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Research Article

A study of cyclic and constacyclic codes over $\mathbb{Z}_4 + u_2\mathbb{Z}_4 + u_3\mathbb{Z}_4 + \ldots + u_t\mathbb{Z}_4$

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Abstract: Constacyclic codes constitute a significant class of linear codes in coding theory and play a crucial role in the construction of optimal codes. Several optimal linear codes have been derived from constacyclic codes. In 2015, Ashraf and Mohammad investigated $(1+2u)$-constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ with $u^2 = 0$. More recently, G. Karthick studied $(1+2u+2v)$-constacyclic codes over the semi-local ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$ under the conditions $u^2 = 3u$, $v^2 = 3v$, and $uv = vu = 0$. In this paper, we generalize their results by examining $(1 + 2u_2 + 2u_3 + \cdots + 2u_t)$-constacyclic codes over the semi-local ring $\mathcal{S} = \mathbb{Z}_4 + u_2\mathbb{Z}_4 + u_3\mathbb{Z}_4 + \cdots + u_t\mathbb{Z}_4$, where $u_i^2 = ku_i$ and $u_iu_j = u_ju_i = 0$ for $2 \leq i \leq t$, $i \neq j$, with $u_1 = 1$ and $k \in \mathbb{Z}_4$. We focus on $(1 + 2u_2 + 2u_3 + \cdots + 2u_t)$-constacyclic codes over $\mathcal{S}$ and establish their structural properties. By introducing new Gray maps, we demonstrate that these constacyclic codes can be transformed into cyclic and quasi-cyclic codes over $\mathbb{Z}_4$. Furthermore, we characterize a generating set for these codes when the code length is odd. Our findings contribute to the database of $\mathbb{Z}_4$ codes and enhance the understanding of constacyclic codes over extended non-chain rings.

Keywords: linear code, constacyclic code, gray map, Quasi-cyclic code.

AMS Subject classification: 94B05, 94B15, 94B60, 94B60

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1. Introduction

Cyclic and constacyclic codes over finite rings have been widely studied due to their strong algebraic structures and their significance in error detection and correction. In 1994, Hammons et al. [7] demonstrated that certain high-performance binary nonlinear codes, such as Kerdock and Preparata codes, are actually the Gray map images of specific linear codes over $\mathbb{Z}_4$. This discovery marked a significant breakthrough in coding theory. Additionally, in 1996, Pless and Qian [14] extensively studied cyclic codes of odd length. These two works sparked significant interest in the study of codes over $\mathbb{Z}_4$. In 2003, Abualrub and Oehmke [1] investigated cyclic codes of a specific even length 2^e over $\mathbb{Z}_4$. Continuing this line of research, Blackford [6] also examined the structure of cyclic codes of oddly even length over $\mathbb{Z}_4$. In 2006, Dougherty and Ling [9] studied cyclic codes over $\mathbb{Z}_4$ of arbitrarily even length and classified all self-dual cyclic codes of length up to 14. Further progress was made by, Yildiz and Aydin [15] investigated the structure of cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and derived generator polynomials for these codes using the generators of cyclic codes over $\mathbb{Z}_4$. Since then, several studies have extended this work to cyclic codes over various extension rings of $\mathbb{Z}_4$; see [2, 5, 8, 12, 13].
Following these studies, researchers explored codes over extensions of $\mathbb{Z}_4$ have extended to its extensions, such as $\mathbb{Z}_4 + u\mathbb{Z}_4$, leading to the discovery of new linear codes with improved parameters. In 2015, Ashraf and Mohammad [4] investigated $(1 + 2u)$-constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ under the condition $u^2 = 0$. Expanding on this, in 2018, Islam and Prakash [10] studied $(1 + 2u)$-constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$ with the conditions $u^2 = v^2 = vu = uv = 0$. In their work, Islam and Prakash also examined the cyclic, quasi-cyclic, and permutation-equivalent quasi-cyclic (QC) codes over $\mathbb{Z}_4$, obtained as the Gray images of $(1+2u)$-constacyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$. More recently, Karthick, G. [11] advanced this research by examining $(1+2u+2v)$-constacyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$ under the conditions $u^2 = 3u$, $v^2 = 3v$, and $uv = vu = 0$. They proved that these codes are equivalent to cyclic and quasi-cyclic codes over $\mathbb{Z}_4$ by defining two Gray maps. Additionally, for codes of odd length, they derived a generating set for $(1+2u+2v)$-constacyclic codes.
Motivated by these recent advancements, we now extend the study of constacyclic codes over the non-chain commutative ring $\mathbb{Z}_4 + u_2\mathbb{Z}_4 + u_3\mathbb{Z}_4 + \cdots + u_t\mathbb{Z}_4$, where $u_i^2 = ku_i$, for $k \in \mathbb{Z}_4$ and $u_iu_j = 0$, for all $i \neq j$. Specifically, we extend the results in [11] by investigating $(1 + 2u_2 + 2u_3 + \cdots + 2u_t)$-constacyclic codes. We establish that these codes can be transformed into quasi-cyclic codes over $\mathbb{Z}_4$ using newly defined Gray maps, thereby broadening existing results. Additionally, for codes of odd length, we construct a generating set and analyze their algebraic properties, leading to new code constructions that enrich the database of $\mathbb{Z}_4$ codes. The remainder of this paper is structured as follows: Section 2 provides the necessary preliminaries and mathematical background. Section 3 introduces and examines the newly defined Gray maps that facilitate the study of $(1 + 2u_2 + 2u_3 + \cdots + 2u_t)$-constacyclic codes. In Section 4, we explore the algebraic properties and generating polynomials of $(1 + 2u_2 + 2u_3 + \cdots + 2u_t)$-constacyclic codes.

2. Preliminaries

Let $\mathcal{R}$ be a ring, and let $\mathcal{C}$ be a nonempty subset of $\mathcal{R}^n$. The set $\mathcal{C}$ is called a linear code of length n over $\mathcal{R}$ if it forms an $\mathcal{R}$-submodule of $\mathcal{R}^n$.

Definition 1. Let β be a unit in $\mathcal{R}$. The β-constacyclic shift is given by

$$\phi_\beta(c_0, c_1, \ldots, c_{n-1}) = (\beta c_{n-1}, c_0, \ldots, c_{n-2}).$$

A code in which every codeword follows this shift is known as a β-constacyclic code. When $\beta = 1$, the β-constacyclic code is referred to as a cyclic code, and ϕ is called the cyclic shift operator. When $\beta = -1$, it is called a negacyclic code.

To analyze the structure of β-constacyclic codes algebraically, we represent their codewords as polynomials over $\frac{\mathcal{R}[x]}{(x^n-\beta)}$. To do so, we define a linear map $\psi : \mathcal{C} \to \frac{\mathcal{R}[x]}{(x^n-\beta)}$, defined by

$$\psi(c_0, c_1, \ldots, c_{n-1}) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}.$$

Thus, a β-constacyclic code can be regarded as a set of polynomials over $\frac{\mathcal{R}[x]}{(x^n-\beta)}$. In fact, the following theorem establishes that any β-constacyclic code can be viewed as an ideal of $\frac{\mathcal{R}[x]}{(x^n-\beta)}$.

Theorem 1. *Let $\mathcal{C}$ be a linear code of length n over $\mathcal{R}$. Then $\mathcal{C}$ is a β-constacyclic over $\mathcal{R}$ if and only if $\psi(\mathcal{C})$ is an ideal of $\frac{\mathcal{R}[x]}{(x^n-\beta)}$.*

Beyond constacyclic codes, an important class of related codes is quasi-cyclic codes, which we define next.

Definition 2. Let $r = (r_0, r_1, \ldots, r_{m-1}) \in \mathcal{R}^{mn}$, where $r_i \in \mathcal{R}^n$ for $i \in \{0, 1, \ldots, m-1\}$. The map $\theta_m : \mathcal{R}^{mn} \to \mathcal{R}^{mn}$ is defined as

$$\theta_m(r_0, r_1, \ldots, r_{m-1}) = (\phi_1(r_0), \phi_1(r_1), \ldots, \phi_1(r_{m-1})),$$

where ϕ_1 denotes the cyclic shift defined earlier. A code that is closed under ϕ_1 is called a quasi-cyclic code of index m.

In certain coding structures, permutations play a crucial role. One such permutation, known as Nechaev's permutation, is defined as follows

Definition 3. Let n is greater than or equal to 1 be an odd positive integer, and consider the permutation

$$\xi = (1, n+1), (3, n+3), \ldots, (2i+1, n+2i+1), \ldots, (n-2, 2n-2)$$

of the set $\{0, 1, \ldots, 2n-1\}$. Nechaev's permutation π is defined as

$$\pi(c_0, c_1, \ldots, c_{2n-2}) = (c_{\xi(0)}, c_{\xi(1)}, \ldots, c_{\xi(2n-1)}).$$

To further explore constacyclic codes, we introduce the ring $\mathcal{S}$, which provides a useful algebraic framework for analyzing these codes. Throughout this paper, we define $\mathcal{S} = \sum_{i=1}^{t} u_i\mathbb{Z}_4$, where $u_1 = 1$, $u_i^2 = ku_i$ for $2 \leq i \leq t$, with $k \in \mathbb{Z}_4$, and $u_iu_j = 0$ for all $i \neq j$. It is evident that $\mathcal{S}$ is a commutative ring with 4^t elements. Furthermore, $\mathcal{S}$ is isomorphic to the quotient ring $\frac{\mathbb{Z}_4[u_2,u_3,\ldots,u_t]}{\langle u_i^2-ku_i, u_iu_j\rangle}$. Every element of $\mathcal{S}$ can be written in the form $a_1 + a_2u_2 + \cdots + a_tu_t$. Note that the set of units in the ring $\mathcal{S}$ varies depending on the value of k, as described in the following lemma.

Lemma 1. *Let $\mathcal{S} = \sum_{i=1}^{t} u_i\mathbb{Z}_4$, where $u_1 = 1$, $u_i^2 = ku_i$ for $2 \leq i \leq t$, with $k \in \mathbb{Z}_4$, and $u_iu_j = 0$ for all $i \neq j$. Denote the set of units in $\mathcal{S}$ by $U(\mathcal{S})$. Then:*

1. *If $k \in \{0, 2\}$, then*

$$U(\mathcal{S}) = \left\{ a_1 + \sum_{i=2}^{t} a_iu_i \mid a_1 \in U(\mathbb{Z}_4) \text{ and } a_i \in \mathbb{Z}_4, \text{ for all } 2 \leq i \leq t \right\}.$$

2. *If $k \in U(\mathbb{Z}_4)$, then*

$$U(\mathcal{S}) = \left\{ a_1 + \sum_{i=2}^{t} a_iu_i \mid a_1 \in U(\mathbb{Z}_4) \text{ and } a_i \in \{0, 2\}, \quad \text{for all } \ 2 \leq i \leq t \right\}.$$

The reason for choosing to study $(1 + 2\sum_{i=2}^{t} u_i)$-constacyclic codes is that $(1 + 2\sum_{i=2}^{t} u_i)$ is always a unit for any $k \in \mathbb{Z}_4$. Henceforth, we set $\beta = (1 + 2\sum_{i=2}^{t} u_i)$.

