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Research Article

D-Distance Magic Labeling of C_n^r

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Abstract: Let $G = (V, E)$ be a graph of order n. Let $D \subseteq \{0, 1, 2, \ldots, \text{diam}(G)\}$ be nonempty. The D-neighborhood $N_D(x)$, of a vertex x is the set of all vertices whose distance from vertex x is an element in D, that is, $N_D(x) = \{y \in V :\ d(x, y) = m, m \in D\}$. A D-distance magic labeling of G is a bijection $f : V \to \{1, 2, \ldots, n\}$ for which there exists a positive integer k, such that $\sum_{x \in N_D(v)} f(x) = k$ for all $v \in V$, where $N_D(v)$ is the D-open neighborhood of v. Let Γ be an abelian group of order n. A (Γ, D)-distance magic labeling of G is a bijection $l : V \to \Gamma$ for which there exists an element $\mu \in \Gamma$, such that $\sum_{x \in N_D(v)} l(x) = \mu$ for all $v \in V$. This paper presents the necessary and sufficient conditions for the existence of D-distance magic labeling for C_n^r for a set D containing elements in arithmetic progression. For the same set D, we also study the (Γ, D)-distance magic labeling of C_n^r for some specific classes of abelian groups Γ.

Keywords: distance magic labeling, group distance magic labeling, circulant graphs.

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1. Introduction

By a graph G, we mean a finite, simple, undirected graph having neither multiple edges nor loops. We write V for the vertex set and E for the edge set of the graph G. By order of the graph, we mean $|V|$, and by the size of the graph, we mean $|E|$. We shall assume that all graphs G considered in the paper are of order n. For graph theoretic terminologies and notations, we refer to Chartrand and Lesniak [1].

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Let $D \subseteq \{0, 1, 2, \ldots, \text{diam}(G)\}$, where $\text{diam}(G)$ represents the diameter of graph G. We define the D-neighborhood $N_D(x)$ of a vertex x to be the set of all vertices whose distance from vertex x is m, where $m \in D$, i.e. $N_D(x) = \{y \in V : \ d(x, y) = m \in D\}$. O'Neil et al. [7] introduced the concept of D-distance magic labeling of graphs. We state its definition below.

Definition 1. A bijection $f : V \to \{1, 2, \ldots, n\}$ is said to be a D-distance magic labeling if there exists a constant k such that for any vertex x, $w(x) = \sum_{y \in N_D(x)} f(y) = k$. The constant k is called D-distance magic constant while the graph G is called D-distance magic graph.

Observe that the distance magic labeling of a graph is a case of D-distance magic labeling when $D = \{1\}$. O'Neil et al. [7] proved the following result.

Theorem 1. *Let $D \subseteq \{0, 1, 2, \ldots, d\}$ and let $D^c = \{0, 1, 2, \ldots, d\} - D$. Then a graph G is D-distance magic if and only if G is D^c-distance magic.*

The r^{th} power of a graph G, denoted by G^r, is defined as the graph having the same vertex set as G, with an edge between two distinct vertices if and only if there exists a path of length at most r between them in G. In this work, we focus on the r^{th} power of a cycle C_n. Observe that C_n^r is a $2r$-regular circulant graph, except in the case when n is even and $r = \frac{n}{2}$. In that exceptional case, $C_n^r \cong K_n$. A circulant graph is a graph on n vertices that admits a cyclic automorphism of order n.

The problem of obtaining necessary and sufficient conditions for the existence of distance magic labeling for the graph C_n^r has been studied by Cichacz [2]. Cichacz obtained the necessary and sufficient conditions for the graph C_n^r to be distance magic when r is odd.

Theorem 2. *[2] If r is odd, the graph C_n^r is distance magic if and only if $2r(r+1) \equiv 0 \pmod{n}$, $n \geq 2r + 2$ and $\frac{n}{gcd(n, r+1)} \equiv 0 \pmod{2}$.*

Theorem 3. *[2] If C_n^r is distance magic, then n is even.*

Godinho et al. [6] obtained the necessary and sufficient conditions for the graph C_n^r to be distance magic when r is even. They proved the following:

Theorem 4. *[6] If $a = \gcd(n, r)$ is even, then C_n^r is distance magic if and only if $a(r+1) \equiv 0 \pmod{n}$.*

In this work, we shall focus on D-distance magic labeling of the graph C_n^r where the elements in the set D are in arithmetic progression. For the sake of completeness, we mention the definition of an arithmetic progression. A sequence of positive integers

$\alpha_1, \alpha_2, \ldots, \alpha_k$, are said to be in arithmetic progression if for any $i = 1, 2, \ldots, k$, $\alpha_i = \alpha_1 + (i-1)d$, for some integer d. This integer d is called the common difference of the arithmetic progression.

Froncek [4] introduced the notion of group distance magic labeling as follows:

Definition 2. For an abelian group Γ and a graph G of the same order, a group distance magic labeling or a Γ-distance magic labeling of G is a bijection $l\colon V \to \Gamma$ such that $\sum_{y \in N(x)} l(y) = \beta \in \Gamma$, for every vertex $x \in V$.

One can notice that if a graph G of order n admits a distance magic labeling, it also admits a Z_n-distance magic labeling, but the converse is not necessarily true. Froncek [4] proved the following theorems:

Theorem 5. *[4] The cartesian product $C_m \Box C_n$, $m, n \geq 3$ is a Z_{mn}-distance magic graph if and only if mn is even.*

Theorem 6. *[4] The graph $C_{2^k} \Box C_{2^k}$, has a Z_2^{2k}-distance magic labeling for $k \geq 2$ and the magic constant $\mu = (0, 0, \ldots, 0)$.*

Cichacz [3] studied the group distance magic labeling of C_n^r for some specific abelian groups Γ. They proved the following theorems:

Theorem 7. *[3] Let $n \geq 2r + 2$ and $\gcd(n, r+1) = d$. If r is even and $n = 2kd$, then C_n^r has a $\mathbb{Z}_\alpha \times \mathcal{A}$-distance magic labeling for any $\alpha \equiv 0 \pmod{2k}$ and any abelian group $\mathcal{A}$ of order $\frac{n}{\alpha}$.*

Theorem 8. *[3] Let $n \geq 2r + 2$ and $\gcd(n, r+1) = d$. If r is odd, $n = 2kd$ and $r \equiv 0 \pmod{k}$ then C_n^r has a $\mathbb{Z}_\alpha \times \mathcal{A}$-distance magic labeling for any $\alpha \equiv 0 \pmod{2k}$ and any abelian group $\mathcal{A}$ of order $\frac{n}{\alpha}$.*

For a subset D of positive integers, if in the Definition 2, we consider the D-neighborhood $N_D(x)$ instead of $N(x)$, then we get (Γ, D)-distance magic labeling of G. Godinho et al. [5] studied the (Γ, D)-distance magic labeling of C_n^r, for some specific abelian groups Γ, when D is a singleton set. In this paper, we shall study the (Γ, D)-distance magic labeling of circulant graphs C_n^r for some specific abelian groups Γ, when the set D contains elements in arithmetic progression.

Before proceeding to the main results, we recall some definitions and notations that will be used throughout this work. An element $g \in \Gamma$ of order two is called an *involution*. It is well known that a non-trivial finite group contains an element of order two if and only if the order of the group is even. The subgroup generated by an element $g \in \Gamma$ will be denoted by $\langle g \rangle$. If H is a subgroup of an abelian group Γ and $g \in \Gamma$, then the set $H + g = \{h + g : h \in H\}$ is a coset of H in Γ.

