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Research Article

Exploring graphs where clique number meets coprime index

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Abstract: In this paper, an algorithmic approach is explored towards vertex coloring, coprime labeling, and coprime index of certain variant of the dot product graphs. The notion of coprime index $\mu(G)$ of a graph G was introduced by Katre et al. This notion has interesting connections with the clique number $\omega(G)$, the intersection number $i(G)$ of G, and the edge-clique covering number $\theta_e(G^{\complement})$ of the complement of the graph. Katre et al. posed a problem to characterize the graphs for which clique number and coprime index are equal, i.e., $\omega(G) = \mu(G)$. In this paper, we provide a broader class of combinatorial graphs $G(R_n)$ satisfying this equality. With a slight modification, these graphs are the dot product graphs introduced by Badawi. The graph $G(R_n)$ is associated to a subset R_n of the first octant of $\mathbb{R}^n$, instead of associating to a ring. This graph generalizes the Kneser graphs, the Boolean graphs, and more generally, the zero-divisor graph of the ring $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_n}$. We first explore the structure of the graph $G(R_n)$ recursively using $G(R_{n-1})$. Then, we utilize it to obtain simple algorithms for the graph labelings such as vertex coloring and coprime labeling of these graphs, and show that these labelings are minimal. The chromatic number $\chi(G(R_n))$ and the coprime index $\mu(G(R_n))$ of $G(R_n)$ are determined. Consequently, we have the class of graphs $G(R_n)$ satisfying the equality: $\omega(G(R_n)) = \mu(G(R_n)) = \chi(G(R_n)) = \theta_e(G(R_n)^{\complement}) = i(G(R_n)^{\complement})$.

Keywords: graph labelings, dot product graphs, chromatic number, coprime index, clique number, zero-divisor graphs of reduced rings, the Boolean graphs.

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1. Introduction

A *graph labeling* is an assignment of integers to the vertices or edges of a graph with a certain rule. Graph labelings play an important role in studying the structural properties of graphs, and also have interesting applications in pattern recognition, communication networks, scheduling, and computer algorithms. A well-known graph labelings are vertex coloring, graceful labeling, prime labeling, coprime labeling, etc. For an extensive survey on graph labelings, see Gallian [5].

A *vertex-coloring* of a graph G is a graph labeling in which vertices of G are labeled by colors (integers) such that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required for the vertex-coloring of G. The notion of prime labeling was introduced by Tout et al. [17] in 1838. A *prime labeling* of a graph having n vertices is a graph labeling to the vertices with first n natural numbers such that the labels of adjacent vertices are coprime. Every graph may not admit a prime labeling. Prime labeling does not exist for complete bipartite graphs $K_{n,n},\ n \geq 3$.

The concepts of coprime labeling and coprime index of graphs were introduced by Katre et al. [8]. They proved that every graph admits a coprime labeling. We now briefly discuss these notions.

Definition 1 ([8]). Let G be a simple graph with the vertex set V. An injection $L : V \to \mathbb{N}\backslash\{1\}$ is a *coprime labeling* of G, if for $u, v \in V$, u is adjacent to v in G if and only if $L(u)$ and $L(v)$ are coprime.

Definition 2 ([8]). Let L be a coprime labeling of a graph G. A prime p is said to be *used* in L, if p divides $L(v)$ for some $v \in V$. Let $\mu(G, L)$ be the number of primes used by L. Then *coprime index* $\mu(G)$ of G is defined by $\mu(G) = \min\{\mu(G, L) : L \text{ is a coprime labeling of } G\}$. A coprime labeling L of G is said to be a *minimal coprime labeling* of G if $\mu(G) = \mu(G, L)$.

Theorem A. [[8]] Let L be a coprime labeling of a graph G and let H be an induced subgraph of G. Then, the restriction map $L|_H$ is a coprime labeling of H. Moreover, $\mu(G) \geq \mu(H)$.

Authors [8] highlighted the importance of the coprime index by giving its interconnection to various graph invariants such as clique number and edge-clique covering number. We now briefly recall these notions. The *clique number* $\omega(G)$ of a graph G is the size of the largest clique (complete subgraph) in G.

Definition 3. Let $E(X)$ denote the edge set of a graph X. A collection of cliques $C_1, C_2, ..., C_k$ is said to be *edge-clique cover* of a graph G, if $E(G) = \bigcup_{i=1}^{k} E(C_i)$. An *edge-clique cover number* $\theta_e(G)$ of G is the minimum cardinality of an edge-clique cover of G.

Erdos et al. [4] obtained the upper bound $\lfloor |V|^2/4 \rfloor$ for $\theta_e(G)$. Let $G^{\complement}$ denotes the complement of a graph G. Interesting connections of both $\omega(G)$ and $\theta_e(G^{\complement})$ with the coprime index $\mu(G)$ are discussed below. Let $\Delta(G)$ be the maximum degree of a vertex in the graph G.

Theorem 1 ([8]). *Let G be a graph with n vertices and $\Delta(G) < n - 1$. Then $\mu(G) = \theta_e(G^{\complement})$.*

For various applications of $\theta_e(G)$, see Fred Roberts [11]. In [11], the author gave a nice relation of the edge clique cover number $\theta_e(G)$ with the intersection number $i(G)$

of a graph G. The *intersection number* $i(G)$ is the minimum size of a set X for which G is the intersection graph of a family of subsets of X.

Theorem 2 ([11]). *For all graphs G, $i(G) = \theta_e(G)$.*

The following observation is immediate from the above two results which connect the coprime index of a graph with the intersection number of the complement of the graph.

Observation 3. For a graph G with n vertices and $\Delta(G) < n-1$, $\mu(G) = i(G^{\complement}) = \theta_e(G^{\complement})$.

Computing $\theta_e(G)$ of a graph G is known to be a NP-hard problem [6], and so computing the coprime index $\mu(G)$ of a graph G is a difficult problem. We have one more relation of the coprime index with the clique number of a graph by Katre et al. [8].