3. Gray Maps from $\mathcal{S}$ to $\mathbb{Z}_4^2$

In this section, we first define three distinct Gray maps from $\mathcal{S}$ to $\mathbb{Z}_4^2$, each constructed based on the condition $u_i^2 = ku_i$, where $k \in \mathbb{Z}_4$. We then explore how these mappings influence the algebraic structure of codes over these rings. In particular, we establish a connection between cyclic and quasi-cyclic codes of length $2n$ over $\mathbb{Z}_4$ and β-constacyclic codes of length n over $\mathcal{S}$, providing insights into their structural properties.
We now formally define these Gray maps as follows.

Definition 4. Let f_i, for $i = 1, 2, 3$, be linear maps defined from $\mathcal{S}$ to $\mathbb{Z}_4^2$ as follows:

$$f_1\left(\sum_{i=1}^{t} u_ia_i\right) = \begin{cases} \left(2a_1 + 3\sum_{i=2}^{t} a_i, 2a_1 + \sum_{i=2}^{t} a_i\right), & \text{if } t \text{ is odd}, \\ \left(a_1 + 3\sum_{i=2}^{t} a_i, 3a_1 + \sum_{i=2}^{t} a_i\right), & \text{if } t \text{ is even}. \end{cases} \tag{3.1}$$

$$f_2\left(\sum_{i=1}^{t} u_ia_i\right) = \begin{cases} \left(a_1 + 3\sum_{i=2}^{t} a_i, a_1 + 3\sum_{i=2}^{t} a_i\right), & \text{if } t \text{ is odd}, \\ \left(a_1 + 3\sum_{i=2}^{t} a_i, 3\sum_{i=1}^{t} a_i\right), & \text{if } t \text{ is even}. \end{cases} \tag{3.2}$$

$$f_3\left(\sum_{i=1}^{t} u_ia_i\right) = \left(a_1 + 2\sum_{i=2}^{t} a_i, 3a_1 + 2\sum_{i=2}^{t} a_i\right). \tag{3.3}$$

Note that the map f_1 is a generalization of the map defined in [11]. The above-defined maps can now be extended to $\mathcal{S}^n$ in two different ways, as described in the following definition.

Definition 5. For $\mathbf{s} \in \mathcal{S}^n$, we write $\mathbf{s} = (s_0, s_1, \ldots, s_{n-1})$, where each s_i can be expressed as $s_i = \sum_{j=1}^{t} u_j a_{(i,j)}$ for $0 \leq i \leq n-1$, with $u_1 = 1$.
Let $f : \mathcal{S} \to \mathbb{Z}_4^2$ be defined as $f(s) = (a, b)$, for any $s = a + ub \in \mathcal{S}$. This f can be extended from $\mathcal{S}^n$ to $\mathbb{Z}_4^{2n}$ in the following two ways:

$$f'(s_0, s_1, \ldots, s_{n-1}) = (a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}),$$

$$f^*(s_0, s_1, \ldots, s_{n-1}) = (a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1}).$$

The map f^* is called the permutation map of f'.

Now, we define one of the important parameters, the Lee weight of linear codes, which plays an important role in error detection and correction. The Lee weight of an element $a \in \mathbb{Z}_4$ is defined as $\min(a, 4 - a)$ and is denoted by $w_L(a)$.
For any element

$$s = \sum_{j=1}^{t} u_j a_j \in \mathcal{S},$$

where $u_1 = 1$, the Lee weight of s is defined as $w_L(s) = w_L(f_i(s))$, f_i are as in definition(4), $i \in \{1, 2, 3\}$. The Lee distance of a code $\mathcal{C}$ is given by

$$d_L(\mathcal{C}) = \min\{w_L(\mathbf{c} - \mathbf{c}') \mid \mathbf{c}, \mathbf{c}' \in \mathcal{C}, \, \mathbf{c} \neq \mathbf{c}'\}.$$

With the above-defined notations, we present some lemmas that are useful in establishing the relationship between cyclic codes over $\mathbb{Z}_4$ and β-constacyclic codes over $\mathcal{S}$, where $\beta = (1 + 2\sum_{i=2}^{t} u_i) \in U(\mathcal{S})$.

Lemma 2. *Let f_1 be the Gray map defined in (3.1), and let σ represent the cyclic shift operator. If $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j \mathbb{Z}_4$ with $u_j^2 = ku_j$ for all $2 \leq j \leq t$ and $k \in U(\mathbb{Z}_4)$, then*

$$\sigma f_1'(\mathbf{s}) = f_1' \phi_\beta(\mathbf{s}), \quad \textit{for all } \mathbf{s} \in \mathcal{S}^n.$$

Proof. We know that $U(\mathbb{Z}_4) = \{1, 3\}$. We provide the proof for the case $u_i^2 = 3u_i$, while the proof for the case $u_i^2 = u_i$ follows similarly.
The map f_1 is defined based on whether t is even or odd. For this, consider the following cases

Case I: Let t is an odd integer

$$\begin{aligned}
\sigma f_1'(\mathbf{s}) &= \sigma f_1'(s_0,s_1,\ldots,s_{n-1})\\
&= \sigma f_1'\left(\sum_{j=1}^{t}u_ja_{(0,j)},\sum_{j=1}^{t}u_ja_{(1,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-1,j)}\right)\\
&= \sigma\Bigg(2a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},2a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},\ldots,\\
&\quad 2a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},\ldots,2a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)}\Bigg)\\
&= \Bigg(2a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},\ldots,\\
&\quad 2a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},\ldots,2a_{(n-2,1)}+\sum_{j=2}^{t}a_{(n-2,j)}\Bigg). \qquad (3.4)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f_1'\phi_\beta(\mathbf{s}) &= f_1'\left(\beta\sum_{j=1}^{t}u_ja_{(n-1,j)},\sum_{j=1}^{t}u_ja_{(0,j)},\sum_{j=1}^{t}u_ja_{(1,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-2,j)}\right)\\
&= f_1'\left(a_{(n-1,1)}+\sum_{j=2}^{t}u_j[3a_{(n-1,j)}+2a_{(n-1,1)}],\sum_{j=1}^{t}u_ja_{(0,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-2,j)}\right)\\
&= \Bigg(2a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},\ldots,2a_{(n-2,1)}+3\sum_{j=2}^{t}a_{(n-2,j)},\\
&\quad 2a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},\ldots,2a_{(n-2,1)}+\sum_{j=2}^{t}a_{(n-2,j)}\Bigg).
\end{aligned} \qquad (3.5)$$

Case II: Let t is an even integer

$$\begin{aligned}
\sigma f_1'(\mathbf{s}) &= \sigma f_1'\left(\sum_{j=1}^{t}u_ja_{(0,j)},\sum_{j=1}^{t}u_ja_{(1,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-1,j)}\right)\\
&= \Bigg(3a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)},a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},\ldots,\\
&\quad a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},3a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},\ldots,3a_{(n-2,1)}+\sum_{j=2}^{t}a_{(n-2,j)}\Bigg). \qquad (3.6)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
f_1'\phi_\beta(\mathbf{s}) &= f_1'\left(\beta\sum_{j=1}^{t}u_ja_{(n-1,j)},\sum_{j=1}^{t}u_ja_{(0,j)},\sum_{j=1}^{t}u_ja_{(1,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-2,j)}\right)\\
&=\Bigg(3a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)},a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},\ldots,\\
&\quad a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},3a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},\ldots,3a_{(n-2,1)}+\sum_{j=2}^{t}a_{(n-2,j)}\Bigg). \qquad (3.7)
\end{aligned}
$$

From equations (3.4), (3.5), (3.6), and (3.7), we conclude the result. □

Lemma 3. *Let f_1 be the Gray map defined in (3.1), and let σ represent the cyclic shift operator. If $\mathcal{S}=\mathbb{Z}_4+\sum_{j=2}^{t}u_j\mathbb{Z}_4$ with $u_j^2=ku_j$, where $k\in U(\mathbb{Z}_4)$ and for all $2\le j\le t$, then*

$$f_1^*(\sigma\mathbf{s})=\sigma^2 f_1^*(\mathbf{s}),\ \textit{for all } \mathbf{s}\in\mathcal{S}^n$$

Proof. We provide the proof for the case $u_i^2=3u_i$, while the proof for the case $u_i^2=u_i$ follows in a similar manner.
If t is an odd integer, consider

$$
\begin{aligned}
f_1^*(\sigma)(\mathbf{s}) =& f_1^*(s_{n-1},s_0,s_1,s_2,\ldots,s_{n-2})\\
=&\Bigg(2a_{(n-1,1)}+3\sum_{j=2}^{n}a_{n-1,j},2a_{(n-1,1)}+\sum_{j=2}^{n}a_{n-1,j},2a_{(0,1)}+3\sum_{j=2}^{n}a_{(0,j)},\ldots\\
&2a_{(n-2,1)}+3\sum_{j=2}^{n}a_{n-2,j},2a_{(n-2,1)}+\sum_{j=2}^{n}a_{(n-2,j)}\Bigg). \qquad (3.8)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\sigma^2 f_1^*(\mathbf{s}) =& \sigma^2 f_1^*(s_0,s_1,s_2,\ldots,s_{n-1})\\
=&\sigma^2\Bigg(2a_{(0,1)}+3\sum_{j=2}^{n}a_{(0,j)},2a_{(0,1)}+\sum_{j=2}^{n}a_{(0,j)},2a_{(1,1)}+3\sum_{j=2}^{n}a_{(1,j)},2a_{(1,1)}+\sum_{j=2}^{n}a_{(1,j)},\ldots,\\
&2a_{(n-1,1)}+3\sum_{j=2}^{n}a_{(n-1,j)},2a_{(n-1,1)}+\sum_{j=2}^{n}a_{(n-1,j)}\Bigg)\\
=&\Bigg(2a_{(n-1,1)}+3\sum_{j=2}^{n}a_{(n-1,j)},2a_{(n-1,1)}+\sum_{j=2}^{n}a_{(n-1,j)},2a_{(0,1)}+3\sum_{j=2}^{n}a_{(0,j)},\ldots\\
&2a_{(n-2,1)}+3\sum_{j=2}^{n}a_{(n-2,j)},2a_{(n-2,1)}+\sum_{j=2}^{n}a_{(n-2,j)}\Bigg). \qquad (3.9)
\end{aligned}
$$

From equations (3.8) and (3.9), we have $f_1^*(\sigma)(\mathbf{s})=\sigma^2 f_1^*(\mathbf{s})$.
For the case where t is an even integer, we can easily verify that $f_1^*(\sigma)(\mathbf{s})=\sigma^2 f_1^*(\mathbf{s})$. □

Corollary 1. *Let $\mathcal{C}$ be a linear code of odd length n over $\mathcal{S}$. Then $\mathcal{C}$ is a cyclic code if and only if $\Phi(\mathcal{C})$ is a β-constacyclic code, where $\Phi:\mathcal{S}^n\to\mathcal{S}^n$ is defined by*

$$\Phi(s_0,s_1,\ldots,s_{n-1})=(s_0,\beta s_1,\beta^2 s_2,\ldots,\beta^{n-1}s_{n-1}).$$

Lemma 4. *Let f_1 be the Gray map defined in (3.1) and let π is Nechaev's permutation. If $\mathcal{S}=\mathbb{Z}_4+\sum_{j=2}^{t}u_j\mathbb{Z}_4$ with $u_j^2=ku_j$, where $k\in U(\mathbb{Z}_4)$ and for all $2\le j\le t$, then*

$$f_1'\Phi(\mathbf{s})=\pi f_1'(\mathbf{s}),\quad \textit{for all } \mathbf{s}\in\mathcal{S}^n$$

where Φ is the map defined in Corollary 1.