2. D-distance magic labeling of C_n^r

Let n and r be positive integers such that $n \geq 3$. The graph C_n^r is a graph on n vertices $\{v_0, v_1, \ldots, v_{n-1}\}$ with the edge set $E(C_n^r) = \{v_i v_{i+j} \;:\; 0 \leq i \leq n-1, 1 \leq j \leq r\}$ where the subscripts i and $i+j$ are taken modulo n. From the above definition it is clear that the graph C_n^r is $2r$-regular having size rn and diameter $\lceil \frac{n-1}{2r} \rceil$, except for the case when n is even and $r = \frac{n}{2}$. When n is even and $r = \frac{n}{2}$, $C_n^r \cong K_n$.
In this section, we shall derive the necessary and sufficient conditions for the existence of D-distance magic labeling of the graph C_n^r when $D = \{\alpha_1, \alpha, \ldots, \alpha_k\} \subseteq \{0, 1, 2, \ldots, \text{diam}(G)\}$, where the elements α_i of D are in arithmetic progression with common difference d. First, we shall introduce the following notation: for a bijection $f : V(C_n^r) \to \{1, 2, \ldots, n\}$ and for $v_i \in V(C_n^r)$, we denote $f(v_i)$ by f_i. The indices i in v_i and f_i are assumed to be taken modulo n. If D is a set having k elements, then since C_n^r is a $2r$-regular graph of order n, if it admits a D-distance magic labeling, then the magic constant must be equal to $kr(n+1)$.

Observation 9. If $D = \{0, 1, 2, \ldots, \text{diam}(G)\}$, then C_n^r is D-distance magic for all n.

Proof. If $D = \{0, 1, 2, \ldots, \text{diam}(G)\}$, then the weight of any vertex of C_n^r is the sum of the labels of all the vertices of C_n^r, which is equal to $\frac{n(n+1)}{2}$, a constant, thus ensuring that C_n^r is D-distance magic. □

Theorem 10. *If $D = \{0, 1, 2, \ldots, p\}$ where $p < \text{diam}(G)$, then C_n^r is not D-distance magic for any n.*

Proof. If C_n^r is D-distance magic with D-distance magic labeling f, then for two vertices x_i and x_{i+1}, $w(x_i) = w(x_{i+1})$ implies $f_{i-pr} = f_{i+pr+1}$, which is a contradiction as f is a bijection. □

Theorem 11. *If $D = \{p, p+1, \ldots, \text{diam}(G)\}$ where $p > 0$, then C_n^r is not D-distance magic for any n.*

Proof. The proof follows from Theorem 10 and Theorem 1. □

Henceforth in this section, we shall assume $D \subseteq \{1, 2, \ldots, \text{diam}(G) - 1\}$.

Lemma 1. *If C_n^r is D-distance magic, then for any $v_j \in V(C_n^r)$ and $\lambda \in \mathbb{Z}$,*

$$\sum_{t=1}^{k} \left(f_{j+(t-1)dr} + f_{j+(\alpha_k+\alpha_1-1+(t-1)d)r+1}\right) = \sum_{t=1}^{k} \left(f_{j+((t-1)d+\lambda)r} + f_{j+(\alpha_k+\alpha_1-1+(t-1)d+\lambda)r+1}\right).$$

Proof. Suppose C_n^r is D-distance magic with a magic labeling f. For $v_j \in V(C_n^r)$, $w(u_j) = w(u_{j+1})$. This implies that,

$$\sum_{t=1}^{k} \big(f_{j-\alpha_t r} + f_{j+(\alpha_t-1)r+1}\big) = \sum_{t=1}^{k} \big(f_{j-(\alpha_t-1)r} + f_{j+\alpha_t r+1}\big).$$

Setting $j = j - \alpha_k r$ we have,

$$\sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t)r} + f_{j+(\alpha_k+\alpha_t-1)r+1}\big) = \sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t+1)r} + f_{j+(\alpha_k+\alpha_t)r+1}\big). \quad (2.1)$$

Equation (2.1) holds for every $v_j \in V(C_n^r)$. Substituting $j + r$ in place of j in (2.1) we get,

$$\sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t+1)r} + f_{j+(\alpha_k+\alpha_t)r+1}\big) = \sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t+2)r} + f_{j+(\alpha_k+\alpha_t+1)r+1}\big).$$

Hence,

$$\sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t)r} + f_{j+(\alpha_k+\alpha_t-1)r+1}\big) = \sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t+2)r} + f_{j+(\alpha_k+\alpha_t+1)r+1}\big).$$

By induction for every $\lambda \in \mathbb{N}$,

$$\sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t)r} + f_{j+(\alpha_k+\alpha_t-1)r+1}\big) = \sum_{t=1}^{k} \big(f_{j+(\alpha_k-\alpha_t+\lambda)r} + f_{j+(\alpha_k+\alpha_t-1+\lambda)r+1}\big).$$

Since the subscript is taken modulo n, this equation holds for all $\lambda \in \mathbb{Z}$. As $\alpha_i = \alpha_1 + (i-1)d$, we have,

$$\sum_{t=1}^{k} \big(f_{j+(t-1)dr} + f_{j+(\alpha_k+\alpha_1-1+(t-1)d)r+1}\big) = \sum_{t=1}^{k} \big(f_{j+((t-1)d+\lambda)r} + f_{j+(\alpha_k+\alpha_1-1+(t-1)d+\lambda)r+1}\big).$$

□

For a bijection $f\colon V(C_n^r) \to \{1, 2, \ldots, n\}$, we denote $\sum_{t=1}^{k} f_{i+(t-1)dr} = g_i$. Therefore from Lemma 1, we have $g_i + g_{i+(\alpha_k+\alpha_1-1)r+1} = g_{i+\lambda r} + g_{i+(\alpha_k+\alpha_1-1+\lambda)r+1}$. We set $\rho = (\alpha_k + \alpha_1 - 1)r + 1$. Then we have $g_i + g_{i+\rho} = g_{i+\lambda r} + g_{i+\rho+\lambda r}$. We denote $g_i + g_{i+\rho} = c_i$ and $a = \gcd(n, r)$.

Observe that if $n < 2\rho$, while calculating the weight of any vertex, the label of at least one vertex is added twice. To avoid this, we assume $n \geq 2\rho$.

Corollary 1. *If C_n^r is D-distance magic then $c_i = c_{i+a}$.*

Proof. Since $\gcd(n, r) = a$, there exists integers x and y such that $a = xn + yr$. Since $c_{i+xn} = c_i$ and $c_{i+yr} = c_i$, hence the result follows. □

Henceforth, we shall assume that the index i in c_i is taken modulo a. For a D-distance magic graph C_n^r we have the following equations:

$$\left.\begin{aligned} g_0 + g_\rho = g_r + g_{\rho+r} = g_{2r} + g_{\rho+2r} = \ldots = g_{(\frac{n}{a}-1)r} + g_{\rho-r} = c_0 \\ g_1 + g_{\rho+1} = g_{r+1} + g_{\rho+r+1} = g_{2r+1} + g_{\rho+2r+1} = \ldots = g_{(\frac{n}{a}-1)r+1} + g_{\rho-r+1} = c_1 \\ \vdots \quad \vdots \quad \vdots \\ g_{a-1} + g_{a-1+\rho} = g_{a-1+r} + g_{a-1+\rho+r} = \ldots = g_{a-1+(\frac{n}{a}-1)r} + g_{a-1+\rho-r} = c_{a-1} \end{aligned}\right\}$$

Lemma 2. *If C_n^r is D-distance magic with a D-distance magic labeling f then $c_0 + c_1 + \ldots + c_{a-1} = ka(n+1)$.*

Proof. Let $u_i \in V(C_n^r)$, then we have,

$$\begin{aligned} w(u_i) &= \sum_{j=1}^{r}(f_{i-(\alpha_1-1)r-j} + f_{i+(\alpha_1-1)r+j} + f_{i-(\alpha_2-1)r-j} + f_{i+(\alpha_2-1)r+j} + f_{i-(\alpha_k-1)r-j} + f_{i+(\alpha_k-1)r+j}) \\ &= \sum_{j=1}^{r}(g_{i-j} + g_{i-j+(\alpha_k+\alpha_1-1)r+1}) = \sum_{j=1}^{r}(g_{i-j} + g_{i-j+\rho}) = \sum_{j=1}^{r} c_{i-j} = \frac{r}{a}(c_0 + c_1 + \ldots + c_{a-1}). \end{aligned}$$