Theorem 4 ([8]). *For any graph G, $\omega(G) \leq \mu(G)$.*

Thus, the notion of the coprime index of a graph has key connections with various graph invariants. Therefore, it is interesting to study the notions of coprime index and coprime labeling. In view of the Theorem 4, authors [8] asked the following problem about the equality of the clique number and the coprime index of a graph.

Problem 1 ([8]). Characterize all graphs G for which the clique number is equal to the coprime index, i.e., $\omega(G) = \mu(G)$.

Patil et al. [10] provided a wider classes of graphs satisfying this equality. They proved this equality for the zero-divisor graphs of some ordered sets and the zero-divisor graphs of the ring $\mathbb{Z}_{p^n}$ (p a prime). In this paper, we provide a certain class of combinatorial graphs $G(R_n)$ associated to a subset R_n of the first octant of $\mathbb{R}^n$ satisfying the equality:

$$\omega(G(R_n)) = \mu(G(R_n)) = \chi(G(R_n)) = \theta_e(G(R_n)^{\complement}) = i(G(R_n)^{\complement}).$$

The graph $G(R_n)$ is a little variant of a well-known class of dot product graphs. Badawi [2] introduced the notion of dot product graphs associated to commutative rings.

Definition 4 ([2]). Let A be a commutative ring with $1 \neq 0$, and let $R := A \times A \times \cdots \times A$ (direct product). The *dot product graph* $\mathcal{D}(R)$ of the ring R is the graph whose vertex set is the set of non-zero elements of R, and vertices x and y are adjacent in $\mathcal{D}(R)$ if $x \cdot y = 0$ in R.

Motivated by Badawi's definition above, we study these graphs with a combinatorial viewpoint instead of its algebraic way of looking at them. Instead of the ring R in the definition, we consider specific finite subset R_n of the first octant of $\mathbb{R}^n$, and associate a variant of the dot product graph, say $G(R_n)$ to it. We state the definition of the graph that suits our exposition.

Definition 5. Let R_n be a subset of the first octant of $\mathbb{R}^n$. The graph $G(R_n)$ is a simple graph with the vertex set $V = \{v \in R_n : v \neq 0$ and the inner product (dot product) $\langle u, v\rangle = 0$ for some non-zero $u \in R_n\}$, and two vertices x and y are adjacent if $\langle x, y\rangle = 0$.

The subset R_n considered in this paper is $R_n := S_{m_1} \times S_{m_2} \times \cdots \times S_{m_n}$ (Cartesian product) such that each S_{m_i} is a m_i-element subset of non-negative reals. Here, S_{m_i}'s are not necessarily the same in sizes or sets. Henceforth we assume $n, m_i > 1$ and $0 \in S_{m_i}$ for all i, since otherwise the $G(R_n)$ is a null graph.

In literature, a special case of this graph when all S_{m_i}'s are equal, is studied to determine various distance-based invariants of the graph [15]. For the spectral properties of the same special case, see [16, Remark 5.4] and [13], where the adjacency-eigenvalues are characterized. Interestingly, the graph $G(R_n)$ serves as a unifying generalization of important graphs from two distinct domains: algebraic graphs (such as zero-divisor graphs of finite reduced rings) and combinatorial graphs (such as Kneser graphs, set graphs or the Boolean graphs). From the construction of $G(R_n)$, it is evident that the graph $G(R_n)$ can be seen as *generalized version* of the Boolean graph or set graph studied in [12], where the special case $R_n := S_2 \times S_2 \times \cdots \times S_2$ with $S_2 = \{0, 1\}$ was considered. The Boolean graph is a combinatorial graph as well as an algebraic graph. For its combinatorial aspects, see Kadu et al. [7] and Shinde et al. [12]. The *Boolean graph* is also known as the zero-divisor graph of the Boolean ring [9]. The notion of a zero-divisor graph was introduced by Beck [3]. Further, this definition was modified by Anderson and Livingston [1] to its present form. The *zero-divisor graph* $\Gamma(R)$ associated to a commutative ring R (with $1 \neq 0$) is a simple graph with the vertex set as the set of non-zero zero-divisors of R, and two vertices x and y are adjacent in $\Gamma(R)$ if $xy = 0$ in R. The graph $G(R_n)$ also generalizes the zero-divisor graphs of finite reduced rings. The graph $G(R_n)$ is isomorphic to the zero-divisor graph of a finite reduced ring if and only if each m_i is a power of a prime. Thus, for the remaining cases when $m_1, m_2, \ldots, m_n$ are not necessarily powers of primes, we can always consider the graph $G(R_n)$ associated with each of these values of m_i, leading to a broader class of graphs $G(R_n)$. This is the class of graphs that we demonstrate in this paper to prove that these graphs satisfy the equality in Problem 1.

This paper is structured as follows: Section 2 introduces a vertex ordering to define the recursive description of the graph $G(R_n)$ with respect to the vertex-adjacency in $G(R_{n-1})$. Section 3 uses this recursive description to obtain algorithms for the graph labelings such as vertex coloring and coprime labeling of the graph $G(R_n)$. Then proved that these labelings are in fact minimal. The chromatic number, clique number, and coprime index of $G(R_n)$ are determined, and proved the equality among these graph invariants.

2. Recursive structure of the graph $G(R_n)$

In this section, the vertex-adjacency in the graph $G(R_n)$ is recursively described using the vertex-adjacency in $G(R_{n-1})$. For this, we use a certain vertex partition of $G(R_n)$. In literature, a recursive approach is used to study determinantal properties of the Boolean graphs [14] and the zero-divisor graph of $\mathbb{F}_q \times \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ [16]. The same recursive approach is utilized to study distance-based invariants of $G(R_n)$ in the case when all S_{m_i}'s are equal [15]. In this section, we extend this recursive approach in a more general setting for the graph $G(R_n)$ where S_{m_i}'s are not necessarily equal, and describe the graph $G(R_n)$ recursively. We utilize the recursive description of $G(R_n)$ in the next section to determine various graph labelings of $G(R_n)$.