Proof. We provide the proof for the case where t is an odd integer, and the case where t is even follows in a similar manner.
Consider

$$\begin{aligned}f_1'\Phi(\mathbf{s})=&f_1'\Phi(s_0,s_1,\ldots,s_{n-1})=f_1'(s_0,\beta s_1,\beta^2 s_2,\ldots,\beta^{n-2}s_{n-2},\beta^{n-1}s_{n-1})\\ =&\Bigg(2a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},2a_{(1,1)}+\sum_{j=2}^{t}a_{(0,j)},2a_{(2,1)}+3\sum_{j=2}^{t}a_{(2,j)},2a_{(3,1)}+\sum_{j=2}^{t}a_{(3,j)},\ldots,\\ &2a_{(n-2,1)}+\sum_{j=2}^{t}a_{(n-2,j)},2a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},2a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},\\ &2a_{(2,1)}+\sum_{j=2}^{t}a_{(2,j)},2a_{(3,1)}+3\sum_{j=2}^{t}a_{(3,j)},\ldots,2a_{(n-2,1)}+3\sum_{j=2}^{t}a_{(n-2,j)},2a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)}\Bigg).\end{aligned}\tag{3.10}$$

On the other hand

$$\begin{aligned}\pi f_1'(\mathbf{s})=&\pi f_1'(s_0,s_1,s_2,\ldots,s_{n-1})\\ =&\pi\Bigg(2a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},2a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},2a_{(2,1)}+3\sum_{j=2}^{t}a_{(2,j)},\ldots,\\ &2a_{(n-2,1)}+3\sum_{j=2}^{t}a_{(n-2,j)},2a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},2a_{(1,1)}+\sum_{j=2}^{t}a_{(1,j)},\\ &2a_{(2,1)}+\sum_{j=2}^{t}a_{(2,j)},\ldots,2a_{(n-2,1)}+\sum_{j=2}^{t}a_{(n-2,j)},2a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)}\Bigg)\\ =&\Bigg(2a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},2a_{(1,1)}+\sum_{j=2}^{t}a_{(0,j)},2a_{(2,1)}+3\sum_{j=2}^{t}a_{(2,j)},2a_{(3,1)}+\sum_{j=2}^{t}a_{(3,j)},\ldots,\\ &2a_{(n-2,1)}+\sum_{j=2}^{t}a_{(n-2,j)},2a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},2a_{(0,1)}+\sum_{j=2}^{t}a_{(0,j)},2a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},\\ &2a_{(2,1)}+\sum_{j=2}^{t}a_{(2,j)},2a_{(3,1)}+3\sum_{j=2}^{t}a_{(3,j)},\ldots,2a_{(n-2,1)}+3\sum_{j=2}^{t}a_{(n-2,j)},2a_{(n-1,1)}+\sum_{j=2}^{t}a_{(n-1,j)}\Bigg).\end{aligned}\tag{3.11}$$

From equations (3.10) and (3.11), we obtain $f_1'\Phi(\mathbf{s})=\pi f_1'(\mathbf{s})$. $\square$

Lemma 5. *Let f_2 be the Gray map defined in (3.2), and let σ represent the cyclic shift operator. If $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \{0, 2\}$ and for all $2 \leq j \leq t$ then*

$$\sigma f_2'(\mathbf{s}) = f_2'\phi_\beta(\mathbf{s}), \quad \textit{for all } \mathbf{s} \in \mathcal{S}^n.$$

Proof. We provide the proof for the case $u_i^2 = 2u_i$, while the proof for the case $u_i^2 = 0$ follows in a similar manner.

Case I: Let t is an odd integer.

Consider

$$\begin{aligned}\sigma f_2'(\mathbf{s}) =&\sigma f_2'\left(\sum_{j=1}^{t} u_j a_{(0,j)}, \sum_{j=1}^{t} u_j a_{(1,j)}, \ldots, \sum_{j=1}^{t} u_j a_{(n-1,j)}\right)\\ =&\sigma\left(a_{(0,1)} + 3\sum_{j=2}^{t} a_{(0,j)}, a_{(1,1)} + 3\sum_{j=2}^{t} a_{(1,j)}, \ldots, a_{(n-1,1)} + 3\sum_{j=2}^{t} a_{(n-1,j)},\right.\\ &\left. a_{(0,1)} + 3\sum_{j=2}^{t} a_{(0,j)}, a_{(1,1)} + 3\sum_{j=2}^{t} a_{(1,j)}, \ldots, a_{(n-1,1)} + 3\sum_{j=2}^{t} a_{(n-1,j)}\right)\\ =&\left(a_{(n-1,1)} + 3\sum_{j=2}^{t} a_{(n-1,j)}, a_{(0,1)} + 3\sum_{j=2}^{t} a_{(0,j)}, \ldots + a_{(n-2,1)} + 3\sum_{j=2}^{t} a_{(n-2,j)},\right.\\ &\left. a_{(n-1,1)} + 3\sum_{j=2}^{t} a_{(n-1,j)}, a_{(0,1)} + 3\sum_{j=2}^{t} a_{(0,j)}, \ldots, a_{(n-2,1)} + 3\sum_{j=2}^{t} a_{(n-2,j)}\right).\end{aligned} \tag{3.12}$$

On the other hand

$$\begin{aligned}f_2'\phi_\beta(\mathbf{s}) =&f_2'\left(\beta\sum_{j=1}^{t} u_j a_{(n-1,j)}, \sum_{j=1}^{t} u_j a_{(0,j)}, \sum_{j=1}^{t} u_j a_{(1,j)}, \ldots, \sum_{j=1}^{t} u_j a_{(n-2,j)}\right)\\ =&f_2'\left(a_{(n-1,1)} + \sum_{j=2}^{t} u_j[a_{(n-1,j)} + 2a_{(n-1,1)}], \sum_{j=1}^{t} u_j a_{(0,j)}, \sum_{j=1}^{t} u_j a_{(1,j)}, \ldots, \sum_{j=1}^{t} u_j a_{(n-2,j)}\right)\\ =&\left(a_{(n-1,1)} + 3\sum_{j=2}^{t} a_{(n-1,j)}, a_{(0,1)} + 3\sum_{j=2}^{t} a_{(0,j)}, \ldots + a_{(n-2,1)} + 3\sum_{j=2}^{t} a_{(n-2,j)},\right.\\ &\left. a_{(n-1,1)} + 3\sum_{j=2}^{t} a_{(n-1,j)}, a_{(0,1)} + 3\sum_{j=2}^{t} a_{(0,j)}, \ldots, a_{(n-2,1)} + 3\sum_{j=2}^{t} a_{(n-2,j)}\right).\end{aligned} \tag{3.13}$$

Case II: Let$t$ is an even integer

$$\begin{aligned}\sigma f_2'(\mathbf{s}) =&\sigma f_2'\left(\sum_{j=1}^{t} u_j a_{(0,j)}, \sum_{j=1}^{t} u_j a_{(1,j)}, \ldots, \sum_{j=1}^{t} u_j a_{(n-1,j)}\right)\\ =&\sigma\left(a_{(0,1)} + 3\sum_{j=2}^{t} a_{(0,j)}, a_{(1,1)} + 3\sum_{j=2}^{t} a_{(1,j)}, \ldots, a_{(n-1,1)} + 3\sum_{j=2}^{t} a_{(n-1,j)},\right.\\ &\left. 3\sum_{j=1}^{t} a_{(0,j)}, 3\sum_{j=1}^{t} a_{(1,j)}, \ldots, 3\sum_{j=1}^{t} a_{(n-1,j)}\right)\end{aligned} \tag{3.14}$$

$$\begin{aligned}&=\Bigg(3\sum_{j=1}^{t}a_{(n-1,j)},a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},\ldots,\\&\quad a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},3\sum_{j=2}^{t}a_{(0,j)},3\sum_{j=2}^{t}a_{(1,j)},\ldots,3\sum_{j=2}^{t}a_{(n-2,j)}\Bigg).\end{aligned}\tag{3.15}$$

On the other hand

$$\begin{aligned}f_2'\phi_\beta(\mathbf{s})=&f_2'\left(\beta\sum_{j=1}^{t}u_ja_{(n-1,j)},\sum_{j=1}^{t}u_ja_{(0,j)},\sum_{j=1}^{t}u_ja_{(1,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-2,j)}\right)\\=&\Bigg(3\sum_{j=1}^{t}a_{(n-1,j)},a_{(0,1)}+3\sum_{j=2}^{t}a_{(0,j)},a_{(1,1)}+3\sum_{j=2}^{t}a_{(1,j)},\ldots,\\&\quad a_{(n-1,1)}+3\sum_{j=2}^{t}a_{(n-1,j)},3\sum_{j=1}^{t}a_{(0,j)},3\sum_{j=1}^{t}a_{(1,j)},\ldots,3\sum_{j=1}^{t}a_{(n-2,j)}\Bigg).\end{aligned}\tag{3.16}$$

From equations (3.12), (3.13), (3.14), and (3.16), we obtain $\sigma f_2'(\mathbf{s})=f_2'\phi_\beta(\mathbf{s})$. □

Lemma 6. *Let f_2 be the Gray map defined in (3.2), and let σ represent the cyclic shift operator. If $\mathcal{S}=\mathbb{Z}_4+\sum_{j=2}^{t}u_j\mathbb{Z}_4$ with $u_j^2=ku_j$, where $k\in\{0,2\}$ and for all $2\le j\le t$, then*

$$f_2^*(\sigma\mathbf{s})=\sigma^2 f_2^*(\mathbf{s}),\ \textit{for all } \mathbf{s}\in\mathcal{S}^n.$$

Proof. Follows in a similar way of Lemma 3 □

Lemma 7. *Let f_2 be the Gray map defined in definition 3.2. If $\mathcal{S}=\mathbb{Z}_4+\sum_{j=2}^{t}u_j\mathbb{Z}_4$ with $u_j^2=ku_j$, where $k\in\{0,2\}$ and for all $2\le j\le t$, then $f_2'\Phi=\pi f_2'$ where π is Nechaev's permutation and Φ is the map defined in Corollary 1.*

Proof. Follows in a similar way of Lemma 4 □

Now, we proceed to prove a theorem that generalizes the result in [11].