Now as $w(u_i) = kr(n+1)$, we have $\frac{r}{a}(c_0 + c_1 + \ldots + c_{a-1}) = kr(n+1)$. Hence we have $c_0 + c_1 + \ldots + c_{a-1} = ka(n+1)$. □

Lemma 3. *A bijection $f : V(C_n^r) \to \{1, 2, \ldots, n\}$ is D-distance magic labeling if and only if $c_i = c_j$ whenever $i \equiv j \pmod{a}$ and $c_0 + c_1 + \ldots + c_{a-1} = ka(n+1)$.*

Proof. Suppose $c_i = c_j$ for $i \equiv j \pmod{a}$ and $c_0 + c_1 + \ldots + c_{a-1} = ka(n+1)$. Then for $u_i \in V(C_n^r)$ we have

$$\begin{aligned} w(u_i) &= \sum_{j=1}^{r}(f_{i-(\alpha_1-1)r-j} + f_{i+(\alpha_1-1)r+j} + f_{i-(\alpha_2-1)r-j} + f_{i+(\alpha_2-1)r+j} + f_{i-(\alpha_k-1)r-j} + f_{i+(\alpha_k-1)r+j}) \\ &= \sum_{j=1}^{r}(g_{i-j} + g_{i-j+(\alpha_k+\alpha_1-1)r+1}) = \sum_{j=1}^{r}(g_{i-j} + g_{i-j+\rho}) = \sum_{j=1}^{r} c_{i-j} = \frac{r}{a}(c_0 + c_1 + \ldots + c_{a-1}) \\ &= \frac{r}{a}ka(n+1) = rk(n+1). \end{aligned}$$

Hence $w(u_i) = rk(n+1)$ for every $u_i \in V(C_n^r)$. Hence, the labeling f is D-distance magic labeling.
The converse follows from Corollary 1 and Lemma 2. □

Lemma 4. *If C_n^r is D-distance magic with D-distance magic labeling f and $\sum_{t=1}^{k} f_{i+(t-1)dr}=g_i$, then for any vertex $u_i \in V(C_n^r)$, the following equations hold,*

$$g_{i+a\rho} = \sum_{j=1}^{a}(-1)^{j-1}c_{i+a-j} + (-1)^a g_i \tag{2.2}$$

and

$$g_{i-a\rho} = \sum_{j=0}^{a-1}(-1)^{j}c_{i-(a-j)} + (-1)^a g_i. \tag{2.3}$$

Proof. Suppose C_n^r is D-distance magic graph with a D-distance magic labeling f and $\sum\limits_{t=1}^{k} f_{i+(t-1)dr} = g_i$, then we have $g_i + g_{i+\rho} = c_i$. Therefore $g_{i+\rho} = c_i - g_i$. Similarly as $g_{i+\rho} + g_{i+2\rho} = c_{i+\rho}$, it follows that $g_{i+2\rho} = c_{i+\rho} - g_{i+\rho} = c_{i+\rho} - c_i + g_i$. Since $\rho - 1 = (\alpha_k + \alpha_1 - 1)r$ and $\gcd(n, r) = a$ it follows that $\rho \equiv 1 \pmod a$. Hence we get $c_{i+\rho} = c_{i+1}$. Thus $g_{i+2\rho} = c_{i+1} - c_i + g_i$. Similarly we obtain $g_{i+3\rho} = c_{i+2} - c_{i+1} + c_i - g_i$. Proceeding in this manner we obtain the expression $g_{i+a\rho} = c_{i+a-1} - c_{i+a-2} + \ldots + c_i - g_i$ if a is odd and the expression $g_{i+a\rho} = c_{i+a-1} - c_{i+a-2} + \ldots - c_i + g_i$ if a is even. This proves equation (2.2).
To prove (2.3), observe that $g_{i-\rho} + g_i = c_{i-\rho}$. Since $i - \rho \equiv i - 1 \pmod a$, we have $g_{i-\rho} + g_i = c_{i-1}$. Hence $g_{i-\rho} = c_{i-1} - g_i$. Similarly $g_{i-2\rho} + g_{i-\rho} = c_{i-2\rho} = c_{i-2}$. From this we get $g_{i-2\rho} = c_{i-2} - c_{i-1} + g_i$. Proceeding in this manner we obtain the expression $g_{i-a\rho} = c_{i-a} - c_{i-(a-1)} + \ldots + c_{i-1} - g_i$ if a is odd and the expression $g_{i-a\rho} = c_{i-a} - c_{i-(a-1)} + \ldots - c_{i-1} + g_i$ if a is even. This proves equation (2.3). □

We now obtain a necessary and sufficient condition for the graph C_n^r to be D-distance magic.

Theorem 12. *Suppose $a = \gcd(n, r)$ is even. Then C_n^r is D-distance magic if and only if $a\rho \equiv 0 \pmod n$.*

Proof. Suppose C_n^r is D-distance magic with D-distance magic labeling f. Without loss of generality, assume g_0 to be the sum of the smallest k labels of f. Substituting $i = 0$ in (2.2) and (2.3) we obtain, $g_{a\rho} = c_{a-1} - c_{a-2} + \ldots + c_1 - c_0 + g_0$ and $g_{n-a\rho} = c_0 - c_1 + \ldots - c_{a-1} + g_0$. Hence $g_{a\rho} = g_0 + A$ and $g_{n-a\rho} = g_0 - A$ where $c_{a-1} - c_{a-2} + \ldots + c_1 - c_0 = A$. If $A \neq 0$, it follows that either $g_{a\rho}$ or $g_{n-a\rho}$ will have value less than g_0, which is a contradiction. Therefore $A = 0$ and $g_{a\rho} = g_0$. Hence $a\rho \equiv 0 \pmod n$.
Conversely, let $a\rho \equiv 0 \pmod n$. We claim that $o(\rho) = a$, where $o(\rho)$ is the order of ρ in $\mathbb{Z}_n$. Since $a\rho \equiv 0 \pmod n$, therefore, $o(\rho)$ divides a. Now, $o(\rho) = \frac{n}{\gcd(n,\rho)}$. Since a divides r, it follows that $\gcd(a, \rho) = 1$. Furthermore, $\gcd(n, \rho)$ divides ρ, hence $\gcd(a, \gcd(n, \rho)) = 1$. Therefore, a divides $o(\rho)$. This proves the claim.
For $0 \leq i \leq \frac{n}{a} - 1$, we define the sequence $\mathcal{A}_i = x_1^i, x_2^i, \ldots, x_{a-1}^i$, where

$$x_j^i = \begin{cases} \frac{jn}{a} + i + 1 & \text{if } j = 0, 2, 4, \ldots, a-2, \\ \frac{(j+1)n}{a} - i & \text{if } j = 1, 3, \ldots, a-1. \end{cases}$$

Then $\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{\frac{n}{a}-1}\}$ is a partition of $\{1, 2, \ldots, n\}$ into $\frac{n}{a}$ subsets and $|\mathcal{A}_i| = a$ for each i.
Next for $0 \leq i \leq \frac{n}{a} - 1$ we have,

$$x_j^i + x_{j+1}^i = \begin{cases} \frac{2(j+1)n}{a} + 1 & \text{if } j = 0, 1, \ldots, a-2, \\ n+1 & \text{if } j = a-1. \end{cases} \tag{2.4}$$

Let $x_j^i + x_{j+1}^i = b_j$. Then we have,

$$\sum_{j=0}^{a-1} b_j = \sum_{j=0}^{a-2} \Big(\frac{2(j+1)n}{a} + 1 \Big) + n + 1 = a(n+1). \tag{2.5}$$

