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Research Article

SDCTD sets in some products of graphs and their application

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Abstract: This paper focuses on three types of product graphs, analyzing the relationships between their ECD sets, SDCTD sets, and factor graphs, while also constructing such ECD sets and SDCTD sets for the product graphs in question. We establish the necessary and sufficient conditions for an ECD set of a hypercube to be an SDCTD set, clarify the conditions under which an SDCTD set constitutes a PD set in a general graph G, and further deduce these conditions specifically for regular graphs and bipartite graphs. By hierarchically partitioning SDCTD sets via tree decomposition, we derive new upper bounds for the domination number of the modular product of two graphs, as well as for that of the modular product of two regular graphs. Additionally, we fully characterize all graph pairs (G, H) for which $\gamma(G \diamond H) = 5$ and prove a general lower bound $\gamma(G \diamond G) \geq \gamma(G) + 2$. The paper concludes by outlining several avenues for potential future research.

Keywords: product graph, ECD set, SDCTD set, PD set, domination number.

AMS Subject classification: 05C07, 05C69, 05C76

1. Introduction

Graph products are fundamental constructs in graph theory, serving as essential tools for constructing complex graph structures. From an algebraic structure perspective, among the 256 distinct types of graph products cataloged by Hamack et al. [4] and Imrich [5], the modular product stands out as the only non-standard product that exhibits both associativity and commutativity. Its edge set definition incorporates characteristics of Cartesian products, direct products, and direct products of complement graphs. This unique edge set structure enables the modular product to act as

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a bridge connecting adjacency and non-adjacency information between factor graphs, thereby serving as a complement to standard graph products such as strong products and lexicographic products. Despite this distinctive property specifically, its ability to complement standard products like Cartesian products and direct products the modular product has long remained at the margins of graph theory research and is often referred to as the forgotten product.

The pioneering work on the modular product by Imrich [5] (1972) explored its algebraic properties, while Shao and Solis-Oba [14] investigated L(2,1)-labelings of modular products in 2013. In recent years, research on modular product graphs has flourished, yielding significant breakthroughs. Sen and Kola [13] proposed upper bounds for the broadcast domination number of modular products in 2022. Kang et al. [6] presented distance formulas and methodologies for deriving the strong metric dimension in their 2024 work, laying essential foundations for studying the metric properties of modular products. For classical graph products, domination theory has been extensively explored [2, 7, 9–12]. However, modular products remained largely overlooked until Sergio Bermudo's 2025 breakthrough [1] on domination numbers this work put forward new research questions and advanced the field of domination number studies for modular product graphs. Here, we build on this research trajectory by focusing specifically on the modular product.

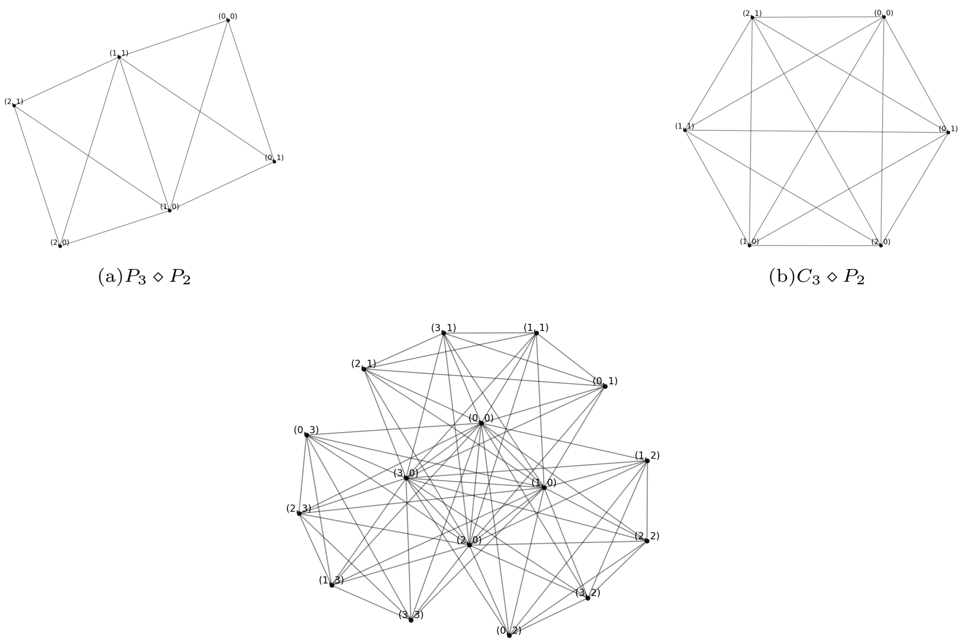
The paper is structured as follows. In the subsequent section, we introduce additional notation and preliminary concepts. In Section 3, we investigate the relationships among SDCTD sets, ECD sets, and PD sets in Cartesian product graphs and direct product graphs, and establish several equalities or inequalities involving their cardinalities. We also characterize the necessary and sufficient conditions for an SDCTD set in a graph G to be a PD set. As applications, we analyze the conditions under which SDCTD sets in regular graphs and bipartite graphs qualify as PD sets. In Section 4, we first leverage the notable phenomenon that modular product graphs often have diameter 2; building on this observation, we derive the total domination number of the complement of graphs with diameter 2. Using tree decomposition and hierarchical construction methods, we obtain upper bounds for the domination number of modular product graphs. We further study upper bounds for the domination number of modular products of regular graphs and resolve two of Bermudo's problems. The paper concludes in Section 5, which outlines several potential directions for future research one of these directions involves the novel concept of SDCTD sets.