We recall the definitions of R_n and the graph $G(R_n)$. For $1 < m_i, n \in \mathbb{Z}$, let

$$R_n = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_n} \text{ (Cartesian product)}$$

where for each i, $S_{m_i} := \{b^i_0 = 0, b^i_1, b^i_2, \ldots, b^i_{m_i-1}\}$ is a m_i-element subset of $\mathbb{R}^+ \cup \{0\}$. Here S_{m_i}'s are not necessarily the same. For $x \in R_n \subset \mathbb{R}^n$, let $x(i)$ be i^{th} coordinate of x. Denote $U_n := \{x \in R_n \ : \ x(i) \neq 0, \ \forall \ i\}$. Note that, the size of R_n given by $|R_n| = \prod_{i=1}^n m_i$, and so $|U_n| = \prod_{i=1}^n (m_i - 1)$. Denote $\mathbf{0_n}$ for the zero element in R_n. Recall that, $G(R_n)$ is a simple graph with vertex set $V_n := R_n \setminus (U_n \cup \{\mathbf{0_n}\})$, and two vertices x and y are adjacent if the inner product $\langle x, y\rangle = 0$. Therefore, $|V_n| = \prod_{i=1}^n m_i - \prod_{i=1}^n (m_i - 1) - 1$. As noted earlier, we assume $n, m_i > 1$ and $0 \in S_{m_i}$ for all i, since otherwise the $G(R_n)$ is a null graph. The graph $G(R_2)$ is the complete bipartite graph K_{m_1-1,m_2-1}.

Recall the set $S_{m_n} := \{b^n_0 = 0, b^n_1, b^n_2, \ldots, b^n_{m_n-1}\}$. For each $k \in \{0, 1, \cdots, m_n - 1\}$, define map

$$\pi_k : R_{n-1} \longrightarrow R_n$$

such that $\pi_k((x_1, x_2, \ldots, x_{n-1})) = (x_1, x_2, \ldots, x_{n-1}, b^n_k)$. That is, for each k, π_k map adjoins an element b^n_k of the set S_{m_n} to the vector $(x_1, x_2, \ldots, x_{n-1}) \in R_{n-1}$. Thus, there are precisely $|S_{m_n}| = m_n$ such π_i maps for each $i \in \{0, 1, 2, ..., m_n\}$. For $X \subset R_{n-1}$, let $\pi_i(X) = \{\pi_i(x) \ : \ x \in X\}$. The maps π_k's are one-one maps, $\bigcup_{k=0}^{k=m_n-1} \pi_k(R_{n-1}) = R_n$, and for every $i \neq j$, $\pi_i(R_{n-1}) \cap \pi_j(R_{n-1}) = \emptyset$. Let $\mathcal{S}_n = \{\pi_1(\mathbf{0_{n-1}}), \pi_2(\mathbf{0_{n-1}}), \cdots, \pi_{m_n-1}(\mathbf{0_{n-1}})\}$. That is, $\mathcal{S}_n \subset R_n$ such that every element of $\mathcal{S}_n$ has n^{th} coordinate positive and the remaining coordinates are zero. We now describe the vertex set V_n recursively from V_{n-1}. In fact, the vertex set V_n is partitioned into $m_n + 2$ parts using V_{n-1}.

$$V_n = \pi_0(U_{n-1}) \cup \mathcal{S}_n \cup \pi_0(V_{n-1}) \cup \pi_1(V_{n-1}) \cup \cdots \cup \pi_{m_n-1}(V_{n-1})$$

That is,

$$V_n = \pi_0(U_{n-1}) \cup \mathcal{S}_n \cup \bigcup_{i=0}^{m_n-1} \pi_i(V_{n-1}).$$

This vertex partition plays an important role in various graph labelings of $G(R_n)$ in Section 3.

Remark 1. The vertex partition V_n of the graph $G(R_n)$ in the case when all m_i's are equal, is used to describe the distance matrix of the graph [15]. The same vertex partition is used to describe an adjacency matrix of the zero-divisor graph of the ring $\mathbb{F}_q \times \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ [16]. The graph studied in [15] is a special case of the graph $G(R_n)$, therefore by [15, Proposition 2.3], the graph $G(R_n)$ generalizes the Boolean graphs and the graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p)$, for p prime. More generally, we have the following result.

Proposition 1. *If A is a finite reduced ring, then there exists a subset $R' \subset \mathbb{R}^n$ for some $n \geq 1$ such that the zero-divisor graph $\Gamma(A)$ is isomorphic to the graph $G(R')$.*

Proof. Let A be a finite reduced ring. Then $A \cong F_1 \times F_2 \times \ldots \times F_n$ as the direct product of finite fields F_i with $|F_i| = q_i^{l_i}$ (say). Set $m_i := q_i^{l_i}$. Therefore m_i's are the power of primes. We list the elements of F_i as $F_i = \{a_0^i = 0, a_1^i, \ldots, a_{m_i-1}^i\}$. Corresponding to each F_i, we define a m_i-element subset $S_{m_i} := \{b_0^i = 0, b_1^i, \ldots, b_{m_i-1}^i\} \subset \mathbb{R}^+ \cup \{0\}$, where $b_l^i \neq b_k^i$, whenever $l \neq k$. Now, set $R' := S_{m_1} \times S_{m_2} \times \ldots \times S_{m_n}$ (Cartesian product), and define the map $\phi : V(\Gamma(A)) \longrightarrow V(G(R'))$ by $\phi\left(a_{j_1}^1, a_{j_2}^2, \ldots, a_{j_n}^n\right) = \left(b_{j_1}^1, b_{j_2}^2, \ldots, b_{j_n}^n\right)$. It is easy to see that x and y adjacent in $\Gamma(A)$ if and only if $\phi(x)$ and $\phi(y)$ are adjacent in the graph $G(R')$. Thus, $\Gamma(A) \cong G(R')$. $\square$