Let $\langle \rho \rangle = \{i\rho : 0 \leq i \leq a-1\}$ be the subgroup generated by ρ in $\mathbb{Z}_n$. Then the set of all cosets $\{\langle \rho \rangle + l \ : \ 0 \leq l \leq \frac{n}{a} - 1\}$ forms a partition of $\mathbb{Z}_n$. For $i \in \mathbb{Z}_n$ we have $i \equiv (\alpha_i \rho + \beta_i) \pmod n$ where $0 \leq \alpha_i \leq a-1$ and $0 \leq \beta_i \leq \frac{n}{a} - 1$. Let $\alpha_i + \beta_i \equiv r_i \pmod a$. Now define $f(v_i) = f_i = x_{r_i}^{\beta_i}$. In what follows, we shall assume the subscript i and the superscript j in x_i^j are taken modulo a and modulo $\frac{n}{a}$ respectively. Clearly f is a bijection from $\{v_0, v_1, \ldots, v_{n-1}\}$ to $\{1, 2, \ldots, n\}$.
Now $\sum_{j=1}^k (f_{i+(j-1)dr} + f_{i+\rho+(j-1)dr}) = c_i$. We claim that for $\lambda \in \mathbb{N}$, $c_i = c_{i+\lambda a}$. Since $i + \rho \equiv \big((\alpha_i + 1)\rho + \beta_i\big) \pmod n$, we have $c_i = \sum_{j=1}^k \big(f_{i+(j-1)dr} + f_{i+\rho+(j-1)dr}\big) = \sum_{j=1}^k \big(x^{\beta_i}_{\alpha_i+\beta_i+(j-1)dr} + x^{\beta_i}_{\alpha_i+\beta_i+(j-1)dr+1}\big) = \sum_{j=1}^k b_{\alpha_i+\beta_i+(j-1)dr} = \sum_{j=1}^k b_{\alpha_i+\beta_i} = k(b_{\alpha_i+\beta_i})$ as $(j-1)dr \equiv 0 \pmod a$. Now suppose that $\lambda a \equiv \big(s\rho + t\big) \pmod a$. Since $\lambda a \equiv 0 \pmod a$ and $\rho \equiv 1 \pmod a$, it follows that $s + t \equiv 0 \pmod a$. Therefore $c_{i+\lambda a} = \sum_{j=1}^k \big(f_{i+\lambda a+(j-1)dr} + f_{i+\lambda a+\rho+(j-1)dr}\big) = \sum_{j=1}^k \big(x^{\beta_i+t}_{\alpha_i+\beta_i+s+t} + x^{\beta_i+t}_{\alpha_i+\beta_i+s+t+1}\big) = \sum_{j=1}^k \big(x^{\beta_i+t}_{\alpha_i+\beta_i} + x^{\beta_i+t}_{\alpha_i+\beta_i+1}\big) = \sum_{j=1}^k b_{\alpha_i+\beta_i} = k(b_{\alpha_i+\beta_i})$. Hence for $i \equiv j \pmod a$, $c_i = c_j$.
Now $k(b_{\alpha_i+\beta_i}) = c_i$. Therefore $\sum\limits_{i=0}^{a-1} c_i = \sum\limits_{i=0}^{a-1} kb_i = ak(n+1)$. Hence, from Lemma 3, the labeling f is D-distance magic. This completes the proof. □

Below, we provide an example of $\{2,4\}$-Distance Magic Labeling of C_{84}^4.

$$\begin{array}{llll} f(v_0) = 1, & f(v_{21}) = 42, & f(v_{42}) = 43, & f(v_{63}) = 84 \\ f(v_{64}) = 2, & f(v_1) = 41, & f(v_{22}) = 44, & f(v_{43}) = 83 \\ f(v_{44}) = 3, & f(v_{65}) = 40, & f(v_2) = 45, & f(v_{23}) = 82 \\ f(v_{24}) = 4, & f(v_{45}) = 39, & f(v_{66}) = 46, & f(v_3) = 81 \\ f(v_4) = 5, & f(v_{25}) = 38, & f(v_{46}) = 47, & f(v_{67}) = 80 \\ f(v_{68}) = 6, & f(v_5) = 37, & f(v_{26}) = 48, & f(v_{47}) = 79 \\ f(v_{48}) = 7, & f(v_{69}) = 36, & f(v_6) = 49, & f(v_{27}) = 78 \\ f(v_{28}) = 8, & f(v_{49}) = 35, & f(v_{70}) = 50, & f(v_7) = 77 \end{array}$$

$$\begin{array}{llll} f(v_8)=9, & f(v_{29})=34, & f(v_{50})=51, & f(v_{71})=76 \\ f(v_{72})=10, & f(v_9)=33, & f(v_{30})=52, & f(v_{51})=75 \\ f(v_{52})=11, & f(v_{73})=32, & f(v_{10})=53, & f(v_{31})=74 \\ f(v_{32})=12, & f(v_{53})=31, & f(v_{74})=54, & f(v_{11})=73 \\ f(v_{12})=13, & f(v_{33})=30, & f(v_{54})=55, & f(v_{75})=72 \\ f(v_{76})=14, & f(v_{13})=29, & f(v_{34})=56, & f(v_{55})=71 \\ f(v_{56})=15, & f(v_{77})=28, & f(v_{14})=57, & f(v_{35})=70 \\ f(v_{36})=16, & f(v_{57})=27, & f(v_{78})=58, & f(v_{15})=69 \\ f(v_{16})=17, & f(v_{37})=26, & f(v_{58})=59, & f(v_{79})=68 \\ f(v_{80})=18, & f(v_{17})=25, & f(v_{38})=60, & f(v_{59})=67 \\ f(v_{60})=19, & f(v_{81})=24, & f(v_{18})=61, & f(v_{39})=66 \\ f(v_{40})=20, & f(v_{61})=23, & f(v_{82})=62, & f(v_{19})=65 \\ f(v_{20})=21, & f(v_{41})=22, & f(v_{62})=63, & f(v_{83})=64 \end{array}$$

Theorem 13. *If $a=\gcd(n,r)$ is odd, the graph C_n^r is D-distance magic if and only if n is even and $a\rho \equiv \frac{n}{2} \pmod n$.*

Proof. Suppose C_n^r is D-distance magic with a D-distance magic labeling f. We claim that $a\rho \not\equiv 0 \pmod n$ and $2a\rho \equiv 0 \pmod n$. If $a\rho \equiv 0 \pmod n$, it follows from (2.2) that $g_i = \frac{c_{a-1}-c_{a-2}+\ldots+c_0}{2}$, for every $v_i \in V(C_n^r)$. This implies $g_i = g_{i+dr}$ which leads to $f_i = f_{i+kdr}$. This is a contradiction since the labeling f is one-one. Therefore $a\rho \not\equiv 0 \pmod n$. Substituting $i=0$ in (2.2) and (2.3) we obtain $g_{a\rho} = c_{a-1}-c_{a-2}+\ldots+c_0-g_0$ and $g_{n-a\rho} = c_{a-1}-c_{a-2}+\ldots+c_0-g_0$ respectively. From these two expressions we obtain $g_{a\rho} = g_{n-a\rho}$. Hence $a\rho \equiv -a\rho \pmod n$ which implies that $2a\rho \equiv 0 \pmod n$. Hence, the claim is proved. As a result, the order of $a\rho$ in $\mathbb{Z}_n$ is 2. Therefore n is even and $a\rho \equiv \frac{n}{2} \pmod n$.
For the converse, let $a\rho \equiv \frac{n}{2} \pmod n$. Therefore the order of ρ in $\mathbb{Z}_n$ is $2a$. For $0 \le i \le \frac{n}{2a}-1$, we define the sequence $\mathcal{B}_i = y_0^i, y_1^i, \ldots, y_{2a-1}^i$ by

$$y_j^i = \begin{cases} \frac{jn}{a}+i+1 & j=0,2,\ldots,a-1, \\ \frac{(j-a)n}{a}+i+1 & j=a+1,a+3,\ldots,2a-2, \\ \frac{(j+1)n}{a}-i & j=1,3,\ldots,a-2, \\ \frac{(j+1-a)n}{a}-i & j=a,a+2,\ldots,2a-1. \end{cases}$$