2. Preliminaries

Throughout this paper, we tacitly assume that all graphs are simple, undirected, and finite. For a graph G, the complement graph $\overline{G}$ has the same vertex set as G, with two vertices adjacent in $\overline{G}$ if and only if they are non-adjacent in G. The distance $d_G(u, v)$ between vertices u and v is the length of the shortest path between them. If no path exists, $d_G(u, v) = \infty$. The diameter of G, denoted $diam(G)$, is the maximum distance between any two vertices in G. The open neighborhood of a vertex v in G, denoted $N_G(v)$, is the set of vertices adjacent to v, while the closed neighborhood $N_G[v] = N_G(v) \cup v$. A vertex is universal if $N_G[v] = V(G)$, and isolated if $N_G(v) = \emptyset$. An edge subset is called a perfect matching if every vertex in E is incident with exactly one edge in this edge subset. In other words, it can pair the vertices in E two by two, and there is an edge in G connecting each pair of vertices. For other useful concepts in this paper, refer to References [3, 4].

2.1. Products of Graphs

Graph products construct a new graph from two graphs G and H with vertex set $V(G) \times V(H)$, distinct products differ in edge definition, each yielding a unique edge set. Two vertices (g, h) and (g', h') are adjacent in the *Cartesian product graph* $G \square H$ if and only if $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$, these edges are called Cartesian edges. Two vertices (g, h) and (g', h') are adjacent in the *Direct product graph* $G \times H$ if and only if $gg' \in E(G)$ and $hh' \in E(H)$, these edges are referred to as direct edges. Two vertices (g, h) and (g', h') are adjacent in the *Modular product graph* $G \diamond H$ if and only if $g = g'$ and $hh' \in E(H)$ (Cartesian edge), or $h = h'$ and $gg' \in E(G)$ (Cartesian edge), or $gg' \in E(G)$ and $hh' \in E(H)$ (direct edge), or $g \neq g'$, $h \neq h'$, $gg' \notin E(G)$ and $hh' \notin E(H)$ (co-direct edge, from $\overline{G} \times \overline{H}$). Modular Product graph $P_3 \diamond P_2$, $C_3 \diamond P_2$, and $K_4 \diamond S_{1,3}$ are depicted in Figure 1.



(a) $P_3 \diamond P_2$ (b) $C_3 \diamond P_2$

(c) $K_4 \diamond S_{1,3}$

Figure 1. Modular Product graph $P_3 \diamond P_2$, $C_3 \diamond P_2$, and $K_4 \diamond S_{1,3}$.