Theorem 2. *Let $\mathcal{C}$ be a linear code over $\mathcal{S}=\mathbb{Z}_4+\sum_{j=1}^{t}u_j\mathbb{Z}_4$. Then*

1. *For $u_j^2=ku_j$, for all $2\le j\le t$ and $k\in U(\mathbb{Z}_4)$*
   - *(a) If $\mathcal{C}$ is a β-constacyclic code, then $f_1'(\mathcal{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*
   - *(b) If $\mathcal{C}$ is a cyclic code of length n, then $f_1^*(\mathcal{C})$ is a two cyclic code of length $2n$ over $\mathbb{Z}_4$.*
   - *(c) If $\mathcal{C}$ is a cyclic code of an odd length n and $\mathcal{T}$=$f_1'(\mathcal{C})$, then $\pi(\mathcal{T})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*
2. *For $u_j^2=ku_j$, for all $2\le j\le t$ and $k\in\{0,2\}$*
   - *(a) If $\mathcal{C}$ is a β-constacyclic code, then $f_2'(\mathcal{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

(b) *If $\mathcal{C}$ is a cyclic code of length n, then $f_2^*(\mathcal{C})$ is a two cyclic code of length $2n$ over $\mathbb{Z}_4$.*

(c) *If $\mathcal{C}$ is a cyclic code of an odd length n and $\mathcal{T}$=$f_2'(\mathcal{C})$, then $\pi(\mathcal{T})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. 1. (a) Let $\mathcal{C}$ be a β-constacyclic code, then for each $\mathbf{c} \in \mathcal{C}$ we have $\phi_\beta(\mathbf{c}) \in \mathcal{C}$. By Lemma 2 we have $\sigma f_1'(\mathcal{C}) = f_1'\phi_\beta(\mathcal{C}) = f_1'(\mathcal{C})$, which implies that $f_1'(\mathcal{C})$ is a cyclic code of length $2n$ over $\mathcal{S}$.

(b) Let $\mathcal{C}$ be a cyclic code of length n. Then $\sigma(\mathbf{c}) \in \mathcal{C}$ for all $\mathbf{c} \in \mathcal{C}$. From Lemma 3, we have $f_1^*\sigma(\mathcal{C}) = f_1^*(\mathcal{C}) = \sigma^2 f_1^*(\mathcal{C})$. Hence $f_1^*(\mathcal{C})$ is a two cyclic code of length $2n$ over $\mathbb{Z}_4$.

(c) Let $\mathcal{C}$ be a cyclic code and $\mathcal{T} = f_1^{'}(\mathcal{C})$. Then from Lemma 4, $\pi f_1^{'}(\mathcal{C}) = \pi(\mathcal{T}) = f_1^{'}\Phi(\mathcal{C})$. From Corollary 1, $\Phi(\mathcal{C})$ is an β-constacyclic code. Hence, by Theorem 2 part 1.a, we have $f_1^{'}\Phi(\mathcal{C})$ is cyclic code of length $2n$ over $\mathbb{Z}_4$, and thus $\pi\mathcal{T}$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.

2. (a) The proof follows from the procedure in 1(a) and Lemma 5.

(b) The proof follows from the procedure in 1(b) and Lemma 6.

(c) The proof follows from the procedure in 1(c), Corollary 1, Lemma 7, and Theorem 2.

□

In Theorem 2, we established a relationship between cyclic codes over $\mathbb{Z}_4$ and β-constacyclic codes over $\mathcal{S}$. Now, to further explore the relationship between quasi-cyclic codes over $\mathbb{Z}_4$ and β-constacyclic codes over $\mathcal{S}$, we first establish few results.

Lemma 8. *Let f_3 be the Gray map defined in (3.3), and $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \leq j \leq t$. Then*

$$\theta_2 f_3^{'}(\mathbf{s}) = f_3^{'}\phi_\beta(\mathbf{s}), \quad \textit{for all} \;\; \mathbf{s} \in \mathcal{S}^n$$

Proof. We provide the proof only for the case $k = 3$, as all other cases follow in a similar manner. Now consider,

$$\begin{aligned}
\theta_2 f_3^{'}(\mathbf{s}) =& \theta_2 f_3^{'}\left(\sum_{j=1}^{t} u_j a_{(0,j)}, \sum_{j=1}^{t} u_j a_{(1,j)}, \ldots, \sum_{j=1}^{t} u_j a_{(n-1,j)}\right)\\
=& \theta_2\left(a_{(0,1)} + 2\sum_{j=2}^{t} a_{(0,j)}, a_{(1,1)} + 2\sum_{j=2}^{t} a_{(1,j)}, \ldots, a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)},\right.\\
&\left. 3a_{(0,1)} + 2\sum_{j=2}^{t} a_{(0,j)}, 3a_{(1,1)} + 2\sum_{j=2}^{t} a_{(1,j)}, \ldots, 3a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)}\right)
\end{aligned}$$

$$=\left(a_{(n-1,1)}+2\sum_{j=2}^{t}a_{(n-1,j)},a_{(0,1)}+2\sum_{j=2}^{t}a_{(0,j)},\ldots,a_{(n-2,1)}+2\sum_{j=2}^{t}a_{(n-2,j)},\right.$$
$$\left.3a_{(n-1,1)}+2\sum_{j=2}^{t}a_{(n-1,j)},3a_{(0,1)}+2\sum_{j=2}^{t}a_{(0,j)},\ldots,3a_{(n-2,1)}+2\sum_{j=2}^{t}a_{(n-2,j)}\right) \tag{3.17}$$

On the other hand

$$\begin{aligned}
f_3^{'}\phi_\beta(\mathbf{s}) =& f_3^{'}\phi_\beta\left(\sum_{j=1}^{t}u_ja_{(0,j)},\sum_{j=1}^{t}u_ja_{(1,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-1,j)}\right)\\
=& f_3^{'}\left(a_{(n-1,1)}+\sum_{j=2}^{t}u_ja_{(n-1,j)},\sum_{j=1}^{t}u_ja_{(0,j)},\sum_{j=1}^{t}u_ja_{(1,j)},\ldots,\sum_{j=1}^{t}u_ja_{(n-2,j)}\right)\\
=& \left(a_{(n-1,1)}+2\sum_{j=2}^{t}a_{(n-1,j)},a_{(0,1)}+2\sum_{j=2}^{t}a_{(0,j)},\ldots,a_{(n-2,1)}+2\sum_{j=2}^{t}a_{(n-2,j)},\right.\\
&\left.3a_{(n-1,1)}+2\sum_{j=2}^{t}a_{(n-1,j)},3a_{(0,1)}+2\sum_{j=2}^{t}a_{(0,j)},\ldots,3a_{(n-2,1)}+2\sum_{j=2}^{t}a_{(n-2,j)}\right).
\end{aligned} \tag{3.18}$$

From the equation (3.17) and (3.18), we get $\theta_2 f_3^{'}(\mathbf{s})=f_3^{'}\phi_\beta(\mathbf{s})$. □

Lemma 9. *Let f_3 be the Gray map defined in (3.3), and let σ represent the cyclic shift operator. If $\mathcal{S}=\mathbb{Z}_4+\sum_{j=2}^{t}u_j\mathbb{Z}_4$ with $u_j^2=ku_j$, where $k\in\mathbb{Z}_4$ and for all $2\leq j\leq t$, then*

$$f_3^*(\sigma\mathbf{s})=\sigma^2 f_3^*(\mathbf{s}),\ \textit{for all } \mathbf{s}\in\mathcal{S}^n$$

Proof. This result follows in a similar way of Lemma 3 □

Lemma 10. *Let f_3 be the Gray map defined in definition 3.3. If $\mathcal{S}=\mathbb{Z}_4+\sum_{j=2}^{t}u_j\mathbb{Z}_4$ with $u_j^2=ku_j$, where $k\in\mathbb{Z}_4$ and for all $2\leq j\leq t$, then $f_3'\Phi=\pi f_3'$ where π is Nechaev's permutation and Φ is the map defined in Corollary 1.*

Proof. This result follows in a similar way of Lemma 4 □

Now, we proceed to prove a theorem that characterizes the relationship between quasi-cyclic codes over $\mathbb{Z}_4$ and β-constacyclic codes over $\mathcal{S}$.

Theorem 3. *Let $\mathcal{C}$ be a linear code over $\mathcal{S}=\mathbb{Z}_4+\sum_{j=1}^{t}u_j\mathbb{Z}_4$ with $u_i^2=ku_i$, where $k\in\mathbb{Z}_4$ and for all $2\leq j\leq t$. Then*

1. *If $\mathcal{C}$ is a β-constacyclic code, then $f_3^{'}(\mathcal{C})$ is a quasi cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*
2. *If $\mathcal{C}$ is a cyclic code of length n, then $f_3^*(\mathcal{C})$ is a two cyclic code of length $2n$ over $\mathbb{Z}_4$.*
3. *If $\mathcal{C}$ is a cyclic code of an odd length n and $\mathcal{T}=f_3'(\mathcal{C})$, then $\pi(\mathcal{T})$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*

Proof. 1. Since $\mathcal{C}$ is β-constacyclic code then $\phi_\beta(\mathbf{s}) \in \mathcal{C}$ for all $\mathbf{s} \in \mathcal{C}$. Then by using Lemma 8 we have $f_3^{'}\phi_\beta(s) = f_3^{'}(\mathcal{C}) = \theta_2 f_3^{'}(\mathcal{C})$. This proves $f_3^{'}(\mathcal{C})$ is a quasi cyclic code of length $2n$ with index 2.

2. Let $\mathcal{C}$ be a cyclic code of length n. Then $\sigma(\mathbf{c}) \in \mathcal{C}$ for all $\mathbf{c} \in \mathcal{C}$. From Lemma 9, we have $f_3^{*}\sigma(\mathcal{C}) = f_3^{*}(\mathcal{C}) = \sigma^2 f_3^{*}(\mathcal{C})$. Hence $f_3^{*}(\mathcal{C})$ is a two cyclic code of length $2n$ over $\mathbb{Z}_4$.

3. The proof follows from the Theorem 2 procedure in 1(c) along with Lemma 10.

□

Remark 1. The Gray maps defined in Definition 4 are not the only maps that yield the above results. We can also consider the following maps that satisfy the given conditions.

1. $\tilde{f}_1\left(\sum_{i=1}^{t} u_i a_i\right) = \left(\sum_{i=1}^{t} k_i a_i, \sum_{i=1}^{t} k_i^{'} a_i\right)$, where $k_1^{'} = k_1 + 2\sum_{i=2}^{t} k_i, k_i^{'} = 3k_i$, $k_1 = k_1^{'} + 2\sum_{i=2}^{t} k_i^{'}$, and $k_i = 3k_i^{'}$ for all $2 \le i \le t$.

2. $\tilde{f}_2\left(\sum_{i=1}^{t} u_i a_i\right) = \left(\sum_{i=1}^{t} k_i a_i, \sum_{i=1}^{t} k_i^{'} a_i\right)$, where $k_1^{'} = k_1 + 2\sum_{i=2}^{t} k_i$, $k_1 = k_1^{'} + 2\sum_{i=2}^{t} k_i^{'}$, and $k_i^{'} = k_i$ for all $2 \le i \le t$.