Remark 2. By Proposition 1, the zero-divisor graph $\Gamma(\mathbb{F}_{{q_1}^{l_1}} \times \mathbb{F}_{{q_2}^{l_2}} \times \ldots \times \mathbb{F}_{{q_n}^{l_n}})$ is isomorphic to $G(R_n)$ if and only if $m_i = {q_i}^{l_i}$ for each i (m_i's are a power of primes). Thus, for the remaining cases when $m_1, m_2, \ldots, m_n$ are not necessarily powers of primes, we have the graph $G(R_n)$ associated with each of these values of m_i, and so this produces a broader class of graphs $G(R_n)$.

The proof of the following lemma is similar to [16, Lemma 2.1]. We extend it for the more general class of graphs $G(R_n)$. Recall the vertex partition V_n. We use it to describe the vertex-adjacency in $G(R_n)$ with respect to the vertex-adjacency in $G(R_{n-1})$.

Lemma 1. *Let $n \geq 2$. Let E_n denotes the edge set of $G(R_n)$. Let V_n be the vertex set of $G(R_n)$, and let π_k $(0 \leq k \leq m_n - 1)$ be maps defined above. Then*

1. *$G(R_n)$ has an induced subgraph say H induced by $\pi_0(V_{n-1})$ such that $G(R_{n-1}) \cong H$.*
2. *If $x \in \pi_0(U_{n-1})$ and $y \in \pi_0(U_{n-1}) \cup \pi_i(V_{n-1}), i \in \{0, 1, 2, ..., m_n - 1\}$ then $(x, y) \notin E_n$.*
3. *If $x \in \mathcal{S}_n$ and $y \in \mathcal{S}_n \cup \pi_i(V_{n-1}), i \in \{1, 2, ..., m_n - 1\}$ then $(x, y) \notin E_n$.*
4. *If $x \in \pi_i(V_{n-1})$ and $y \in \pi_j(V_{n-1})$ for $i, j \in \{1, 2, ..., m_n - 1\}$, then $(x, y) \notin E_n$.*

5. *If* $x \in \mathcal{S}_n$ *and* $y \in \pi_0(U_{n-1}) \cup \pi_0(V_{n-1})$ *then* $(x, y) \in E_n$.

6. *Let* $x, y \in V_{n-1}$. *Then* $(x, y) \in E_{n-1}$ *if and only if* $\Big(\pi_0(x), \pi_i(y)\Big)$ *and* $\Big(\pi_0(y), \pi_i(x)\Big)$ *are in* E_n *for* $i \in \{0, 1, 2, ..., m_n - 1\}$.

Proof. (**1**) Let H be the subgraph of $G(R_n)$ induced by $\pi_0(V_{n-1})$. Then the map $\phi : G(R_{n-1}) \to H$ such that $\phi(x) = \pi_0(x)$ is graph isomorphism. Therefore, H is an induced subgraph of $G(R_n)$ isomorphic to $G(R_{n-1})$.

(**2**) If $x \in \pi_0(U_{n-1})$ and $y \in \pi_0(U_{n-1}) \cup \pi_i(V_{n-1}), i \in \{0, 1, 2, \ldots, m_n - 1\}$ then all the coordinates of x are positive except n-th coordinate which is zero and y has atleast one positive coordinate except n-th coordinate and remaining coordinates of y are nonnegative. Therefore, $\langle x, y\rangle \neq 0$. Hence, $(x, y) \notin E_n$.

(**3**) If $x \in \mathcal{S}_n$ and $y \in \mathcal{S}_n \cup \pi_i(V_{n-1}), i \in \{1, 2, ..., m_n - 1\}$ then n-th coordinates of x and y are nonzero and remaining coordinates are nonnegative. Therefore, $\langle x, y\rangle \neq 0$. Hence, $(x, y) \notin E_n$.

(**4**) Let $x \in \pi_i(V_{n-1})$ and $y \in \pi_j(V_{n-1})$ for $i, j \in \{1, 2, \ldots, m_n - 1\}$. Then both x and y have their n^{th} coordinates nonzero and the remaining coordinates are nonnegative. This implies $\langle x, y\rangle \neq 0$. Therefore, $(x, y) \notin E_n$.

(**5**) If $x \in \mathcal{S}_n$ and $y \in \pi_0(U_{n-1}) \cup \pi_0(V_{n-1})$ then except the n-th coordinate of x, all are zeros and y has n-th coordinate zero. Therefore, $\langle x, y\rangle = 0$. Hence, $(x, y) \in E_n$.

(**6**) Let $x, y \in V_{n-1}$ such that $x = (x_1, x_2, \ldots, x_{n-1})$ and $y = (y_1, y_2, ..., y_{n-1})$ then $\pi_0(x) = (x_1, x_2, ..., x_{n-1}, 0)$ and $\pi_i(y) = (y_1, y_2, ..., y_{n-1}, b_i^n)$, where for each i, b_i^n is i^{th} element of $S_{m_n} := \{b_0^n = 0, b_1^n, b_2^n, \ldots, b_{m_n-1}^n\}$. Now, it follows that, $(x, y) \in E_{n-1}$ if and only if $x_j y_j = 0, \ \forall\ j = 1, 2, ..., n-1$ if and only if $\langle \pi_0(x), \pi_i(y)\rangle = 0$ if and only if $\Big(\pi_0(x), \pi_i(y)\Big)$ and $\Big(\pi_0(y), \pi_i(x)\Big)$ are in E_n. □