Based on the definition of the modular product graph $G \diamond H$, note that the three sets whose union constitutes the edge sets $E(G \diamond H)$ are pairwise disjoint. Thus, the edge set of the modular product is

$$E(G \diamond H) = E(G \Box H) \cup E(G \times H) \cup E(\overline{G} \times \overline{H}).$$

From the definition, the closed neighborhood of a vertex in the modular product is immediately given by

$$N_{G \diamond H}[(g,h)] = (N_G[g] \times N_H[h]) \cup (\overline{N_G[g]} \times \overline{N_H[h]}).$$

2.2. Key Vertex Sets and Their Cardinalities

Several specialized vertex sets play a central role in graph domination theory. This subsection mainly introduces the concepts of vertex subsets such as the dominating set, total dominating set (TD set), efficient closed dominating set (ECD set), and simultaneously dominating and complement total dominating set (SDCTD set).
A vertex subset $D \subseteq V(G)$ is a dominating set if every vertex not in D has at least one neighbor in D. The minimum cardinality of a dominating set is the domination number, denoted by

$$\gamma(G) = \min\{|D| : D \text{ is a dominating set of } G\}.$$

A vertex subset $D \subseteq V(G)$ is a total dominating set (TD set) if every vertex in G is adjacent to some vertex in D. The minimum cardinality of a TD set is the total domination number, denoted by

$$\gamma_t(G) = \min\{|D| \mid D \text{ is a } TD \text{ set of } G\}.$$

If a dominating set D of a graph G satisfies that the family of its closed neighborhoods $\{N_G[V] : v \in D\}$ is a partition of the vertex set $V(G)$, then D is called an efficiently closed dominating set (ECD set), also known as a 1-perfect code, and the graph G is called an ECD graph. Additionally, G has no 2-ECD set, that is, for any $u, v \in V(G)$, $N_G[u] \cup N_G[v] \neq V(G)$ or $N_G[u] \cap N_G[v] \neq \emptyset$.
A set $D \subseteq V(G)$ is a Simultaneously Dominating and Complement TD Set (SDCTD set) of a graph G if and only if D is both a dominating set of G and a TD set of the complement graph $\overline{G}$. The minimum cardinality of an SDCTD set of G is called the SDCTD number of G, denoted by

$$\overline{\gamma}(G) = \min\{|D| : D \text{ is an } SDCTD \text{ set of } G\}.$$

It is straightforward to observe that the SDCTD number exists if and only if $\overline{G}$ has no isolated vertex, which is equivalent to G having no universal vertex.

A set D of vertices in a graph G is a paired domination set(PD set) if every vertex of G is adjacent to a vertex in D and the subgraph induced by D contains a perfect matching (not necessarily as an induced subgraph). Necessarily, the paired domination number of a graph is an even integer. The minimum cardinality of a PD set of G is the paired domination number, denoted by

$$\gamma_{pr}(G) = \min\{|D| \mid D \text{ is a } PD \text{ set of } G\}.$$

These sets form a foundation for analyzing the structural properties of graphs and their products, particularly in the context of the modular product's relationships among vertex sets and its cardinality explored in subsequent sections.

3. On the Relationships Among Vertex Sets

3.1. SDCTD set and PD set

To develop network vertex sets with both coverage and symmetry, this subsection establishes some conditions under which an SDCTD set of a graph qualifies as a PD set. Subsequently, in a general graph G, we first establish the conditions for an SDCTD set to be a PD set, and then specifically derive such conditions for regular graphs and bipartite graphs.

Theorem 1. *An SDCTD set of a graph is a PD set if and only if $|D|$ is even and the induced subgraph $G[D]$ contains a perfect matching.*

Proof. ($\Leftarrow$) Assume D is even and $G[D]$ contains a perfect matching. Since D is even, let $D = \{v_1, v_2, \ldots, v_{2k}\}$. The perfect matching in $G[D]$ implies there exist k edges $e_1 = (v_{i_1}, v_{j_1}), e_2 = (v_{i_2}, v_{j_2}), \ldots, e_k = (v_{i_k}, v_{j_k})$ that cover all vertices of D with no shared vertices. From the definition of a PD set, the vertex pairs $\{(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}), \ldots, (v_{i_k}, v_{j_k})\}$ form a partition of D, where each pair is adjacent in G. Thus, D is a PD set.
($\Rightarrow$) First, prove $G[D]$ is even. Given D is a PD set, it partitions into disjoint vertex pairs $\{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$ with $u_i \sim v_i$ in G for $i = 1, 2, \ldots, k$. Since these pairs are disjoint and cover D, we have $|D| = 2k$, so D is even.
Next, prove $G[D]$ contains a perfect matching. For the partition $\{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$, $u_i \sim v_i$ in G implies $(u_i, v_i) \in E(G[D])$ by the definition of an induced subgraph. Let $M = \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$. Then M is a perfect matching in $G[D]$, as each vertex in D is incident to exactly one edge in M. □