3. $\tilde{f}_3\left(\sum_{i=1}^{t} u_i a_i\right) = \left(\sum_{i=1}^{t} k_i a_i, \sum_{i=1}^{t} k_i^{'} a_i\right)$, where $k_1^{'} = k_1 + 2\sum_{i=2}^{t} k_i$, and $k_1 = k_1^{'} + 2\sum_{i=2}^{t} k_i^{'}$.
   (a) If $k \in \{0, 2\}$, then $k_i \in \mathbb{Z}_4$.
   (b) If $k \in U(\mathbb{Z}_4)$ then $k_i^{'} = 3k_i^{'}$, and $k_i = 3k_i$ for all $2 \le i \le t$.

However, note that these are the only possible maps that can be considered; apart from these, no other map can be used to derive the above results.

4. Gray Maps from $\mathcal{S}$ to $\mathbb{Z}_4^3$

In this section, we introduce two new Gray maps from $\mathcal{S}$ to $\mathbb{Z}_4^3$ and analyze their role in the algebraic structure of codes over these rings.
We now proceed to define these Gray maps.

Definition 6. Let g_i, for $i = 1, 2$, be linear maps defined from $\mathcal{S}$ to $\mathbb{Z}_4^3$ as follows:

$$g_1\left(\sum_{j=1}^{t} u_j a_j\right) = \left(a_1 + 2\sum_{j=2}^{t} a_j, 2\sum_{j=2}^{t} a_j, a_1\right). \tag{4.1}$$

$$g_2\left(\sum_{j=1}^{t} u_j a_j\right) = \left(3a_1 + 2\sum_{j=2}^{t} a_j, 3a_1 + 2\sum_{j=2}^{t} a_j, 3a_1 + 2\sum_{j=2}^{t} a_j\right). \tag{4.2}$$

Observe that g_1 generalizes the map from [11]. These maps extend to $\mathcal{S}^n$ in two ways, as defined below.

Definition 7. If $g : \mathcal{S} \to \mathbb{Z}_4^3$ is defined as $f(s) = (a, b, c)$, then g can be extended from $\mathcal{S}^n$ to $\mathbb{Z}_4^{3n}$ in the following two ways:

$$g'(s_0, s_1, \ldots, s_{n-1}) = (a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1}),$$

$$g^*(s_0, s_1, \ldots, s_{n-1}) = (a_0, b_0, c_0, a_1, b_1, c_1, \ldots, a_{n-1}, b_{n-1}, c_{n-1}).$$

The map g^* is called the permutation map of g'.

Now, we prove some lemmas related to g_1' and g_1^* that help for establish the relationship between quasi-cyclic codes of length $3n$ over $\mathbb{Z}_4$ and β-connsycyclic codes of length n over $\mathcal{S}$.

Lemma 11. *Let g_1 be the Gray map defined in (4.1), and $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \leq j \leq t$. Then*

$$\theta_3 g_1'(\mathbf{s}) = g_1'\phi_\beta(\mathbf{s}) \text{ for all } \mathbf{s} \in \mathcal{S}^n$$

Proof. We provide the proof for the case $u_i^2 = 3u_i$, while the proof for other cases follows in a similar manner.
Now consider,

$$\begin{aligned}
\theta_3 g_1'(\mathbf{s}) =& \theta_3 g_1'(s_0, s_1, s_2, \ldots, s_{n-1}) \\
=& \theta_3 g_1'\left(\sum_{j=1}^{t} u_j a_{(0,j)}, \sum_{j=1}^{t} u_j a_{(1,j)} \ldots, \sum_{j=1}^{t} u_j a_{(n-1,j)}\right) \\
=& \theta_3 \Bigg(a_{(0,1)} + 2\sum_{j=2}^{t} a_{(0,j)}, a_{(1,1)} + 2\sum_{j=2}^{t} a_{(1,j)}, \ldots, a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)}, \\
& 2\sum_{j=2}^{t} a_{(0,j)}, 2\sum_{j=2}^{t} a_{(1,j)}, \ldots, 2\sum_{j=2}^{t} a_{(n-1,j)}, a_{(0,1)}, a_{(1,1)}, \ldots, a_{(n-1,1)}\Bigg) \\
=& \Bigg(a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)}, a_{(0,1)} + 2\sum_{j=2}^{t} a_{(0,j)}, \ldots, a_{(n-2,1)} + 2\sum_{j=2}^{t} a_{(n-2,j)}, \\
& 2\sum_{j=2}^{t} a_{(n-1,j)}, 2\sum_{j=2}^{t} a_{(0,j)}, \ldots, 2\sum_{j=2}^{t} a_{(n-2,j)}, a_{(n-1,1)}, a_{(0,1)}, \ldots, a_{(n-2,1)}\Bigg).
\end{aligned}$$

On the other hand

$$
\begin{aligned}
g_1'\phi_\beta(\mathbf{s}) =& g_1'\phi_\beta(s_0, s_1, \ldots, s_{n-1})\\
=& g_1'(\beta s_{n-1}, s_0, s_1, \ldots, s_{n-2})\\
=& \Bigg(a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)}, a_{(0,1)} + 2\sum_{j=2}^{t} a_{(0,j)}, \ldots, a_{(n-2,1)} + 2\sum_{j=2}^{t} a_{(n-2,j)},\\
& 2\sum_{j=2}^{t} a_{(n-1,j)}, 2\sum_{j=2}^{t} a_{(0,j)}, \ldots, 2\sum_{j=2}^{t} a_{(n-2,j)}, a_{(n-1,1)}, a_{(0,1)}, \ldots, a_{(n-2,1)} \Bigg)
\end{aligned}
$$

Hence, $\theta_3 g_1'(\mathbf{s}) = g_1'\phi_\beta(\mathbf{s})$ for all $\mathbf{s} \in \mathcal{S}^n$. $\square$

Lemma 12. *Let g_1 be the Gray map defined in (4.1), and let σ represent the cyclic shift operator. If $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \le j \le t$, then*

$$
g_1^*\sigma(\mathbf{s}) = \sigma^3 g_1^*(\mathbf{s}), \ \textit{for all } \mathbf{s} \in \mathcal{S}^n
$$

Proof. We provide the proof for the case $u_i^2 = 3u_i$, while the proof for other cases follows in a similar manner.
Now consider,

$$
\begin{aligned}
g_1^*\sigma(\mathbf{s}) =& g_1^*\sigma(s_0, s_1, \ldots, s_{n-1})\\
=& g_1^*\sigma\left(\sum_{j=1}^{t} u_j a_{(0,j)}, \sum_{j=1}^{t} u_j a_{(1,j)}, \ldots, \sum_{j=1}^{t} u_j a_{(n-1,j)}\right)\\
=& g_1^*\left(\sum_{j=1}^{t} u_j a_{(n-1,j)}, \sum_{j=1}^{t} u_j a_{(0,j)}, \ldots, \sum_{j=1}^{t} u_j a_{(n-2,j)}\right)\\
=& \Bigg(a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)}, 2\sum_{j=2}^{t} a_{(n-1,j)}, a_{(n-1,1)}, a_{(0,1)}\\
& +2\sum_{j=2}^{t} a_{(0,j)}, 2\sum_{j=2}^{t} a_{(0,j)}, a_{(0,1)}, \ldots, a_{(n-2,1)} + 2\sum_{j=2}^{t} a_{(n-2,j)}, 2\sum_{j=2}^{t} a_{(n-2,j)}, a_{(n-2,1)} \Bigg).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\sigma^3 g_1^*(\mathbf{s}) =& \sigma^3 \Bigg(a_{(0,1)} + 2\sum_{j=2}^{t} a_{(0,j)}, 2\sum_{j=2}^{t} a_{(0,j)}, a_{(0,1)}, a_{(1,1)} + 2\sum_{j=2}^{t} a_{(1,j)}, 2\sum_{j=2}^{t} a_{(1,j)}, a_{(1,1)},\\
& \ldots, a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)}, 2\sum_{j=2}^{t} a_{(n-1,j)}, a_{(n-1,1)} \Bigg)\\
=& \Bigg(a_{(n-1,1)} + 2\sum_{j=2}^{t} a_{(n-1,j)}, 2\sum_{j=2}^{t} a_{(n-1,j)}, a_{(n-1,1)}, a_{(0,1)}\\
& +2\sum_{j=2}^{t} a_{(0,j)}, 2\sum_{j=2}^{t} a_{(0,j)}, a_{(0,1)}, \ldots, a_{(n-2,1)} + 2\sum_{j=2}^{t} a_{(n-2,j)}, 2\sum_{j=2}^{t} a_{(n-2,j)}, a_{(n-2,1)} \Bigg).
\end{aligned}
$$

Hence, $g_1^*\sigma(\mathbf{s}) = \sigma^3 g_1^*(\mathbf{s})$, for all $\mathbf{s} \in \mathcal{S}^n$. $\square$

Lemma 13. *Let g_1 be the Gray map defined in definition 4.1. If $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \leq j \leq t$, then*

$$g_1'\Phi(\mathbf{s}) = \pi g_1'(\mathbf{s}) \quad \textit{for all} \quad \mathbf{s} \in \mathcal{S}^n$$

where π is Nechaev's permutation and Φ is the map defined in Corollary 1.

Proof. Follows in a similar way of Lemma 4 □

We now proceed to prove the relationship between quasi-cyclic codes and β-constacyclic codes.

Theorem 4. *Let $\mathcal{C}$ be a linear code over $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=1}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \leq j \leq t$. Then*

1. *If $\mathcal{C}$ is a β-constacyclic code, then $g_1'(\mathcal{C})$ is a quasi cyclic code of length $3n$ over $\mathbb{Z}_4$.*
2. *If $\mathcal{C}$ is a cyclic code of length n, then $g_1^*(\mathcal{C})$ is a three cyclic code of length $3n$ over $\mathbb{Z}_4$.*
3. *If $\mathcal{C}$ is a cyclic code of an odd length n and $\mathcal{T}=g_1'(\mathcal{C})$, then $\pi(\mathcal{T})$ is a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$.*

Proof. Proof follows in a similar way of Theorem 2 □

Now, we state some results related to g_2' and g_2^*, which can be proved similarly to the results in Section 3 or by direct computations. Using these results, we establish the relationship between cyclic and β-constacyclic codes.

Lemma 14. *Let g_2 be the Gray map defined in (4.2), and let σ represent the cyclic shift operator. If $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \leq j \leq t$, then*

$$\sigma g_2'(\mathbf{s}) = g_2'\phi_\beta(\mathbf{s}), \quad \textit{for all } \mathbf{s} \in \mathcal{S}^n.$$

Proof. Proof follows in a similar way of Lemma 2 □

Lemma 15. *Let g_2 be the Gray map defined in (4.2), and let σ represent the cyclic shift operator. If $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \leq j \leq t$, then*

$$g_2^*(\sigma\mathbf{s}) = \sigma^3 g_2^*(\mathbf{s}), \textit{ for all } \mathbf{s} \in \mathcal{S}^n$$

Proof. Proof follows in a similar way of Lemma 12. □

Lemma 16. *Let g_2 be the Gray map defined in definition 4.2. If $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in \mathbb{Z}_4$ and for all $2 \leq j \leq t$, then*

$$g_2'\Phi(\mathbf{x}) = \pi g_2'(\mathbf{s}) \quad \textit{for all } \mathbf{s} \in \mathcal{S}^n$$

where π is Nechaev's permutation and Φ is the map defined in Corollary 1.