Remark 3. Lemma 1 describes the structure of $G(R_n)$ recursively using the vertex-adjacency in $G(R_{n-1})$. Figure 1 describes a general idea to construct the graph $G(R_n)$. In Figure 1, all the $m_n + 2$ parts of the vertex partition V_n are represented by an elliptical shape. Note that, the sizes of the last m_n parts of V_n equal to the size $|V_{n-1}|$. That is, $|V_{n-1}| = |\pi_0(V_{n-1})| = |\pi_1(V_{n-1})| = \cdots = |\pi_{m_n-1}(V_{n-1})|$. The subgraph induced by the first two parts $\pi_0(U_{n-1}) \cup \mathcal{S}_n$ is a complete bipartite graph. For example, in Figure 1, a sample vertex $\pi_0(u)$ from the first part $\pi_0(U_{n-1})$ is adjacent to all the vertices of the second part $\mathcal{S}_n$. Also, a sample vertex $\pi_j(\mathbf{0}_{n-1})$ of the second part is adjacent to all the vertices in the first part. The subgraph induced by $\pi_0(V_{n-1})$ is the graph $H \cong G(R_{n-1})$. Every vertex in the part $\mathcal{S}_n$ is adjacent to all the vertices of H. Consider a sample edge, say (x, y) of the graph $G(R_{n-1})$. This implies $\Big(\pi_0(x), \pi_0(y)\Big)$ is an edge in H. Then the effect of this pair of adjacent vertices is that $\Big(\pi_0(x), \pi_i(y)\Big)$ and $\Big(\pi_0(y), \pi_i(x)\Big)$ are edges in $G(R_n)$ for each i. By Lemma 1, the cases discussed above cover all possibilities of having the pairs of adjacent vertices in the graph $G(R_n)$. The pairs of non-adjacent vertices in the graph $G(R_n)$ can be easily seen from Proposition 1(2-4).

For large n, the size of the vertex set V_n is exponential. Inductively, one can determine the size of the edge set of the graph, which is greater than the size of V_n. Thus, the graph $G(R_n)$

has more complexity due to the exponential sizes of both the vertex set and the edge set of the graph.

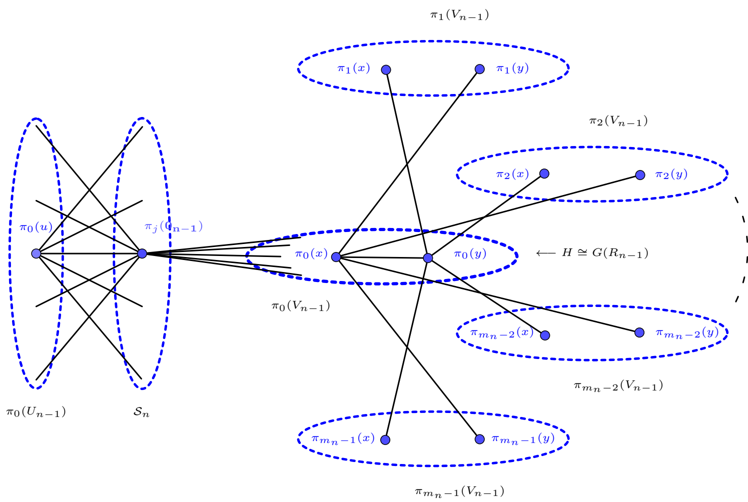

Figure 1. A recursive description of the graph $G(R_n)$ using the graph $G(R_{n-1})$.

In the next section, the recursive description of $G(R_n)$ in Lemma 1 and Remark 3 is utilized to obtain algorithms of various colorings of the graph.

3. Vertex coloring and coprime index of $G(R_n)$

In this section, we show that the class of graphs $G(R_n)$ satisfies the equality $\omega(G(R_n)) = \mu(G(R_n))$ in Problem 1 posed by Katre et al. [8]. By utilizing the recursive description of $G(R_n)$ from Section 2, we obtain simple algorithms for the minimal coprime labeling and the minimal vertex coloring of the graph $G(R_n)$. As a Consequence, we show that,

$$\omega(G(R_n)) = \mu(G(R_n)) = \chi(G(R_n)) = \theta_e(G(R_n)^{\complement}) = i(G(R_n)^{\complement}).$$

Proposition 2. *The clique number $\omega(G(R_n))$ of the graph $G(R_n)$ is n, and the diameter diam$(G(R_n))$ of the graph $G(R_n)$ is at most 3.*

Proof. For each $i \in \{1, 2, ..., n\}$, there exists a vertex say w_i of $G(R_n)$ whose i^{th} coordinate is non-zero, and the remaining coordinates are zero. Then the subgraph induced by $C := \{w_1, w_2, ..., w_n\} \subset V_n$ is a clique of order n in the graph $G(R_n)$. The vertex set V_n is a subset of $\mathbb{R}^n$, and the set $C \subset V_n$ is a maximal orthogonal subset in the vector space $\mathbb{R}^n$. This implies C is a maximal clique in $G(R_n)$. Therefore, the clique number $\omega(G(R_n))$ is n. Let u and v be arbitrary vertices in V_n. Then u and v are adjacent to some vertices of the clique induced by C. Therefore, the distance between u and v in $G(R_n)$ is at most 3. Hence, the diameter diam$(G(R_n))$ is at most 3. □

Recall Lemma 1. We utilize it to obtain a recursive algorithm for vertex coloring of $G(R_n)$.

Theorem 5. *Let $c_1, c_2, ..., c_{k-1}, c_k$ be distinct colors. If ψ_{n-1} is a $(k-1)$-coloring of $G(R_{n-1})$ with colors $c_1, c_2, ..., c_{k-1}$. Then the map ψ_n is a k-coloring of $G(R_n)$ with colors $c_1, c_2, ..., c_{k-1}, c_k$, where*

$$\psi_n(v) = \begin{cases} \psi_{n-1}(\pi_i^{-1}(v)), & \text{if } v \in \pi_i(V_{n-1}) \text{ for some } i \in \{0, 1, 2, ..., m_n - 1\}; \\ c_k, & \text{if } v \in \mathcal{S}_n; \\ c, & \text{if } v \in \pi_0(U_{n-1}), \end{cases}$$

where c is any color used in ψ_{n-1}.