Corollary 1. *If the SDCTD set D of G is a clique and $|D|$ is even, then D be a PD set.*

Theorem 2. *Let D is an SDCTD set of G. Then D is a PD set if and only if $\forall S \subseteq D$, $o(G[D \setminus S]) \leq |S|$.*

Proof. Suppose D is a PD set. By definition, the induced subgraph $G[D]$ contains a perfect matching. Since $H = G[D]$ has a perfect matching, Tutte's Theorem guarantees that for any subset $S \subseteq V(H) = D$, $o(H - S) = o(G[D \setminus S]) \leq |S|$, where $o(H - S)$ denotes the number of odd connected components in $H - S$.
Conversely, assume that for every $S \subseteq D$, $o(G[D \setminus S]) \leq |S|$. The premise ensures that for any $S \subseteq D$, the inequality $o(G[D \setminus S]) \leq |S|$ holds, which is exactly Tutte's condition for H. By Tutte's Theorem, $G[D]$ contains a perfect matching. Thus, D can be partitioned into adjacent vertex pairs in G, confirming D is a PD set. □

The following research proposes conditions for the existence of PD sets in regular graph and bipartite graph.

Theorem 3. *Let G be a k-regular graph and D be its SDCTD set. If $|D|$ is even and $\delta(\overline{G}[D]) \geq 2$, then D be a PD set.*

Proof. Let G be a k-regular graph and D be its SDCTD set with $|D| = 2k$ ($k \in \mathbb{N}$) and $\delta(\overline{G}[D]) \geq 2$. For any vertex $v \in D$, the degree in $G[D]$ satisfies $d_{G[D]}(v) = |D| - 1 - d_{\overline{G}[D]}(v) \leq 2k - 3$. In the complement subgraph $\overline{G}[D]$, each vertex has at least 2 neighbors, so each vertex in $G[D]$ has at most $|D| - 3$ neighbors. Assume for contradiction there exists $S \subseteq D$ with $o(G[D \setminus S]) = t > |S|$, so $t \geq |S| + 1$. Each odd component C_i ($|C_i|$ odd) has a vertex $v_i \in C_i$ adjacent to S in $\overline{G}[D]$. The t odd components require at least t edges from S in $\overline{G}[D]$, but $|S| \cdot (|D| - 1) \geq t \geq |S| + 1$, so $|S| \cdot (2k - 2) \geq 1$. Since $k \geq 1$, $|S| \geq \frac{1}{2k-2}$ holds trivially, leading to a contradiction. By Tutte's Theorem, $G[D]$ has a perfect matching. Hence, D partitions into disjoint vertex pairs adjacent in G, D is a PD set. □

Theorem 4. *Let $G = (X, Y, E)$ be a bipartite graph and $D \subseteq X$ an SDCTD set. Then D is a PD set if and only if $|D|$ is even and for any subset $S \subseteq D$, the non-neighborhood $|\overline{N}(S)| \geq |S|$.*

Proof. If D is a PD set, $G[D]$ contains a perfect matching, so $|D| = 2k$ is even. Suppose for contradiction there exists $S \subseteq D$ with $|\overline{N}(S)| < |S|$. Then $|N(S)| = |Y| - |\overline{N}(S)| > |Y| - |S|$. We construct an auxiliary bipartite graph $H = (D, Y, E')$ with E' as edges between D and Y in G. For D to be PD set, each pair $(u, v) \subseteq D$ share a common neighbor $y \in Y$, i.e., $uy, vy \in E'$. By Hall's theorem on H, a matching covering D requires $|N_H(S)| \geq |S|$ for all $S \subseteq D$. Since D is paired set, each $y \in Y$ can match at most two vertices in D, so, $|Y| \geq |N_H(S)| \geq \frac{|S|}{2}$. If $|N_H(S)| \leq |S|$, then $|N_H(S)| \geq |Y| - |S|$, which leads to $|Y| > |Y| - |S| + \frac{|S|}{2}$, so $0 > \frac{|S|}{2}$ is a contradiction. Therefore, we have $|\overline{N}(S)| \geq |S|$ for all $S \subseteq D$.