Proof. Follows in a similar way of Lemma 4. □

Theorem 5. *Let $\mathcal{C}$ be a linear code over $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=1}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$ where $k \in \mathbb{Z}_4$. Then*

1. *If $\mathcal{C}$ is a β-constacyclic code, then $g_2'(\mathcal{C})$ is a cyclic code of length $3n$ over $\mathbb{Z}_4$.*
2. *If $\mathcal{C}$ is a cyclic code of length n, then $g_2^*(\mathcal{C})$ is a three cyclic code of length $3n$ over $\mathbb{Z}_4$.*
3. *If $\mathcal{C}$ is a cyclic code of an odd length n and $\mathcal{T}=g_2'(\mathcal{C})$, then $\pi(\mathcal{T})$ is a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$.*

Proof. Proof follows in a similar way of Theorem 2. □

Remark 2. The Gray maps defined in Definition 6 are not the only maps that yield the above results. We can also consider the following maps that satisfy the given conditions.

1. $\tilde{g_1}\ \left(\sum_{i=1}^{t} u_i a_i\right) = \left(\sum_{i=1}^{t} k_i a_i, \sum_{i=1}^{t} k_i^{'} a_i, \sum_{i=1}^{t} k_i^{''} a_i\right)$, where $k_1 = k_1 + 2\sum_{i=2}^{t} k_i$, $k_1^{'} = k_1^{'} + 2\sum_{i=2}^{t} k_i^{'}$, and $k_1^{''} = k_1^{''} + 2\sum_{i=2}^{t} k_i^{''}$.
   (a) If $k \in \{0, 2\}$, then $k_i \in \mathbb{Z}_4$.
   (b) If $k \in \{1, 3\}$, then $k_i = 3k_i$ and $k_i^{'} = 3k_i^{'}$ for all $2 \leq i \leq t$, $k_i^{''} = 3k_i^{''}$ for all $2 \leq i \leq t$.

2. $\tilde{g_2}\ \left(\sum_{i=1}^{t} u_i a_i\right) = (\sum_{i=1}^{t} k_i a_i, \sum_{i=1}^{t} k_i^{'} a_i, \sum_{i=1}^{t} k_i^{''} a_i)$, where $k_1 = k_1^{'} + 2\sum_{i=2}^{t} k_i^{'}$, $k_1^{'} = k_1^{''} + 2\sum_{i=2}^{t} k_i^{''}$, $k_1^{''} = k_1 + 2\sum_{i=2}^{t} k_i$, and $k_i = k_i^{'} = k_i^{''}$ for all $2 \leq i \leq t$. (a) If $k \in \{0, 2\}$, then $k_i^{''} = k_i^{'} = k_i$ for all $2 \leq i \leq t$.
   (b) If $k \in U(\mathbb{Z}_4)$, then $k_i^{''} = 3k_i$ and $k_i = 3k_i^{'}$ for all $2 \leq i \leq t$, $k_i^{'} = 3k_i^{''}$ for all $2 \leq i \leq t$.

We should note that these are the only possible maps that can be considered. There are no other maps can be used to obtain the above results.

Remark 3. As in [11], the ring $\mathcal{S}_2 = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$ is considered, and Gray maps from $\mathcal{S}_2$ to $\mathbb{Z}_4^2$ and $\mathbb{Z}_4^3$ are defined and studied. Similarly, we can define and study Gray maps from $\mathcal{S}$ to $\mathbb{Z}_4^4, \mathbb{Z}_4^5, \ldots, \mathbb{Z}_4^{t-1}$. For instance, one can easily verify that the following maps from $\mathcal{S}$ to $\mathbb{Z}_4^{t-1}$ satisfies similar results as in Sections 3 and 4.

1. Let $F : \mathcal{S} \to \mathbb{Z}_4^{t-1}$ be a map defined as
   $F\ \left(\sum_{i=1}^{t} u_i a_i\right) = \left(\sum_{i=1}^{t} k_i a_i, \sum_{i=1}^{t} k_i^{'} a_i, \sum_{i=1}^{t} k_i^{''} a_i, \ldots, \sum_{i=1}^{t} k_i^{t-1} a_i\right)$, where
   $k_1 = k_1^{'} + 2\sum_{i=2}^{t} k_i^{'}$, $k_1^{'} = k_1^{''} + 2\sum_{i=2}^{t} k_i^{''}$, $k_1^{''} = k_1^{'''} + 2\sum_{i=2}^{t} k_i^{'''}$, $\ldots$, $k_1^{t-1} = k_1 + 2\sum_{i=2}^{t} k_i$.
   (a) If $k \in \{0, 2\}$, then $k_i^{t-1} = k_i^{t-2} = \ldots = k_i^{'} = k_i$ for all $2 \leq i \leq t$.
   (b) If $k \in \{1, 3\}$, then $k_i = 3k_i^{'}$, $k_i^{'} = 3k_i^{''}$, $\ldots$, $k_i^{t-1} = 3k_i$ for all $2 \leq i \leq t$.

2. Let $G : \mathcal{S} \to \mathbb{Z}_4^{t-1}$ be a map defined as
$G\left(\sum_{i=1}^{t} u_i a_i\right) = \left(\sum_{i=1}^{t} k_i a_i, \sum_{i=1}^{t} k_i^{'} a_i, \sum_{i=1}^{t} k_i^{''} a_i\right)$, where
$k_1 = k_1 + 2\sum_{i=2}^{t} k_i$, $k_1^{'} = k_1^{'} + 2\sum_{i=2}^{t} k_i^{'}$, $k_1^{''} = k_1^{''} + 2\sum_{i=2}^{t} k_i^{''}$, $\ldots$,
$k_1^{t-1} = k_1^{t-1} + 2\sum_{i=2}^{t} k_i^{t-1}$.
(a) If $k \in \{0, 2\}$, then $k_i \in \mathbb{Z}_4$.
(b) If $k \in U(\mathbb{Z}_4)$, then $k_i = 3k_i$, $k_i^{'} = 3k_i^{'}$, $k_i^{''} = 3k_i^{''}$, $\ldots$, $k_i^{t-1} = 3k_i^{t-1}$ for all $2 \le i \le t$.

With the help of the Gray maps F and G, we establish the relationship between β-constacyclic codes and cyclic and quasi-cyclic codes, respectively.

5. Structure of β-constacyclic code

This section is devoted to studying the structure of cyclic and β-constacyclic codes over $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in U(\mathbb{Z}_4)$ and $2 \le j \le t$.
Define $e_1 = 1 - \sum_{j=2}^{t} ku_j$ and $e_j = ku_j$, for all $2 \le j \le t$. These elements satisfy the following properties:

$$e_i e_j = 0 \quad \text{for } i \neq j, \quad e_j^2 = e_j, \quad \text{and} \quad \sum_{j=1}^{t} e_j = 1. \tag{5.1}$$

Consequently, every element in $\mathcal{S}$ has a unique representation as $\sum_{j=1}^{t} r_j e_j$, where $r_1 = a_1$ and $r_j = 3ka_1 + ka_j$ for all $2 \le j \le t$.
Let $\mathcal{A}_i$, for $i = 1, 2, \ldots, n$, be a sequence of nonempty sets. We define

$$\bigoplus_{i=1}^{n} \mathcal{A}_i = \left\{\sum_{i=1}^{n} a_i \mid a_i \in \mathcal{A}_i\right\} \text{ and } \bigotimes_{i=1}^{n} \mathcal{A}_i = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathcal{A}_i\}.$$

Let $\mathcal{C}$ be a linear code over $\mathcal{S}$, and define $\mathcal{C}_i = \left\{a_i \mid \sum_{j=1}^{t} a_j e_j \in \mathcal{C}\right\}$, for all $i = 1, 2, \ldots, t$. Then, we have $\mathcal{C} = \bigoplus_{i=1}^{t} \mathcal{C}_i$. Moreover, if $\mathcal{C}$ is linear over $\mathcal{S}$, then each $\mathcal{C}_i$ is linear over $\mathbb{Z}_4$.
With the notations defined above, we now examine the structure of β-constacyclic codes in the following result.

Theorem 6. *Let $\mathcal{C}$ be a linear code over $\mathcal{S}$ then $\mathcal{C}$ is cyclic code of length n over $\mathcal{S}$ if and only if $\mathcal{C}_i$ are cyclic codes over $\mathbb{Z}_4$ for all $i = 1, 2, \ldots, t$.*

Proof. Let $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ where $c_i = a_{(1,i)}e_1 + a_{(2,i)}e_2 + \ldots + a_{(t,i)}e_t$ and $\mathcal{C}$ be a cyclic code over $\mathcal{S}$ then $\sigma(c) = \sigma(a_1e_1) + \sigma(a_2e_2) + \ldots + \sigma(a_te_t) \in \mathcal{C}$. This implies that $\mathcal{C}_i$ are cyclic codes over $\mathbb{Z}_4$ for all $i = 1, 2, \ldots t$.

Conversely, suppose $\mathcal{C}_1$ be a cyclic code. Therefore $\sigma(a_1) \in \mathcal{C}_1$. This implies that $\sigma(a_1)e_1 + a_2e_2 + a_3e_3 + \ldots + a_te_t \in \mathcal{C}$ and hence $e_1(\sigma(a_1)e_1 + a_2e_2 + a_3e_3 + \ldots + a_te_t) = \sigma(a_1)e_1 \in \mathcal{C}$ for some $a_i \in \mathcal{C}_i$ for $i = 2, 3, \ldots, t$. Similarly, $\sigma(a_i)e_i \in \mathcal{C}$ for $i = 2, 3, \ldots, t$. By using the linearity, we get $\sigma(a_1e_1 + \sigma(a_2e_2 + \ldots + \sigma(a_te_t) = \sigma(\mathcal{C}) \in \mathcal{C}$. Hence, $\mathcal{C}$ is cyclic code over $\mathcal{S}$. □

Lemma 17. *[3] Let $\mathcal{C}$ be a cyclic code of length n over $\mathbb{Z}_4$. If n is odd then $\mathbb{Z}_4[x]/(x^n - 1)$ is a principal ideal ring and $\mathcal{C} = (p(x), 2q(x)) = (p(x) + 2q(x))$ where $p(x)$ and $q(x)$ generate cyclic codes with $q(x) \mid p(x) \mid (x^n - 1) \pmod 4$.*

Theorem 7. *Let $\mathcal{C}$ be a cyclic code of odd length n. Then there exist $p(x)$ such that $\mathcal{C} = \langle p(x) \rangle$.*

Proof. Let $\mathcal{C}$ be a cyclic code. By Theorem 6, $\mathcal{C}_i$ is cyclic for $i = 1, 2, \ldots, t$. By Lemma 17, we have $\mathcal{C}_i = \langle p_i(x) \rangle$. Hence, for any element in $e_i\mathcal{C}_i$, we obtain $e_ia_i(x)p_i(x) \in e_i\mathcal{C}_i$, for $a_i(x) \in \mathbb{Z}_4[x]$. Using the representation of $\mathcal{C}$ in $\mathcal{S}$, we get

$$\sum_{i=1}^{t} e_ia_i(x)p_i(x) \in \mathcal{C}.$$

Multiplying both sides by e_i, we obtain $\langle e_ip_i(x) \rangle \subseteq \mathcal{C}$. Thus, the generator of $\mathcal{C}$ is given by $p(x) = \displaystyle\sum_{i=1}^{t} e_ip_i(x)$. □