Proof. Let ψ_{n-1} be a $(k-1)$-vertex coloring of $G(R_{n-1})$ with distinct colors say, $c_1, c_2, ..., c_{k-1}$. Let c_k be a color different from the colors used in ψ_{n-1}. Let $\psi_n : V_n \longrightarrow \{c_1, c_2, ..., c_{k-1}, c_k\}$ be map defined above. We claim to show that ψ_n is a k-coloring of $G(R_n)$. It is sufficient to show, $\psi_n(x) \neq \psi_n(y)$, whenever x is adjacent to y in $G(R_n)$. Lemma 1(5) and 1(6) gives all possible pairs of adjacent vertices in $G(R_n)$. Therefore it is sufficient to make only two cases as follows. Case 1: If $x \in \mathcal{S}_n$ and $y \in \pi_0(U_{n-1}) \cup \pi_0(V_{n-1})$. Then by Lemma 1(5), x is adjacent to y in $G(R_n)$. From the definition of ψ_n, $\psi_n(x) = c_k$ and $\psi_n(y) = \psi_{n-1}(\pi_0^{-1}(y)) \in \{c_1, c_2, ..., c_{k-1}\}$. Therefore, in this case, it is proved that $\psi_n(x) \neq \psi_n(y)$ whenever x is adjacent to y in $G(R_n)$. Case 2: If $x, y \in \bigcup_{i=0}^{m_n-1} \pi_i(V_{n-1})$ then there exist vertices say, z, w of $G(R_{n-1})$ such that $x = \pi_i(z)$ and $y = \pi_j(w)$ for some $i, j \in \{0, 1, 2, ..., m_n - 1\}$. Now, if x is adjacent to y in $G(R_n)$ then by Lemma 1(6), vertex z is adjacent to w in $G(R_{n-1})$. As ψ_{n-1} is a vertex coloring of $G(R_{n-1})$ we have, $\psi_{n-1}(z) \neq \psi_{n-1}(w)$, i.e., $\psi_{n-1}(\pi_i^{-1}(x)) \neq \psi_{n-1}(\pi_i^{-1}(y))$. Thus, $\psi_n(x) = \psi_{n-1}(\pi_i^{-1}(x)) \neq \psi_{n-1}(\pi_i^{-1}(y)) = \psi_n(y)$. That is $\psi_n(x) \neq \psi_n(y)$ whenever x is adjacent to y in $G(R_n)$. From both the cases above, we conclude that ψ_n is a k-coloring of the graph $G(R_n)$. □

We now show that the map ψ_n in Theorem 5 is in fact a minimal vertex-coloring of $G(R_n)$.

Theorem 6. *Let $1 < m_i, n \in \mathbb{N}$. The chromatic number, $\chi(G(R_n)) = n$.*

Proof. The proof is by induction on n. For the base case $n = 2$, $\chi(G(R_2)) = 2$ because $G(R_2)$ is a complete bipartite graph. Assume that the result is true for $n = k - 1$. That is, assume that $\chi(G(R_{k-1})) = k - 1$. By Lemma 1(1), $G(R_{k-1}) \cong H$, where H is an induced subgraph of $G(R_k)$ induced by $\pi_0(V_{k-1})$. Therefore, $\chi(G(R_k)) \geq k - 1$. By Lemma 1(5), every vertex in $\mathcal{S}_k$ is adjacent to all the vertices in H, and by Lemma 1(3) every pair of vertices in $\mathcal{S}_k$ are non-adjacent. Therefore, the graph $G(R_k)$ has an induced subgraph say H' induced by $\pi_0(V_{k-1}) \cup \mathcal{S}_k$ such that the chromatic number $\chi(H') = k$. Therefore, $\chi(G(R_k)) \geq k$. The equality holds if there exists a k-coloring of the graph $G(R_k)$. By induction hypothesis, $G(R_{k-1})$ has a $(k-1)$-coloring, and so by Lemma 5, there exist a k-coloring of $G(R_k)$. Therefore, we have the equality $\chi(G(R_k)) = k$. Thus, by induction principle, $\chi(G(R_n)) = n$ for all $n \geq 2$. □

Recall Definition 1. We obtain a recursive algorithm for a coprime labeling of the graph $G(R_n)$ using Lemma 1.

Theorem 7. *Let $p_1, p_2, ..., p_{k-1}, p_k$ be distinct primes. If L_{n-1} is a coprime labeling of $G(R_{n-1})$ using primes $p_1, p_2, ..., p_{k-1}$ then L_n is a coprime labeling of $G(R_n)$ with primes $p_1, p_2, ..., p_{k-1}, p_k$ where,*

$$L_n(v) = \begin{cases} L_{n-1}(\pi_0^{-1}(v)), & \text{if } v \in \pi_0(V_{n-1}); \\ L_{n-1}(\pi_i^{-1}(v)) \cdot p_k^i, & \text{if } v \in \pi_i(V_{n-1}) \text{ for some } i \in \{1, 2, ..., m_n - 1\}; \\ p_k^i, & \text{if } v \in \mathcal{S}_n \text{ and } v = \pi_i(\mathbf{0}_{n-1}),\ i \in \{1, 2, ..., m_n - 1\}; \\ (p_1 \cdot p_2 \cdots p_{k-1})^j, & \text{if } v \in \pi_0(U_{n-1}) \text{ and } v \text{ is } j^{th} \text{ element of } U_{n-1}. \end{cases}$$

Proof. The map L_n above is an injective map that follows from the definition of the map L_n and the injective nature of the map L_{n-1}. It remains to prove the statement, 'x is adjacent to y in $G(R_n)$ if and only if $L_n(x)$ and $L_n(y)$ are coprime.'