Then $\{\mathcal{B}_0,\mathcal{B}_1,\ldots,\mathcal{B}_{\frac{n}{2a}-1}\}$ is a partition of $\{1,2,\ldots,n\}$ into $\frac{n}{2a}$ subsets and $|\mathcal{B}_i| = 2a$ for each i. For $0 \le i,l \le \frac{n}{2a}-1$ and $0 \le j \le 2a-1$ we have $y_j^i + y_{j+1}^i = y_j^l + y_{j+1}^l$. Also for $0 \le j \le a-1$, $y_j^i + y_{j+1}^i = y_{j+a}^i + y_{j+a+1}^i$. We have

$$y_j^i + y_{j+1}^i = \begin{cases} \frac{2(j+1)n}{a}+1 & 0,1,\ldots,a-2, \\ n+1 & j=a-1. \end{cases} \tag{2.6}$$

Let $y_j^i + y_{j+1}^i = b_j$. Then we have $b_{j+a} = b_a$ and $\sum_{j=0}^{a-1} b_j = a(n+1)$. The index j in b_j is assumed to be taken modulo a.

For each i in $\mathbb{Z}_n$ we have $i \equiv \big(\alpha_i\rho + \beta_i\big)(\text{mod } n)$ with $0 \leq \alpha_i \leq 2a-1$ and $0 \leq \beta_i \leq \frac{n}{2a} - 1$. Let $\alpha_i + \beta_i \equiv r_i \pmod{2a}$. We define $f(v_i) = f_i = y_{r_i}^{\beta_i}$. In what follows, we shall assume the subscript i and the superscript j in y_i^j are taken modulo $2a$ and modulo $\frac{n}{2a}$ respectively. Clearly f is a bijection from $\{v_0, v_1, \ldots, v_{n-1}\}$ to $\{1, 2, \ldots, n\}$.
Using a similar argument as in the proof of Theorem 12, it follows that for $\lambda \in \mathbb{N}$, $c_i = c_{i+\lambda a}$. Hence for $i \equiv j \pmod{a}$, $c_i = c_j$. Furthermore we have $c_0 + c_1 + \ldots + c_{a-1} = ak(n+1)$. Hence by Lemma 3, the labeling f is D-distance magic. □

Below, we show an example of $\{2,4\}$-Distance magic labeling of C_{96}^3.

$f(v_{0}) = 1,$	$f(v_{16}) = 64,$	$f(v_{32}) = 65,$	$f(v_{48}) = 32,$	$f(v_{64}) = 33,$	$f(v_{80}) = 96,$
$f(v_{81}) = 2,$	$f(v_{1}) = 63,$	$f(v_{17}) = 66,$	$f(v_{33}) = 31,$	$f(v_{49}) = 34,$	$f(v_{65}) = 95,$
$f(v_{66}) = 3,$	$f(v_{82}) = 62,$	$f(v_{2}) = 67,$	$f(v_{18}) = 30,$	$f(v_{34}) = 35,$	$f(v_{50}) = 94,$
$f(v_{51}) = 4,$	$f(v_{67}) = 61,$	$f(v_{83}) = 68,$	$f(v_{3}) = 29,$	$f(v_{19}) = 36,$	$f(v_{35}) = 93,$
$f(v_{36}) = 5,$	$f(v_{52}) = 60,$	$f(v_{68}) = 69,$	$f(v_{84}) = 28,$	$f(v_{4}) = 37,$	$f(v_{20}) = 92,$
$f(v_{21}) = 6,$	$f(v_{37}) = 59,$	$f(v_{53}) = 70,$	$f(v_{69}) = 27,$	$f(v_{85}) = 38,$	$f(v_{5}) = 91,$
$f(v_{6}) = 7,$	$f(v_{22}) = 58,$	$f(v_{38}) = 71,$	$f(v_{54}) = 26,$	$f(v_{70}) = 39,$	$f(v_{86}) = 90,$
$f(v_{87}) = 8,$	$f(v_{7}) = 57,$	$f(v_{23}) = 72,$	$f(v_{39}) = 25,$	$f(v_{55}) = 40,$	$f(v_{71}) = 89,$
$f(v_{72}) = 9,$	$f(v_{88}) = 56,$	$f(v_{8}) = 73,$	$f(v_{24}) = 24,$	$f(v_{40}) = 41,$	$f(v_{56}) = 88,$
$f(v_{57}) = 10,$	$f(v_{73}) = 55,$	$f(v_{89}) = 74,$	$f(v_{9}) = 23,$	$f(v_{25}) = 42,$	$f(v_{41}) = 87,$
$f(v_{42}) = 11,$	$f(v_{58}) = 54,$	$f(v_{74}) = 75,$	$f(v_{90}) = 22,$	$f(v_{10}) = 43,$	$f(v_{26}) = 86,$
$f(v_{27}) = 12,$	$f(v_{43}) = 53,$	$f(v_{59}) = 76,$	$f(v_{75}) = 21,$	$f(v_{91}) = 44,$	$f(v_{11}) = 85,$
$f(v_{12}) = 13,$	$f(v_{28}) = 52,$	$f(v_{44}) = 77,$	$f(v_{60}) = 20,$	$f(v_{76}) = 45,$	$f(v_{92}) = 84,$
$f(v_{93}) = 14,$	$f(v_{13}) = 51,$	$f(v_{29}) = 78,$	$f(v_{45}) = 19,$	$f(v_{51}) = 46,$	$f(v_{67}) = 83,$
$f(v_{68}) = 15,$	$f(v_{94}) = 50,$	$f(v_{14}) = 79,$	$f(v_{30}) = 18,$	$f(v_{46}) = 47,$	$f(v_{52}) = 82,$
$f(v_{53}) = 16,$	$f(v_{69}) = 49,$	$f(v_{95}) = 80,$	$f(v_{15}) = 17,$	$f(v_{31}) = 48,$	$f(v_{47}) = 81,$

3. (Γ, D)-Distance Magic Labeling of C_n^r

In this section, we study the (Γ, D)-distance magic labeling of C_n^r for some abelian groups Γ. We assume $D = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq \{1, 2, \ldots, \text{diam}(G) - 1\}$, where the elements α_i of D are in arithmetic progression with common difference d. As in the previous section, we set $\rho = (\alpha_k + \alpha_1 - 1)r + 1$ and assume $n \geq 2\rho$. Let $C_n = v_0v_1v_2\ldots v_{n-1}$. For a vertex v_i in C_n, its D-neighborhood in C_n^r is

$$N_D(v_i) = \bigcup_{j=1}^{r} \left(\bigcup_{t=1}^{k} \left\{ v_{i-(\alpha_t-1)r-j}, v_{i-(\alpha_t-1)r-j+\rho} \right\} \right)$$

where the subscripts are taken modulo n.