Theorem 8. *Let $\mathcal{C}$ be a linear code. Then $\mathcal{C}$ is a β-constacyclic code over $\mathcal{S} = \mathbb{Z}_4 + \sum_{j=2}^{t} u_j\mathbb{Z}_4$ with $u_j^2 = ku_j$, where $k \in U(\mathbb{Z}_4)$ and $2 \leq j \leq t$ if and only if $\mathcal{C}_1$ is cyclic and $\mathcal{C}_i$ are negacyclic code for all $i = 2, 3, \ldots, t$.*

Proof. Let $\mathcal{C}$ be a β-constacyclic code over $\mathcal{S}$, and let $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ where $c_i = a_{(1,i)}e_1 + a_{(2,i)}e_2 + \ldots + a_{(t,i)}e_t$ and $a_i = (a_{(i,0)}, a_{(i,1)}, a_{(i,2)}, \ldots, a_{(i,n-1)}) \in \mathcal{C}_i$ for all $i = 1, 2, \ldots, t$ then $\sum_{i=1}^{t} a_ie_i \in \mathcal{S}$. Since $\mathcal{C}$ is β-constacyclic code

$$\begin{aligned}
\phi_\beta(c_0, c_1, \ldots, c_{n-1}) =& (\beta c_{n-1}, c_0, \ldots, c_{n-2}) \\
=& ((1 + 2\sum_{j=2}^{t} u_j)(a_{(1,n-1)}e_1 + \sum_{i=2}^{t} a_{(i,n-1)}e_i), \sum_{i=1}^{t} a_{(i,0)}e_i, \sum_{i=1}^{t} a_{(i,1)}e_i, \ldots, \\
& \sum_{i=1}^{t} a_{(i,n-2)}e_i) \\
=& (a_{(1,n-1)}(1 - (3k-2)\sum_{j=2}^{t} u_j) + a_{(2,n-1)}3ku_2 + \ldots + a_{(t,n-1)}3ku_t, \\
& \sum_{i=1}^{t} a_{(i,0)}e_i, \sum_{i=1}^{t} a_{(i,1)}e_i, \ldots, \sum_{i=1}^{t} a_{(i,n-2)}e_i) \\
=& (a_{(1,n-1)}e_1 - \sum_{i=2}^{t} a_{(i,n-1)}e_i, \sum_{i=1}^{t} a_{(i,0)}e_i, \sum_{i=1}^{t} a_{(i,1)}e_i, \ldots, \sum_{i=1}^{t} a_{(i,n-2)}e_i).
\end{aligned}$$

$$=(a_{(1,n-1)}, a_{(1,0)}, \ldots, a_{(1,n-2)})e_1 + (-a_{(2,n-1)}, a_{(2,0)}, \ldots, a_{(2,n-2)})e_2 + \ldots + (-a_{(t,n-1)}, a_{(t,0)}, \ldots, a_{(t,n-2)})e_t$$

This implies, $\phi_{-1}(a_{(j)}) \in \mathcal{C}_j$ for all $j = 2, 3, \ldots, t$ and $\sigma(a_1) \in \mathcal{C}_1$.
Hence, $\mathcal{C}_1$ is cyclic and $\mathcal{C}_j$ are negacyclic code for all $i = 2, 3, \ldots, t$.
Conversely, assume that $\mathcal{C}_1$ is cyclic and $\mathcal{C}_j$ are negacyclic code for all $j = 2, 3, \ldots, t$. Let $(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, where $c_i = a_{(1,i)}e_1 + a_{(2,i)}e_2 + \ldots + a_{(t,i)}e_t$. Since $\mathcal{C}_1$ is cyclic and $\mathcal{C}_j$ are negacyclic $\phi_{-1}(a_{(j)}) \in \mathcal{C}_j$ for all $j = 2, 3, \ldots, t$ and $\sigma(a_1) \in \mathcal{C}_1$, we have $\sigma(a_1e_1 + \sum_{j=2}^{t} \phi_{-1}(a_j)e_j) \in \mathcal{C}$. That is $(\beta c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathcal{C}$. Hence $\mathcal{C}$ is β-constacyclic code.

□

In the following results, we derive the generating set for β-constacyclic codes.

Theorem 9. *Let n be an odd integer. Then the map $\tau : \mathcal{S}[x]/\langle x^n - 1\rangle \to \mathcal{S}[x]/\langle x^n - \beta\rangle$ defined by $\tau(p(x)) = p(\beta x)$ is a ring isomorphism.*

Proof. Let $p(x) = h(x)$ in $\mathcal{S}[x]/\langle x^n - 1\rangle$.
Then, we have $p(x) \equiv h(x) \pmod{(x^n - 1)}$. Replacing x with βx in the equation above gives

$$p(\beta x) - h(\beta x) \equiv 0 \pmod{(x^n\beta^n - 1)}.$$

This implies that

$$p(\beta x) - h(\beta x) \equiv 0 \pmod{\beta^n(x^n - \beta)}.$$

Since $\beta^n = \beta$ for an odd integer n, it follows that

$$p(\beta x) = h(\beta x) \text{ in } \mathcal{S}[x]/\langle x^n - \beta\rangle.$$

Thus, τ is an injective and well-defined map. Moreover, $\mathcal{S}[x]/\langle x^n - 1\rangle$ and $\mathcal{S}[x]/\langle x^n - \beta\rangle$ are finite rings with the same number of elements. Since τ is injective, it must also be surjective. It is easy to see that τ is a ring homomorphism. Hence, τ is a ring isomorphism. □

Corollary 2. *Let $\mathcal{C}$ be a linear code of odd length n over $\mathcal{S}$. Then $\mathcal{C}$ is a cyclic code if and only if $\tau(\mathcal{C})$ is an β-constacyclic code over $\mathcal{S}$.*

Theorem 10. *Let $\mathcal{C}$ be a β-constacyclic code over $\mathcal{S}$ then there exist a polynomial $p(x)$ such that $\mathcal{C} = \langle p(x)\rangle$.*

Remark 4. Let $a_1(x) + \sum_{i=2}^{t} u_i a_i(x) = \sum_{i=1}^{t} e_i p_i(x)$. Then $a_1(x) = p_1(x), a_i(x) = 3kp_1(x) + kp_i(x)$ for all $i = 2, 3, \ldots, t$.

Theorem 11. *Let $f_1^{'}$ be the gray map defined in the previous sections. Consider $\mathcal{C} = \langle p_1(x) + (\sum_{i=2}^{t} 3kp_1(x) + kp_i(x))u_i\rangle$ be a β-constacyclic code with $u_j^2 = ku_j$, where $k \in U(\mathbb{Z}_4)$. Then $f_1^{'}(\mathcal{C})$ is a cyclic code over $\mathbb{Z}_4$ and $f_1^{'}(\mathcal{C}) = \langle p(x)\rangle$.*

1. *If $t \equiv 0 \pmod 4$, then $p(x) = (3k + 1)p_1(x) + x^n(k + 3)p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x)$.*

2. *If* $t \equiv 1 \pmod 4$*, then* $p(x) = 2p_1(x) + x^n 2p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x)$.
3. *If* $t \equiv 2 \pmod 4$*, then* $p(x) = (k+1)p_1(x) + x^n(3k+3)p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x)$.
4. *If* $t \equiv 3 \pmod 4$*, then* $p(x) = (2k+1)p_1(x) + x^n(2k+3)p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x)$.

Proof. For $r(x) \in \mathcal{C}$ there exists $h_i(x) \in \mathbb{Z}_4[x]$ such that

$$\begin{aligned} r(x) =& p(x)h(x) \\ =& \left(h_1(x)p_1(x) + \sum_{i=2}^{t} (3kh_1(x)p_1(x) + kh_i(x)p_i(x) \right) u_i. \end{aligned}$$

Now consider the following cases:

1. If $t \equiv 0 \pmod 4$, then

$$\begin{aligned} f_1^{'}(r(x)) =& \left((3k+1)p_1(x)h_1(x) + 3k \sum_{i=2}^{t} p_i(x)h_i(x), (k+3)p_1(x)h_1(x) \right. \\ & \left. + k \sum_{i=2}^{t} p_i(x)h_i(x) \right) \\ =& h_1(x)((3k+1)p_1(x), (k+3)p_1(x)) + \sum_{i=2}^{t} h_i(x)(3kp_i(x), kp_i(x)). \end{aligned}$$

Hence, $f_1^{'}(r(x)) \in \frac{\mathbb{Z}_4}{(x^n-1)} \times \frac{\mathbb{Z}_4}{(x^n-1)}$, using the fact $a, b \in \frac{\mathbb{Z}_4}{(x^n-1)} \times \frac{\mathbb{Z}_4}{(x^n-1)}$ implies $a + x^n b \in \frac{\mathbb{Z}_4}{(x^{2n}-1)}$, we have that $f_1^{'}(\mathcal{C}) = \langle (3k+1)p_1(x) + x^n(k+3)p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x) \rangle$ is a cyclic code over $\frac{\mathbb{Z}_4}{(x^{2n}-1)}$.

2. If $t \equiv 1 \pmod 4$, then

$$\begin{aligned} f_1^{'}(r(x)) =& \left(2p_1(x)h_1(x) + 3k \sum_{i=2}^{t} p_i(x)h_i(x), 2p_1(x)h_1(x) + k \sum_{i=2}^{t} p_i(x)h_i(x) \right) \\ =& h_1(x)2p_1(x), 2p_1(x)) + \sum_{i=2}^{t} h_i(x)(3kp_i(x), kp_i(x)). \end{aligned}$$

This gives, $f_1^{'}(\mathcal{C}) = \langle 2p_1(x) + x^n 2p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x) \rangle$ is a cyclic code over $\frac{\mathbb{Z}_4}{(x^{2n}-1)}$.

3. If $t \equiv 2 \pmod 4$, then

$$\begin{aligned} f_1^{'}(r(x)) =& \left((k+1)p_1(x)h_1(x) + 3k \sum_{i=2}^{t} p_i(x)h_i(x), (3k+3)p_1(x)h_1(x) \right. \\ & \left. + k \sum_{i=2}^{t} p_i(x)h_i(x) \right) \\ =& h_1(x)((k+1)p_1(x), (3k+3)p_1(x)) + \sum_{i=2}^{t} h_i(x)(3kp_i(x), kp_i(x)). \end{aligned}$$

This gives, $f_1^{'}(\mathcal{C}) = \langle (k+1)p_1(x) + x^n(3k+3)p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x)\rangle$ is a cyclic code over $\frac{\mathbb{Z}_4}{(x^{2n}-1)}$.