Case 1. Let $x, y \in \pi_0(V_{n-1})$. Then, x is adjacent to y in $G(R_n) \iff \pi_0^{-1}(x)$ and $\pi_0^{-1}(y)$ are adjacent in $G(R_{n-1})$ (by Lemma 1(6)) $\iff L_{n-1}(\pi_0^{-1}(x))$ and $L_{n-1}(\pi_0^{-1}(y))$ are coprime (since L_{n-1} is coprime labeling of $G(R_{n-1})$) $\iff L_n(x)$ and $L_n(y)$ are coprime.

Case 2. Let $x \in \pi_0(V_{n-1})$ and $y \in \pi_i(V_{n-1})$ for some $i \in \{1, 2, \ldots, m_n - 1\}$. Then, x is adjacent to y in $G(R_n) \iff \pi_0^{-1}(x)$ and $\pi_i^{-1}(y)$ are adjacent in $G(R_{n-1})$ (by Lemma 1(6)) $\iff L_{n-1}(\pi_0^{-1}(x))$ and $L_{n-1}(\pi_i^{-1}(y))$ are coprime (since L_{n-1} is coprime labeling of $G(R_{n-1})$) $\iff L_{n-1}(\pi_0^{-1}(x))$ and $L_{n-1}(\pi_i^{-1}(y)) \cdot p_k^i$ are coprime (since the prime p_k is not used in labeling L_{n-1}) $\iff L_n(x)$ and $L_n(y)$ are coprime.

Case 3. Let $x \in \pi_0(V_{n-1})$ and $y \in \mathcal{S}_n$. Then by Lemma 1(5), x is adjacent to y in $G(R_n)$. Therefore, it is enough to show $L_n(x)$ and $L_n(y)$ are coprime. $L_n(x) = L_{n-1}(\pi_0^{-1}(x))$. Therefore, all the prime factors of $L_n(x)$ lies in the set $\{p_1, p_2, ..., p_{k-1}\}$. Whereas $L_n(y) = p_k^i$, for some $i \in \{1, 2, \ldots, m_n - 1\}$. This implies $L_n(x)$ and $L_n(y)$ have no common factor, and so they are coprime.

Case 4. Let $x \in \pi_0(V_{n-1})$ and $y \in \pi_0(U_{n-1})$. Then by Lemma 1(2), x is not adjacent to y in $G(R_n)$. Here, $L_n(x) = L_{n-1}(\pi_0^{-1}(x))$, so every prime factor of $L_n(x)$ lies in $\{p_1, p_2, ..., p_{k-1}\}$. Now $L_n(y) = (p_1 \cdot p_2 \cdots p_{k-1})^i$, for some i. This implies, $L_n(x)$ and $L_n(y)$ are not coprime.

Case 5. Let $x, y \in \pi_i(V_{n-1}) \cup \mathcal{S}_n,\ i \in \{1, 2, \ldots, m_n - 1\}$. Then by Lemma 1(3) and 1(4), x is not adjacent to y in $G(R_n)$. Further, by the definition of the map L_n above, $L_n(x)$ and $L_n(y)$ have a common prime factor p_k. Therefore, $L_n(x)$ and $L_n(y)$ are not coprime.

Case 6. Let $x \in \pi_i(V_{n-1})$ for some $i \in \{1, 2, \ldots, m_n - 1\}$ and $y \in \pi_0(U_{n-1})$. Then by Lemma 1(2), x is not adjacent to y in $G(R_n)$. Further, $L_n(x) = L_{n-1}(\pi_i^{-1}(x)) \cdot p_k^i$, and so there exist a prime factor say p_l of $L_n(x)$ such that $p_l \in \{p_1, p_2, \ldots, p_{k-1}\}$. Also, $L_n(y) = (p_1 \cdot p_2 \cdots p_{k-1})^j$, for some j. This implies, $L_n(x)$ and $L_n(y)$ have a common prime factor p_l. Therefore, $L_n(x)$ and $L_n(y)$ are not coprime.

Case 7. Let $x \in \mathcal{S}_n$ and $y \in \pi_0(U_{n-1})$. Then by Lemma 1(5), x is adjacent to y in $G(R_n)$. Also, $L_n(x) = p_k^i$ and $L_n(y) = (p_1 \cdot p_2 \cdots p_{k-1})^j$ for some i and j. This implies, $L_n(x)$ and $L_n(y)$ are coprime.

Case 8. Let $x, y \in \pi_0(U_{n-1})$. Then by Lemma 1(2), x is not adjacent to y in $G(R_n)$. Further, $L_n(x)$ and $L_n(y)$ have a common factor p_1. Therefore, $L_n(x)$ and $L_n(y)$ are not coprime.

Thus, by the above cases, it is proved that, if x is adjacent (not adjacent) to y in $G(R_n)$ then $L_n(x)$ and $L_n(y)$ are coprime (not coprime). Therefore, the map L_n is a coprime labeling of the graph $G(R_n)$. □

Recall Definition 2. We show that the map L_n is a minimal coprime labeling of $G(R_n)$.