Conversely, let $|D| = 2k$ and $|\overline{N}(S)| \geq |S|$ for all $S \subseteq D$, we need to show that D is a PD set. We duplicate each vertex $y \in Y$ to two vertices y_1 and y_2, forming a new vertex set $Y' = \{y_1, y_2 \mid y \in Y,\ y_1 \text{ and } y_2 \text{ are duplicated from } y\}$. Define the auxiliary bipartite graph $H' = (D, Y', E'')$, where an edge $uy_i \in E''$ if and only if

$uy \in E$ for $i = 1, 2$. For any subset $S \subseteq D$, we have $|N_{H'}(S)| = 2|N_H(S)|$, where $N_H(S)$ is the neighborhood of S in the auxiliary graph $H = (D, Y, E)$.

For any $S \subseteq D$, from the condition $|\overline{N}(S)| \geq |S|$, we have $|N_H(S)| = |Y| - |\overline{N}(S)| \leq |Y| - |S|$. Thus $|N_{H'}(S)| = 2|N_H(S)| \leq 2(|Y| - |S|)$. We aim to show $|N_{H'}(S)| \geq |S|$.

Case 1. From $|N_{H'}(S)| = 2|N_H(S)| \leq 2(|Y| - |S|)$, we need $2(|Y| - |S|) \geq |S|$, which simplifies to $|Y| \geq \frac{3|S|}{2}$. Since $|S| \leq \frac{2|Y|}{3}$, multiplying both sides by $\frac{3}{2}$ gives $|Y| \geq \frac{3|S|}{2}$. Thus, $2(|Y| - |S|) \geq 2(\frac{3|S|}{2} - |S|) = |S|$, so $|N_{H'}(S)| \geq |S|$.

Case 2. By the condition $|\overline{N}(S)| \geq |S|$ and since $|S| > \frac{2|Y|}{3}$, $|N_H(S)| = |Y| - |\overline{N}(S)| \geq |Y| - |S| > |Y| - \frac{2|Y|}{3} = \frac{|Y|}{3}$. Also, $D \subseteq X$ in bipartite graph $G = (X, Y, E)$, so $|D| = 2k \leq 2|Y|$. Otherwise, no matching covering D could exist in G. Thus, $|S| \leq |D| \leq 2|Y|$, implying $\frac{|S|}{2} \leq |Y|$. So, $|N_H(S)| \geq |Y| - |S| > \frac{|Y|}{3} \geq \frac{|S|}{2}$. However, based on our assumption $|\overline{N}(S)| \geq |S|$, we have $|Y| = |\overline{N}(S)| + |N_H(S)| > |N_H(S)| + |S|$. Substituting the inequality $|N_H(S)| > \frac{|Y|}{3}$ into the above expression, we obtain $|Y| > \frac{|Y|}{3} + |S|$. Rearranging this inequality gives $\frac{2|Y|}{3} > |S|$, which contradicts the prior condition $|S| > \frac{2|Y|}{3}$.

By Hall's theorem, H' has a matching covering D. Each $y \in Y$ matches at most 2 vertices in D, so D partitions into k pairs $\{u_1, v_1\}, \cdots, \{u_k, v_k\}$, where each pair shares a common neighbor in Y. This partition forms a perfect matching in $G[D]$. Thus, D is a PD set. □

Corollary 2. *For the complete bipartite graph $K_{m,n}$, an SDCTD set D is a PD set if and only if $|D|$ is even and $|D| \leq 2\min(m, n)$.*

Proof. If D is a PD set, by Theorem 4, $|D|$ must be even. Let $|D| = 2k$, where k is a positive integer.
In $K_{m,n}$, each pair of paired vertices must span the bipartition, so D contains k vertices from X and k from Y. Thus, $k \leq \min(m, n)$, so$|D| = 2k \leq 2\min(m, n)$.