Lemma 5. *For $n \geq 2\rho$, if C_n^r is (Γ, D)-distance magic for an abelian group Γ, then n is even.*

Proof. Let $l\colon V(C_n^r) \to \Gamma$ be a (Γ, D)-distance magic labeling and $\mu \in \Gamma$ be the magic constant. Then it is easy to check that for any natural number γ; we have

$w(v_{i+\alpha_k r}) = w(v_{i+\alpha_k r+\gamma\rho})$. Therefore,

$$\sum_{t=1}^{k}\Big(\sum_{q=0}^{r-1} l(v_{i+(\alpha_k-\alpha_t)r+q})\Big) = \sum_{t=1}^{k}\Big(\sum_{q=0}^{r-1} l(v_{i+(\alpha_k-\alpha_t)r+q+2\gamma\rho})\Big).$$

Suppose that n is odd; then $\frac{n}{\gcd(n,\rho)} \equiv 1(\text{mod } 2)$. Thus $\gcd(n,\rho) = \gcd(n,2\rho)$ and $< 2\rho >=< \rho >$. Hence $\rho = 2c\rho$ for some $c \geq 1$. Set $\gamma = c$, $i = 0, 1$ and obtain respectively:

$$\sum_{t=1}^{k}\Big(\sum_{q=0}^{r-1} l(v_{(\alpha_k-\alpha_t)r+q})\Big) = \sum_{t=1}^{k}\Big(\sum_{q=0}^{r-1} l(v_{(\alpha_k-\alpha_t)r+q+\rho})\Big),$$

$$\sum_{t=1}^{k}\Big(\sum_{q=1}^{r} l(v_{(\alpha_k-\alpha_t)r+q})\Big) = \sum_{t=1}^{k}\Big(\sum_{q=1}^{r} l(v_{(\alpha_k-\alpha_t)r+q+\rho})\Big).$$

Since $N_D(v_i) = \bigcup_{j=1}^{r}\Big(\bigcup_{t=1}^{k}\{v_{i-(\alpha_t-1)r-j}, v_{i-(\alpha_t-1)r-j+\rho}\}\Big)$ and C_n^r is (Γ, D)-distance magic, we obtain:

$$2\sum_{t=1}^{k}\Big(\sum_{q=0}^{r-1} l(v_{(\alpha_k-\alpha_t)r+q})\Big) = \mu,$$

$$2\sum_{t=1}^{k}\Big(\sum_{q=1}^{r} l(v_{(\alpha_k-\alpha_t)r+q})\Big) = \mu.$$

Therefore, $2\Big(\sum_{t=1}^{k} l(v_{(\alpha_k-\alpha_t)r}) - \sum_{t=1}^{k} l(v_{(\alpha_k-\alpha_t)r+r})\Big) = 0$. Note that n being odd implies that there does not exist an element $g \neq 0$, $g \in \Gamma$ such that $2g = 0$. Thus $\sum_{t=1}^{k} l(v_{(\alpha_k-\alpha_t)r}) = \sum_{t=1}^{k} l(v_{(\alpha_k-\alpha_t)r+r})$. This leads to $\sum_{t=1}^{k} l(v_{(\alpha_k-\alpha_t)r}) = \sum_{t=1}^{k} l(v_{(\alpha_k-\alpha_{t-1})r})$, which implies $l(v_0) = l(v_{\alpha_k r})$ and we obtain a contradiction as $n \geq 2\rho$. □

Theorem 14. *Let $n \geq 2\rho$ and $\gcd(n,\rho) = d$. If r is even and $n = 2qd$, then C_n^r has a $(\mathbb{Z}_\alpha \times \mathcal{A}, D)$-distance magic labeling for any $\alpha \equiv 0 \pmod{2q}$ and any abelian group $\mathcal{A}$ of order $\frac{n}{\alpha}$.*

Proof. Let $\frac{n}{\alpha} = p$. Let $\mathcal{A} = \{a_0, a_1, \ldots, a_{p-1}\}$ such that $a_0 = 0$. Since $\Gamma = \mathbb{Z}_\alpha \times \mathcal{A}$, every element $g \in \Gamma$ can be written in the form $g = (j, a_i)$ with $j \in \mathbb{Z}_\alpha$ and $a_i \in \mathcal{A}$.

Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ be the vertex set of C_n^r. Let $X = \langle \rho \rangle$ be the subgroup of Z_n of order $2q$. Let $\{X+1, \ldots, X+(d-1)\}$ be the set of cosets of X in $\mathbb{Z}_n$. For $j = 1, 2, \ldots, d-1$, let X_j denote the set of all vertices whose subscripts belong to the coset $X + j$. Notice that $\alpha = 2qh$ for some positive integer h. Let $H = \langle 2h \rangle$ be the subgroup of Z_α of order q.

We shall define a (Γ, D)-distance magic labeling $l : V = \{v_0, v_1, \ldots, v_{n-1}\} \to \mathbb{Z}_\alpha \times \mathcal{A}$ such that $l(v_i) = (l_1(v_i), l_2(v_i))$ where l_1 and l_2 are maps from V into $\mathbb{Z}_\alpha$ and $\mathcal{A}$ respectively. First label the vertices of X as follows:

$$l(v_{2i\rho}) = (2ih, a_0), \quad l(v_{(2i+1)\rho}) = (-2ih-1, -a_0), \; i = 0, 1, \ldots, k-1.$$

If the subscript m in v_m belongs to the coset $X + j$, then denote it by m_j. Notice that a vertex v_{m_j} belongs to X_j then the vertex $v_{m_j - \rho - 1}$ belongs to X_{j-1}. We label $X_1, X_2, \ldots, X_{d-1}$ recursively in the following manner:

$$l_1(x_{v_j}) = \begin{cases} l_1(v_{m_j-\rho-1}) + 1, & \text{if } l_1(v_{m_j - j(\rho+1)}) \equiv 0 \pmod{2h}, \\ l_1(v_{m_j-\rho-1}) - 1, & \text{if } l_1(v_{m_j - j(\rho+1)}) \not\equiv 0 \pmod{2h}. \end{cases}$$

$$l_2(v_{m_j}) = \begin{cases} a_{\lfloor j/h \rfloor} & l_1(v_{m_j}) \equiv 0 \pmod{2}, \\ -a_{\lfloor j/h \rfloor} & l_1(v_{m_j}) \equiv 1 \pmod{2}. \end{cases}$$

Clearly l is a bijection and satisfies the relation $l(v_{2i}) + l(v_{2i+\rho}) = (-1, 0)$ and $l(v_{2i+1}) + l(v_{2i+\rho+1}) = (2h-1, 0)$ for any i.

Recall that $N_D(v_i) = \bigcup_{j=1}^{r} \left(\bigcup_{t=1}^{k} \{v_{i-(\alpha_t-1)r-j}, v_{i-(\alpha_t-1)r-j+\rho}\} \right)$. Since r is even, it implies that, for any i,

$$\begin{aligned} w(v_i) &= \sum_{j=1}^{r} \left[\sum_{t=1}^{k} \Big(l(v_{i-(\alpha_t-1)r-j}) + l(v_{i-(\alpha_t-1)r-j+\rho}) \Big) \right] \\ &= \frac{kr}{2}(2h-2, 0). \end{aligned}$$

Hence l is $(\Gamma, \{d\})$-distance magic with magic constant $\frac{kr}{2}(2h-2, 0)$. This completes the proof. □

Below, we show an example of $\{2,4\}$-Distance magic labeling of C_{44}^2 using $\mathbb{Z}_4 \times \mathbb{Z}_{11}$.

$$\begin{array}{cccc}
l(v_0) = (0, a_0), & l(v_{11}) = (3, -a_0), & l(v_{22}) = (2, a_0), & l(v_{33}) = (1, -a_0), \\
l(v_{12}) = (1, -a_1), & l(v_{23}) = (2, a_1), & l(v_{34}) = (3, -a_1), & l(v_1) = (0, a_1), \\
l(v_{24}) = (2, a_2), & l(v_{35}) = (1, -a_2), & l(v_2) = (0, a_2), & l(v_{13}) = (3, -a_2), \\
l(v_{36}) = (3, -a_3), & l(v_3) = (0, a_3), & l(v_{14}) = (1, -a_3), & l(v_{25}) = (2, a_3), \\
l(v_4) = (0, a_4), & l(v_{15}) = (3, -a_4), & l(v_{26}) = (2, a_4), & l(v_{37}) = (1, -a_4), \\
l(v_{16}) = (1, -a_5), & l(v_{27}) = (2, a_5), & l(v_{38}) = (3, -a_5), & l(v_5) = (0, a_5), \\
l(v_{28}) = (2, a_6), & l(v_{39}) = (1, -a_6), & l(v_6) = (0, a_6), & l(v_{17}) = (3, -a_6), \\
l(v_{40}) = (3, -a_7), & l(v_7) = (0, a_7), & l(v_{18}) = (1, -a_7), & l(v_{29}) = (2, a_7), \\
l(v_8) = (0, a_8), & l(v_{19}) = (3, -a_8), & l(v_{30}) = (2, a_8), & l(v_{41}) = (1, -a_8), \\
l(v_{20}) = (1, -a_9), & l(v_{31}) = (2, a_9), & l(v_{42}) = (3, -a_9), & l(v_9) = (0, a_9), \\
l(v_{32}) = (2, a_{10}), & l(v_{43}) = (1, -a_{10}), & l(v_{10}) = (0, a_{10}), & l(v_{21}) = (3, -a_{10}).
\end{array}$$