Theorem 8. *Let $1 < m_i, n \in \mathbb{N}$. The coprime index, $\mu(G(R_n)) = n$.*

Proof. We prove the result by induction on n. For $n = 2$, the graph $G(R_2)$ is isomorphic to the complete bipartite graph $K_{m_1-1,\, m_2-1}$ with the bipartition $\pi_0(U_1) \cup \mathcal{S}_2$. This implies, $\mu(G(R_2)) \neq 1$, and hence $\mu(G(R_2)) \geq 2$. The equality holds since $G(R_2)$ has a coprime labeling using two distinct primes. This labeling can be obtained using distinct primes say p_1 and p_2 as follows. For each i and j assign the label p_1^i to i^{th} vertex of $\mathcal{S}_2$, and assign the label p_2^j to j^{th} vertex in $\pi_0(U_1)$. Thus, $\mu(G(R_2)) = 2$. Now, assume that the result is true for $n = k-1$, i.e., $\mu(G(R_{k-1})) = k-1$. By Lemma 1(1), the graph $G(R_k)$ has an induced subgraph say H induced by the vertex set $V(H) := \pi_0(V_{k-1})$ such that $H \cong G(R_{k-1})$. Therefore, $\mu(H) = \mu(G(R_{k-1})) = k-1$. Then by Lemma A we have, $\mu(G(R_k)) \geq k-1$. Now, by Lemma 1(5), every vertex of H is adjacent to all the vertices of $\mathcal{S}_k$. Therefore, for an induced subgraph say H' of the graph $G(R_k)$ induced by $V(H) \cup \mathcal{S}_k$ we have, $\mu(H') > k-1$, i.e., $\mu(H') \geq k$. Again by Lemma A, $\mu(G(R_k)) \geq k$. For the equality, it is sufficient to show the existence of a coprime labeling of $G(R_k)$ using k distinct primes. By the induction hypothesis, the graph $G(R_{k-1})$ has a coprime labeling using $k-1$ distinct primes, therefore by Theorem 7 there exists a coprime labeling of $G(R_k)$ with k distinct primes. Hence, we have the equality $\mu(G(R_k)) = k$. Thus, by induction principle, $\mu(G(R_n)) = n$ for all $2 \leq n \in \mathbb{N}$. □

Thus, the class of graphs $G(R_n)$ satisfies the equality posed in Problem 1.

Theorem 9. *The graph $G(R_n)$ satisfies the following equality:*

$$\omega(G(R_n)) = \mu(G(R_n)) = \chi(G(R_n)) = \theta_e(G(R_n)^{\complement}) = i(G(R_n)^{\complement}) = n.$$

Proof. The proof follows by Proposition 2, Theorem 8, Theorem 6 and Observation 3. □

As a consequence of the above result, we can determine the clique number, the chromatic number, and the coprime index of some well-known algebraic graphs that are special cases of $G(R_n)$.

Corollary 1. *Let p be a prime. Consider the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n terms), $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (n terms) and $\mathbb{F}_q \times \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ (n terms), where $\mathbb{F}_q$ is a finite field with q elements. Let Γ be the zero-divisor graph of any one of these rings. Then the graph Γ satisfies the equality:*

$$\omega(\Gamma) = \mu(\Gamma) = \chi(\Gamma) = \theta_e(\Gamma^{\complement}) = i(\Gamma^{\complement}) = n.$$

More generally, if $\Gamma(R)$ is the zero-divisor graph of the ring $R := \mathbb{F}^{n_1}_{{p_1}^{l_1}} \times \mathbb{F}^{n_2}_{{p_2}^{l_2}} \times \cdots \times \mathbb{F}^{n_k}_{{p_k}^{l_k}}$ (k terms), where for each i, $\mathbb{F}^{n_i}_{{p_i}^{l_i}} = \mathbb{F}_{{p_i}^{l_i}} \times \mathbb{F}_{{p_i}^{l_i}} \times \cdots \times \mathbb{F}_{{p_i}^{l_i}}$ (n_i terms) and, $p_1^{l_1}, p_2^{l_2}, \ldots p_k^{l_k}$ are power of primes which are not necessarily same. Then the graph $\Gamma(R)$ satisfies the equality:

$$\omega(\Gamma(R)) = \mu(\Gamma(R)) = \chi(\Gamma(R)) = \theta_e(\Gamma(R)^{\complement}) = i(\Gamma(R)^{\complement}) = \textstyle\sum_{i=1}^{k} n_i.$$

Proof. The equality about the graph Γ follows from Remark 1 and Theorem 9. For the second part, set $m_i = p_i^{l_i}$, for each i. Thus, m_i's are power of primes, therefore by Remark 2, $\Gamma(R) \cong G(\mathcal{R})$ where $\mathcal{R} = S^{n_1}_{m_1} \times S^{n_2}_{m_2} \times \cdots \times S^{n_k}_{m_k}$ and for each i, $S^{n_i}_{m_i} = S_{m_i} \times S_{m_i} \times \cdots \times S_{m_i}$ (n_i terms). Further, the graph $G(\mathcal{R})$ is the graph $G(R_n)$ where each factor S_{m_i} is repeated n_i times, and the total number of factors in the expression of $\mathcal{R}$ is $n = n_1 + n_2 + \cdots + n_k$. Hence, the result follows from Theorem 9. □

4. Conclusion

In this paper, an algorithmic approach is developed towards a vertex coloring and a coprime labeling of the variant graph $G(R_n)$ of the dot product graphs introduced by Badawi [2]. Instead of being associated with a ring, this graph is defined combinatorially over a subset R_n of the first octant of the space $\mathbb{R}^n$. The graph $G(R_n)$ studied in this paper is an unifying generalization of important graphs from two distinct domains (algebra and combinatorics) and brings them together under a single umbrella. The special cases include Kneser graphs, set graphs or the Boolean graphs, and zero-divisor graphs of finite reduced rings. These graphs have more complexity due to exponential sizes of their vertex sets and the edge sets. An algorithmic description of the graph $G(R_n)$ is obtained using the graph $G(R_{n-1})$. This description is utilized to determine a simple algorithms for a vertex coloring and a coprime labeling of the graph. Further, it is proved that these labelings are minimal. Consequently, it is proved that, the class of these graphs satisfies the equality in the problem posed by Katre et al. [8]. That is, the equality between the clique number, the chromatic number, and the coprime index of the graph is proved. Computing an edge-clique cover number of a graph is known to be NP-hard problem. This number is determined for the complement of the graph. The algorithmic description of the graph can be utilized to determine the distance-based invariants and various graph labelings of the graph and its special cases.

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