Conversely, let $|D| = 2k \leq 2\min(m, n)$. Construct D with k vertices in X and k in Y. For any $S \subseteq D$ with s vertices from X and t from Y, the non-neighborhood $\overline{N}(S)$ consists of uncovered vertices in the opposite partitions, totaling $(m - k) + (n - k)$. Since $m \geq k$ and $n \geq k$, we have$|\overline{N}(S)| \geq |S|$. By Theorem 4, D is a PD set. □

Corollary 3. *For the complete bipartite graph $K_{m,n}$, $\gamma_{pr}(K_{m,n}) \leq 2\min(m, n)$.*

3.2. SDCTD set and ECD set

In some common graphs, there exist sets that are both ECD sets and SDCTD sets. For example, in the complete graph $K_n (n \geq 3)$, taking $D = V(K_n)$ makes D both an ECD set and an SDCTD set of K_n. For the cycle graph $C_{3k} (k \in N^+)$, the constructed vertex subset $D = \{v_1, v_4, \ldots, v_{3k-2}\}$ is both an ECD set and an SDCTD set. In the complete bipartite graph $K_{m,n} (m, n \geq 2)$, let the two vertex partition sets of $K_{m,n}$ be X and Y, with $|A| = m$ and $|B| = n$. Taking $D = A$ (or $D = B$) makes D both an ECD set and an SDCTD set of $K_{m,n}$. We now address the relationships between SDCTD sets and ECD sets in Cartesian product graphs and direct product graphs.

Theorem 5. *If G and H are ECD graphs and their ECD sets D_G, D_H are Orthogonal Projection, then $G\Box H$ is an ECD graph with ECD set $D = D_G \times D_H$.*

Proof. For any $(g, h) \in V(G\Box H)$, since D_G is a dominating set of G, there exists $g' \in D_G$ such that $g \in N_G[g']$. Similarly, there exists $h' \in D_H$ such that $h \in N_H[h']$. So $(g, h) \in N_G[g'] \times N_H[h']$, meaning D dominates $G\Box H$. Suppose there exist $(g_1, h_1) \in D$ and $(g_2, h_2) \in D$ such that $(N_G[g_1] \times N_H[h_1]) \cap (N_G[g_2] \times N_H[h_2]) \neq \emptyset$. Then there exist $g \in N_G[g_1] \cap N_G[g_2]$ and $h \in N_H[h_1] \cap N_H[h_2]$. Since D_G is an ECD set of G, $N_G[g_1] \cap N_G[g_2] \neq \emptyset$ if and only if $g_1 = g_2$. Similarly, $h_1 = h_2$. Thus, $(g_1, h_1) = (g_2, h_2)$, indicating that distinct closed neighborhoods are disjoint. Therefore, $\{N_{G\Box H}[(g, h)] | (g, h) \in D\}$ is a partition of $V(G\Box H)$, so D is an ECD set. □

Theorem 6. *Let $diam(G) \leq 2$ and $diam(H) \leq 2$. If both G and H have SDCTD sets, then $G\Box H$ has an SDCTD set D such that $|D| \leq \overline{\gamma}(G) \cdot \overline{\gamma}(H)$.*

Proof. Let D_G, D_H be the minimum SDCTD sets of G and H, respectively. Define $D = D_G \times D_H$. For any $(g, h) \in V(G\Box H)$:

Case 1. If $g \in D_G$, since D_G being a dominating set of G, there exists $g' \in D_G$ with $gg' \in E(G)$. By the definition of the Cartesian product graph, $(g, h) \sim (g', h) \in G\Box H$.

Case 2. If $h \in D_H$, analogously, there exists $h' \in D_H$ such that $(g, h) \sim (g, h')$. Thus, D is a dominating set of $G\Box H$.

In the complement graph $\overline{G}\Box\overline{H}$, $(g, h) \sim (g', h')$ if and only if $gg' \notin E(G)$ and $h = h'$, or $hh' \notin E(H)$ and $g = g'$. Since $diam(G) \leq 2$ and D_G is an SDCTD set, every $g \in D_G$ has a neighbor in D_G within $\overline{G}$. Similarly, because $diam(H) \leq 2$ and D_H is an SDCTD set of H, for any $h \in D_H$, h has a neighbor in $D_H \in \overline{H}$. Thus, each $(g, h) \in D$ has a neighbor in D, which means D is a TD set of $\overline{G}\Box\overline{H}$. Therefore, $G\Box H$ has an SDCTD set $D = D_G \times D_H$ with $|D| \leq \overline{\gamma}(G) \cdot \overline{\gamma}(H)$. □

Corollary 4. *If $\gamma(G) = \overline{\gamma}(G)$ and $\gamma(H) = \overline{\gamma}(H)$, then $G\Box H$ has SDCTD set D such that $|D| \leq \gamma(G)\gamma(H)$.*

According to the above corollary, if both G and H that satisfying the conditions are bipartite graphs, then the SDCTD set D of $G\Box H$ induces a bipartite graph.
It is also straightforward to observe that the following conclusion holds for hypercubes.

