\mathrm{D}(G)$ and the bipartite double cover $\mathrm{DC}(G)$. These constructions are frequently encountered in spectral graph theory and network modeling, particularly in contexts where symmetry, duplication, or bipartite structure is essential. The example focuses on a small base graph G, for which the vertex energies are explicitly calculated and displayed. In the double graph $\mathrm{D}(G)$, each vertex of G is replicated in two disjoint copies, and the edge structure is preserved within each copy. The energies of the corresponding vertices in both copies are shown to be identical to those in the original graph, thereby confirming the vertex energy invariance asserted in Theorem 1. Likewise, in the bipartite double cover DC(G), each vertex of G is split into two parts, which are then connected according to the adjacency relations in G, but across the two parts, ensuring a bipartite structure. The energies of the resulting vertices are also computed and shown to match those in the original graph, in agreement with Theorem 2. This example serves as an instructive verification of the theoretical results and provides visual and computational support for the invariance of vertex energy under these transformations. All calculations and visualizations were carried out using the open-source mathematical software SAGEMATH, and the corresponding code is made available for reproducibility.

Example 1. In Figure 1, the graph G, its double graph D(G), and the double bipartite cover DC(G) are illustrated. For each vertex $v \in V(G)$, the vertex energy $\mathscr{E}_G(v)$ is displayed inside the circle corresponding to that vertex. In D(G), the graph consists of two copies of G, denoted as G' and G''. For each vertex $v \in V(G)$, the corresponding vertices $v' \in V(G')$ and $v'' \in V(G'')$ have vertex energies:

$$\mathscr{E}_{\mathrm{D}(G)}(v') = \mathscr{E}_G(v) \quad \text{and} \quad \mathscr{E}_{\mathrm{D}(G)}(v'') = \mathscr{E}_G(v).$$

Similarly, in DC(G), each vertex $v \in V(G)$ is replaced by two vertices x_i and y_i. For each $v_i \in V(G)$, the vertex energies in DC(G) satisfy:

$$\mathscr{E}_{\mathrm{DC}(G)}(x_i) = \mathscr{E}_G(v_i) \quad \text{and} \quad \mathscr{E}_{\mathrm{DC}(G)}(y_i) = \mathscr{E}_G(v_i).$$

This demonstrates that the vertex energy is preserved in the double graph D(G) and the double cover of bipartites DC(G). Please check the SAGEMATH [11] code available in the github page[1] for verification of this example and other examples.

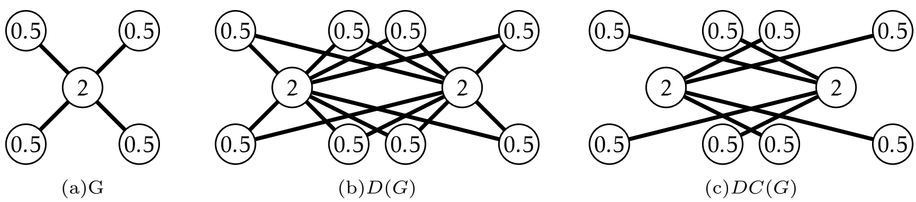

Figure 1. A graph G, its double graph D(G) and its double cover of bipartites DC(G).

[1] https://github.com/cahitdede/vertex_energy

3. Conclusion

This paper explores the vertex energy of graphs under the transformations of the double graph and bipartite double cover. We have demonstrated that the vertex energy remains invariant for both operations, preserving the energy-related spectral properties of the original graph. By analyzing the eigenvalue structure and corresponding eigenvectors, we provided rigorous proofs for the invariance of vertex energy in these transformed graphs.
These findings not only deepen our understanding of energy-based invariants in spectral graph theory but also highlight the robust nature of vertex energy under graph operations. The results have potential applications in areas such as network theory, chemistry, and physics, where energy measures play a crucial role. Future research could investigate how vertex energy behaves under other graph transformations or extend these concepts to directed or weighted graphs to further explore their theoretical and practical implications.

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Statements and Declarations

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References

[1] O. Arizmendi, J.F. Hidalgo, and O. Juarez-Romero, *Energy of a vertex*, Linear Algebra Appl. **557** (2018), 464–495. https://doi.org/10.1016/j.laa.2018.08.014.
[2] O. Arizmendi, B.C. Luna-Olivera, and M.R. Ibáñez, *Coulson integral formula for the vertex energy of a graph*, Linear Algebra Appl. **580** (2019), 166–183. https://doi.org/10.1016/j.laa.2019.06.009.
[3] O. Arizmendi and S. Sigarreta, *The change of vertex energy when joining trees*, Linear Algebra Appl. **687** (2024), 117–131. https://doi.org/10.1016/j.laa.2024.01.014.
[4] D.M. Cvetković, P. Rowlinson, and S. Simić, *An introduction to the theory of graph spectra*, vol. 75, Cambridge university press Cambridge, 2010.
[5] H.A. Ganie, S. Pirzada, and A. Iványi, *Energy, Laplacian energy of double graphs and new families of equienergetic graphs*, Acta Univ. Sapientiae, Informatica **6** (2014), no. 1, 89–116.

[6] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forsch-ungszentram Graz. **103** (1978), 1–22.

[7] I. Gutman and B. Furtula, *Vertex energies of graphs- A tutorial*, MATCH Commun. Math. Comput. Chem. **93** (2025), no. 3, 691–698.
https://doi.org/10.46793/match.93-3.691G.

[8] X. Li, Y. Shi, and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[9] V. Nikiforov, *The energy of graphs and matrices*, J. Math. Anal. Appl. **326** (2007), no. 2, 1472–1475.
https://doi.org/10.1016/j.jmaa.2006.03.072.

[10] H.S. Ramane, S.Y. Chowri, T. Shivaprasad, and I. Gutman, *Energy of vertices of subdivision graphs*, MATCH Commun. Math. Comput. Chem. **93** (2025), no. 3, 701–711.
https://doi.org/10.46793/match.93–3.701R.

[11] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 9.1.0)*, 2021, `https://www.sagemath.org`.

[12] S.K. Vaidya and K.M. Popat, *Some new results on energy of graphs*, MATCH commun. Math. Comput. Chem. **77** (2017), no. 4, 589–594.