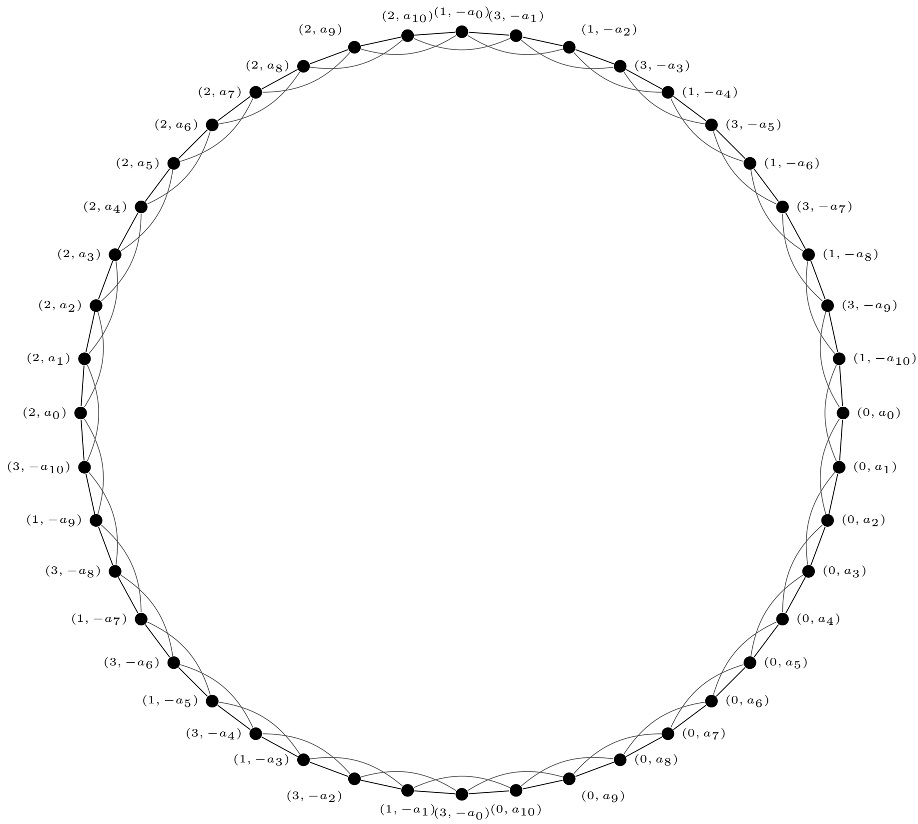

Figure 1. **$\{2,4\}$-Distance magic labeling of C_{44}^2 using $\mathbb{Z}_4 \times \mathbb{Z}_{11}$.**

Theorem 15. *Let $n \geq 2\rho$ and $\gcd(n, \rho) = d$. If r is odd, $n = 2qd$ and $r \equiv 0 \pmod{q}$, then C_n^r has a $(\mathbb{Z}_\alpha \times \mathcal{A}, D)$-distance magic labeling for any $\alpha \equiv 0 \pmod{2q}$ and any abelian group $\mathcal{A}$ of order $\frac{n}{\alpha}$.*

Proof. Let $\frac{n}{\alpha} = p$. Let $\mathcal{A} = \{a_0, a_1, \dots, a_{p-1}\}$ such that $a_0 = 0$. Since $\Gamma = \mathbb{Z}_\alpha \times \mathcal{A}$, every element $g \in \Gamma$ can be written in the form $g = (j, a_i)$ with $j \in \mathbb{Z}_\alpha$ and $a_i \in \mathcal{A}$.

Let $V = \{v_0, v_1, \dots, v_{n-1}\}$ be the vertex set of C_n^r. Let $X = \langle \rho \rangle$ be the subgroup of Z_n of order $2q$. Let $\{X + 1, \dots, X + (d-1)\}$ be the set of cosets of X in $\mathbb{Z}_n$. For $j = 1, 2, \dots, d-1$, let X_j denote the set of all vertices whose subscripts belong to the coset $X + j$. Notice that $\alpha = 2qh$ for some positive integer h. Let $H = \langle 2h \rangle$ be the subgroup of Z_α of order q.

We shall define a (Γ, D)-distance magic labeling $l : V = \{v_0, v_1, \dots, v_{n-1}\} \to \mathbb{Z}_\alpha \times \mathcal{A}$ such that $l(v_i) = (l_1(v_i), l_2(v_i))$ where l_1 and l_2 are maps from V into $\mathbb{Z}_\alpha$ and $\mathcal{A}$ respectively. First label the vertices of X as follows:
If $k = 1$, then $l(v_0) = (0, a_0)$, $l(v_\rho) = (-1, -a_0)$.
If $k = 3$, then $l(v_0) = (0, a_0)$, $l(v_\rho) = (-2, -a_0)$, $l(v_{2\rho}) = (2, a_0)$, $l(v_{3\rho}) = (-3, -a_0)$, $l(v_{4\rho}) = (1, a_0)$, $l(v_{5\rho}) = (-1, -a_0)$.
For $k \geq 5$ let $l(v_0) = (0, a_0)$, $l(v_{2\rho}) = (2, a_0)$, $l(v_{4\rho}) = (4, a_0)$, $\dots$, $l(v_{2i\rho}) = (2i, a_0)$, $\dots$, $l(v_{(q-3)\rho}) = (q-3, a_0)$, $l(v_{(q-1)\rho}) = (q-1, a_0)$, $l(v_{(q+1)\rho}) = (q-2, a_0)$, $l(v_{(q+3)\rho}) = (q-4, a_0)$, $l(v_{(q+5)\rho}) = (q-6, a_0)$, $\dots$,$l(v_{(2q-4)\rho}) = (3, a_0)$, $l(v_{(2q-2)\rho}) = (1, a_0)$ and $l(v_\rho) = (-2, -a_0)$, $l(v_{3\rho}) = (-4, -a_0)$, $\dots$, $l(v_{(2i+1)\rho}) = (-2i-2, -a_0)$, $\dots$, $l(v_{(q-4)\rho}) = (-q+3, -a_0)$, $l(v_{(q-2)\rho}) = (-q+1, -a_0)$, $l(v_{q\rho}) = (-q, -a_0)$, $l(v_{(q+2)\rho}) = (-q+2, -a_0)$,$\dots$, $l(v_{(2q-3)\rho}) = (-3, -a_0)$, $l(v_{(2q-1)\rho}) = (-1, -a_0)$.