4. If $t \equiv 3 \pmod 4$, then

$$\begin{aligned} f_1^{'}(r(x)) =& \Bigg((2k+1)p_1(x)h_1(x) + 3k\sum_{i=2}^{t} p_i(x)h_i(x), (2k+3)p_1(x)h_1(x) \\ & +k\sum_{i=2}^{t} p_i(x)h_i(x)\Bigg) \\ =& h_1(x)((2k+1)p_1(x), (2k+3)p_1(x)) + \sum_{i=2}^{t} h_i(x)(3kp_i(x), kp_i(x)). \end{aligned}$$

This gives, $f_1^{'}(\mathcal{C}) = \langle (2k+1)p_1(x) + x^n(2k+3)p_1(x), 3kp_2(x) + x^n kp_2(x), 3kp_3(x) + x^n kp_3(x), \ldots, 3kp_t(x) + x^n kp_t(x)\rangle$ is a cyclic code over $\frac{\mathbb{Z}_4}{(x^{2n}-1)}$.

Hence, we conclude the proof. □

Theorem 12. *Let $f_3^{'}$ be the gray map defined in the previous sections. Consider $\mathcal{C} = \langle p_1(x) + (\sum_{i=2}^{t} 3kp_1(x) + kp_i(x))u_i\rangle$ be a β-constacyclic code with $u_j^2 = ku_j$, where $k \in U(\mathbb{Z}_4)$. Then $f_3^{'}(\mathcal{C})$ is a quasi cyclic code over $\mathbb{Z}_4$ and $f_3^{'}(\mathcal{C}) = \langle p(x)\rangle$.*

1. *If $t \equiv 0, 2 \pmod 4$, then $p(x) = (2k+1)p_1(x) + x^n(2k+3)p_1(x), 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \ldots, 2kp_t(x) + x^n 2kp_t(x)$.*
2. *If $t \equiv 1, 3 \pmod 4$, then $p(x) = p_1(x) + x^n 3p_1(x), 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \ldots, 2kp_t(x) + x^n 2kp_t(x)$.*

Proof. For $r(x) \in \mathcal{C}$, there exists $h_i(x) \in \mathbb{Z}_4[x]$ such that

$$\begin{aligned} r(x) =& p(x)h(x) \\ =& \left(h_1(x)p_1(x) + \sum_{i=2}^{t} (3kh_1(x)p_1(x) + kh_i(x)p_i(x)\right) u_i. \end{aligned}$$

Now consider the following cases:

1. If $t \equiv 0, 2 \pmod 4$, then

$$\begin{aligned} f_3^{'}(r(x)) =& \Bigg((2k+1)p_1(x)h_1(x) + 2k\sum_{i=2}^{t} p_i(x)h_i(x), (2k+3)p_1(x)h_1(x) \\ & +2k\sum_{i=2}^{t} p_i(x)h_i(x)\Bigg) \\ =& h_1(x)((2k+1)p_1(x), (2k+3)p_1(x)) + \sum_{i=2}^{t} h_i(x)(2kp_i(x), 2kp_i(x)). \end{aligned}$$

This gives, $f_1^{'}(\mathcal{C}) = \langle (2k+1)p_1(x) + x^n(2k+3)p_1(x), 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \ldots, 2kp_t(x) + x^n 2kp_t(x)\rangle$ is a cyclic code over $\frac{\mathbb{Z}_4}{(x^{2n}-1)}$.

2. If $t \equiv 1,3 \mod 4$, then

$$f_1^{'}(r(x)) = \left(p_1(x)h_1(x) + 2k\sum_{i=2}^{t} p_i(x)h_i(x), 3p_1(x)h_1(x) + 2k\sum_{i=2}^{t} p_i(x)h_i(x)\right)$$
$$= h_1(x)(p_1(x), 3p_1(x)) + \sum_{i=2}^{t} h_i(x)(2kp_i(x), 2kp_i(x)).$$

This gives, $f_1^{'}(\mathcal{C}) = \langle p_1(x) + x^n 3p_1(x), 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \dots, 2kp_t(x) + x^n 2kp_t(x)\rangle$ is a cyclic code over $\frac{\mathbb{Z}_4}{(x^{2n}-1)}$.

□

Theorem 13. *Let g_1' be the gray map defined in the previous sections. Consider $\mathcal{C} = \langle p_1(x)+(\sum_{i=2}^{t} 3kp_1(x)+kp_i(x))u_i\rangle$ be a β-constacyclic code with $u_j^2 = ku_j$, where $k \in U(\mathbb{Z}_4)$. Then $g_1^{'}(\mathcal{C})$ is a quasi cyclic code over $\mathbb{Z}_4$ and $g_1^{'}(\mathcal{C}) = \langle p(x)\rangle$.*

1. *If $t \equiv 0,2 \pmod 4$, then $p(x) = (2k+1)p_1(x) + x^n 2kp_1(x) + x^{2n}p_1, 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \dots, 2kp_t(x) + x^n 2kp_t(x)$.*
2. *If $t \equiv 1,3 \pmod 4$, then $p(x) = p_1(x) + x^{2n}p_1(x), 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \dots, 2kp_t(x) + x^n 2kp_t(x)$.*

Theorem 14. *Let g_2' be the gray map defined in the previous sections. Consider $\mathcal{C} = \langle p_1(x)+(\sum_{i=2}^{t} 3kp_1(x)+kp_i(x))u_i\rangle$ be a β-constacyclic code with $u_j^2 = ku_j$, where $k \in U(\mathbb{Z}_4)$. Then $g_2^{'}(\mathcal{C})$ is a cyclic code over $\mathbb{Z}_4$ and $g_2^{'}(\mathcal{C}) = \langle p(x)\rangle$.*

1. *If $t \equiv 0,2 \pmod 4$, then $p(x) = (2k+3)p_1(x) + x^n(2k+3)p_1(x) + x^{2n}(2k+3)p_1, 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \dots, 2kp_t(x) + x^n 2kp_t(x)$.*
2. *If $t \equiv 1,3 \pmod 4$, then $p(x) = 3p_1(x) + x^n 3p_1 + x^{2n}3p_1(x), 2kp_2(x) + x^n 2kp_2(x), 2kp_3(x) + x^n 2kp_3(x), \dots, 2kp_t(x) + x^n 2kp_t(x)$.*

Remark 5. If $k \in \{0,2\}$, then the ring $\mathcal{S}$ contains only trivial idempotent elements. Hence, there does not exist any subset in $\mathcal{S}$ satisfying the conditions in (5.1).

6. Conclusion

In this paper, we have generalized previous results on constacyclic codes by investigating $(1 + 2u_2 + 2u_3 + \cdots + 2u_t)$-constacyclic codes over the semi-local ring $\mathcal{S} = \mathbb{Z}_4 + u_2\mathbb{Z}_4 + \cdots + u_t\mathbb{Z}_4$. We have established their structural properties and demonstrated how these codes can be transformed into cyclic and quasi-cyclic codes over $\mathbb{Z}_4$ using newly defined Gray maps.
Furthermore, we characterized a generating set for these constacyclic codes when the code length is odd. Our findings contribute to the study of constacyclic codes over extended non-chain rings and expand the database of $\mathbb{Z}_4$ codes. Future work may explore different β-constacyclic codes over $\mathcal{S}$, such as those where $\beta = 1 + 2u_j$ for

some $2 \leq j \leq t$ or $\beta = 1 + 2\sum_{i=2}^{l} u_i$ for $2 \leq l < t$, as mentioned in Lemma 1. Both of these choices for β are units for all $k \in \mathbb{Z}_4$.
Additionally, one can investigate the generating set for the case where $\beta = 1 + 2\sum_{j=2}^{t} u_j$ and $k \in \{0, 2\}$, as discussed in Remark 5. Notably, the methodology used in this paper is not applicable to this case, suggesting the need for alternative techniques.

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References

[1] T. Abualrub and R. Oehmke, *Cyclic codes of length* 2^e *over* $\mathbb{Z}_4$, Discrete Appl. Math. **128** (2003), no. 1, 3–9.
https://doi.org/10.1016/S0166-218X(02)00432-8.

[2] T. Abualrub and I. Siap, *Cyclic codes over the rings* $\mathbb{Z}_2+u\mathbb{Z}_2$ *and* $\mathbb{Z}_2+u\mathbb{Z}_2+u^2\mathbb{Z}_2$, Des. Codes Cryptogr. **42** (2007), no. 3, 273–287.
https://doi.org/10.1007/s10623-006-9034-5.

[3] ———, *Reversible cyclic codes over* $\mathbb{Z}_4$, Australas. J. Combin. **38** (2007), 195–205.

[4] M. Ashraf and G. Mohammad, $(1+2u)$*-constacyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4$, arXiv preprint arXiv:1504.03445,
https://doi.org/10.48550/arXiv.1504.03445.

[5] R.K. Bandi and M. Bhaintwal, *A note on cyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4$, Discrete Math. Algorithms Appl. **8** (2016), no. 1, 1650017.
https://doi.org/10.1142/S1793830916500178.

[6] T. Blackford, *Cyclic codes over* $\mathbb{Z}_4$ *of oddly even length*, Discrete Appl. Math. **128** (2003), no. 1, 27–46.
https://doi.org/10.1016/S0166-218X(02)00434-1.

[7] A.R. Calderbank, A.R. Hammons Jr, P.V. Kumar, N.J.A. Sloane, and P. Solé, *The* $\mathbb{Z}_4$*-linearity of kerdock, preparata, goethals and related codes*, IEEE Trans. Inf. Theory **40** (1994), no. 2, 301–319.

[8] Y. Cao and Q. Li, *Cyclic codes of odd length over* $\mathbb{Z}_4[u]/\langle u^k\rangle$, Cryptogr. Commun. **9** (2017), no. 5, 599–624.
https://doi.org/10.1007/s12095-016-0204-7.

[9] S. T. Dougherty and S. Ling, *Cyclic codes over $\mathbb{Z}_4$ of even length*, Des. Codes Cryptogr. **39** (2006), no. 2, 127–153.
https://doi.org/10.1007/s10623-005-2773-x.

[10] H. Islam and O. Prakash, *A study of cyclic and constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$*, Int. J. Inf. Coding Theory **5** (2018), no. 2, 155–168.
https://doi.org/10.1504/IJICOT.2018.095017.

[11] G. Karthick, *A study on structure of codes over $\mathcal{F}_4 + u\mathcal{F}_4 + v\mathcal{F}_4$*, Commun. Comb. Optim. **9** (2024), no. 3, 567–578.
https://doi.org/10.22049/cco.2023.28011.1503.

[12] M. Özen, N.T. Özzaim, and N. Aydin, *Cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$*, Turkish J. Math. **41** (2017), no. 5, P1235.
https://doi.org/10.3906/mat-1602-35.

[13] M. Özen, F.Z. Uzekmek, N. Aydin, and N.T. Özzaim, *Cyclic and some constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^2 - 1\rangle$*, Finite Fields Appl. **38** (2016), 27–39.
https://doi.org/10.1016/j.ffa.2015.12.003.

[14] V.S. Pless and Z. Qian, *Cyclic codes and quadratic residue codes over $\mathbb{Z}_4$*, IEEE Trans. Inf. Theory **42** (1996), no. 5, 1594–1600.
https://doi.org/10.1109/18.532906.

[15] B. Yildiz and N. Aydin, *On cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their $\mathbb{Z}_4$–images*, Int. J. Inf. Coding Theory **2** (2014), no. 4, 226–237.
https://doi.org/10.1504/IJICOT.2014.066107.