If the subscript m in v_m belongs to the coset $X + j$, then denote it by m_j. Notice that a vertex v_{m_j} belongs to X_j if the vertex $v_{m_j-\rho-1}$ belongs to X_{j-1}. We label vertices in $X_1, X_2, \dots, X_{h-1}$ recursively as follows:

$$l(v_{m_j}) = \begin{cases} \big(l_1(v_{m_j+\rho-1}) + k, a_0\big) & \text{if } m_j + j(\rho-1) \equiv 0 \pmod{2\rho}, \\ \big(l_1(v_{m_j+\rho-1}) - k, -a_0\big) & \text{if } m_j + j(\rho-1) \not\equiv 0 \pmod{2\rho}. \end{cases}$$

Notice that a vertex v_{m_j} belongs to X_j if the vertex $v_{m_j+h(\rho-1)}$ belongs to X_{j-h}. We label vertices in $X_h, X_{h+1}, \dots, X_{d-1}$ recursively as follows:

$$l(v_{m_j}) = \begin{cases} \big(l_1(v_{m_j+h(\rho-1)}), a_{\lfloor \frac{j}{h} \rfloor}\big) & \text{if } l_1(v_{m_j+h(\rho-1)}) < \frac{\alpha}{2}, \\ \big(l_1(v_{m_j+h(\rho-1)}), -a_{\lfloor \frac{j}{h} \rfloor}\big) & \text{if } l_1(v_{m_j+h(\rho-1)}) > \frac{\alpha}{2}. \end{cases}$$

Obviously l is a bijection and observe that if $q = 1$, then $l(v_i) + l(v_{i+\rho}) = (-1, 0)$ for any i, whereas for $q > 1$ since $r \equiv 0 \pmod{q}$:

$$
\begin{aligned}
&l(v_{iq+0}) + l(v_{iq+\rho}) = (-2, 0)\\
&l(v_{iq+1}) + l(v_{iq+\rho+1}) = (0, 0)\\
&l(v_{iq+2}) + l(v_{iq+\rho+2}) = (-2, 0)\\
&l(v_{iq+3}) + l(v_{iq+\rho+3}) = (0, 0)\\
&\vdots\\
&l(v_{(i+1)q-3}) + l(v_{(i+1)q+\rho-3}) = (-2, 0)\\
&l(v_{(i+1)q-2}) + l(v_{(i+1)q+\rho-2}) = (0, 0)\\
&l(v_{(i+1)q-1}) + l(v_{(i+1)q+\rho-1}) = (-1, 0)
\end{aligned}
$$

for $i = 0, 1, 2, \ldots, \frac{r}{q} - 1$.

Furthermore, because $N_D(v_i) = \bigcup_{j=1}^{r} \left(\bigcup_{t=1}^{k} \left\{ v_{i-(\alpha_t - 1)r - j}, v_{i-(\alpha_t-1)r-j+\rho} \right\} \right)$ and $\frac{r}{q}$ is odd, we obtain $w(v_i) = k(-r, 0)$ for any i. □

Below, we show an example of $\{2, 4\}$-Distance magic labeling of C_{96}^{3} using $\mathbb{Z}_{24} \times \mathbb{Z}_4$.

$l(v_0) = (0, a_0)$, $l(v_{16}) = (22, -a_0)$, $l(v_{32}) = (2, a_0)$, $l(v_{48}) = (21, -a_0)$, $l(v_{64}) = (1, a_0)$, $l(v_{80}) = (23, -a_0)$,
$l(v_{17}) = (5, a_0)$, $l(v_{33}) = (18, -a_0)$, $l(v_{49}) = (4, a_0)$, $l(v_{65}) = (20, -a_0)$, $l(v_{81}) = (3, a_0)$, $l(v_1) = (19, -a_0)$,
$l(v_{34}) = (7, a_0)$, $l(v_{50}) = (17, -a_0)$, $l(v_{66}) = (6, a_0)$, $l(v_{82}) = (16, -a_0)$, $l(v_2) = (8, a_0)$, $l(v_{18}) = (15, -a_0)$,
$l(v_{51}) = (9, a_0)$, $l(v_{67}) = (13, -a_0)$, $l(v_{83}) = (11, a_0)$, $l(v_3) = (12, -a_0)$, $l(v_{19}) = (10, a_0)$, $l(v_{35}) = (14, -a_0)$,
$l(v_{68}) = (2, a_1)$, $l(v_{84}) = (21, -a_1)$, $l(v_4) = (1, a_1)$, $l(v_{20}) = (23, -a_1)$, $l(v_{36}) = (0, a_1)$, $l(v_{52}) = (22, -a_1)$,
$l(v_{85}) = (4, a_1)$, $l(v_5) = (20, -a_1)$, $l(v_{21}) = (3, a_1)$, $l(v_{37}) = (19, -a_1)$, $l(v_{53}) = (5, a_1)$, $l(v_{69}) = (18, -a_1)$,
$l(v_6) = (6, a_1)$, $l(v_{22}) = (16, -a_1)$, $l(v_{38}) = (8, a_1)$, $l(v_{54}) = (15, -a_1)$, $l(v_{70}) = (7, a_1)$, $l(v_{86}) = (17, -a_1)$,
$l(v_{23}) = (11, a_1)$, $l(v_{39}) = (12, -a_1)$, $l(v_{55}) = (16, a_1)$, $l(v_{71}) = (14, -a_1)$, $l(v_{87}) = (9, a_1)$, $l(v_7) = (13, -a_1)$,
$l(v_{40}) = (1, a_2)$, $l(v_{56}) = (23, -a_2)$, $l(v_{72}) = (0, a_2)$, $l(v_{88}) = (22, -a_2)$, $l(v_8) = (2, a_2)$, $l(v_{24}) = (21, -a_2)$,
$l(v_{57}) = (3, a_2)$, $l(v_{73}) = (19, -a_2)$, $l(v_{89}) = (5, a_2)$, $l(v_9) = (18, -a_2)$, $l(v_{25}) = (4, a_2)$, $l(v_{41}) = (20, -a_2)$,
$l(v_{74}) = (8, a_2)$, $l(v_{90}) = (15, -a_2)$, $l(v_{10}) = (7, a_2)$, $l(v_{26}) = (17, -a_2)$, $l(v_{42}) = (6, a_2)$, $l(v_{58}) = (16, -a_2)$,
$l(v_{91}) = (10, a_2)$, $l(v_{11}) = (14, -a_2)$, $l(v_{27}) = (9, a_2)$, $l(v_{43}) = (13, -a_2)$, $l(v_{59}) = (11, a_2)$, $l(v_{75}) = (12, -a_2)$,
$l(v_{12}) = (0, a_3)$, $l(v_{28}) = (22, -a_3)$, $l(v_{44}) = (2, a_3)$, $l(v_{60}) = (21, -a_3)$, $l(v_{76}) = (1, a_3)$, $l(v_{92}) = (23, -a_3)$,
$l(v_{29}) = (5, a_3)$, $l(v_{45}) = (18, -a_3)$, $l(v_{61}) = (4, a_3)$, $l(v_{77}) = (20, -a_3)$, $l(v_{93}) = (3, a_3)$, $l(v_{13}) = (19, -a_3)$,
$l(v_{46}) = (7, a_3)$, $l(v_{62}) = (17, -a_3)$, $l(v_{78}) = (6, a_3)$, $l(v_{94}) = (16, -a_3)$, $l(v_{14}) = (8, a_3)$, $l(v_{30}) = (15, -a_3)$,
$l(v_{63}) = (9, a_3)$, $l(v_{79}) = (13, -a_3)$, $l(v_{95}) = (11, a_3)$, $l(v_{15}) = (12, -a_3)$, $l(v_{31}) = (10, a_3)$, $l(v_{47}) = (14, -a_3)$.

4. Conclusion and Scope

In this paper, we have studied D-distance magic labeling of circulant graphs C_n^r when the elements in the set D are in arithmetic progression. For such a set D, we have also studied (Γ, D)-distance magic labeling of C_n^r for several classes of abelian groups Γ. The existence of group distance magic labeling for various other classes of abelian groups are yet to be explored. The following question naturally arises.

Problem 1. Classify the sets $D \subset \{0, 1, 2, \ldots, \text{diam}(G)\}$ for which a distance magic labeling exists for graph C_n^r.

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