Theorem 7. *A hypercube Q_n has an ECD set D is an SDCTD set if and only if $n = 2^m - 1 (m \geq 1)$.*

Proof. In Part 4 of Reference [8], the necessary and sufficient conditions for the existence of perfect codes in hypercubes have been discussed, and the proof of necessity is omitted. When $n = 2^m - 1$, construct the perfect code D corresponding to the Hamming code for $n = 2^m - 1 (m \geq 1)$ is the set of all binary vectors in $\{0, 1\}^n$ with even Hamming weight, which satisfies D is an ECD set of Q_n (the closed neighborhoods partition the vertex set). The distance between any two vertices in $D \in Q_n$ in has minimum distance of 3, so they are adjacent in $\overline{Q_n}$. Thus, each vertex in D has neighbors in $\overline{Q_n}$, meaning D is an SDCTD set. □

We introduce a new concept of orthogonal projection for the Cartesian product graph $G \square H$. If the vertex set of the Cartesian product graph $G \square H$ can be expressed as $V(G \square H) = \bigcup_{(g,h) \in D_G \times D_H} (N_G[g] \times N_H[h])$ and the closed neighborhoods $N_G[g] \times N_H[h]$ are pairwise disjoint, then D_G and D_H are said to be Orthogonal Projections in the Cartesian product graph $G \square H$. In this case, $D = D_G \times D_H$ forms an ECD set of $G \square H$.

For example, the ECD set $D_G = D_H = \{(000), (111)\}$ of Q_3 is the Hamming code. In the Cartesian product, define

$$D = D_G \times D_H = \{(000, 000), (000, 111), (111, 000), (111, 111)\}.$$

Then $D = D_G \times D_H$ is an ECD set for $Q_3 \square Q_3$, confirming it is an ECD graph.

Theorem 8. *If the closed neighborhoods of the ECD sets D_G and D_H of graphs G and H form partitions of $V(G)$ and $V(H)$, respectively, then the ECD set of $G \times H$ is the set $D_G \times D_H$, and the closed neighborhoods of $D_G \times D_H$ form a partition of $V(G \times H)$.*

Proof. Take any $(g, h) \in V(G \times H)$. Since D_G is a dominating set of G, there exists $g' \in D_G$ such that $g \in N_G[g']$. Similarly, there exists $h' \in D_H$ such that $h \in N_H[h']$. By the definition of the direct product, if $g = g'$ and $h = h'$, then (g, h) is adjacent to or equal to (g', h'), so $(g, h) \in N_{G \times H}[(g', h')]$. Thus, D is a dominating set of $G \times H$. Suppose there exist $(g_1, h_1), (g_2, h_2) \in D$ with $(g_1, h_1) \neq (g_2, h_2)$ such that $N_{G \times H}[(g_1, h_1)] \cap N_{G \times H}[(g_2, h_2)] \neq \emptyset$. Then there exists $(g, h) \in V(G \times H)$ satisfying $g \in N_G[g_1] \cap N_G[g_2]$ and $h \in N_H[h_1] \cap N_H[h_2]$. By the ECD set properties of G and H, $N_G[g_1] \cap N_G[g_2] = \emptyset$, if $g_1 \neq g_2$, and $N_H[h_1] \cap N_H[h_2] = \emptyset$, if $h_1 \neq h_2$, implying $g_1 = g_2$ and $h_1 = h_2$, a contradiction. Hence, distinct closed neighborhoods are pairwise disjoint.

By dominance, all closed neighborhoods of vertices in D cove $V(G \times H)$. By disjointness, they form a partition of $V(G \times H)$. Therefore, D is an ECD set of $G \times H$. □

Theorem 9. *If $\gamma_t(\overline{G}) = \overline{\gamma}(G)$ and $\gamma_t(\overline{H}) = \overline{\gamma}(H)$, then an SDCTD set of $G \times H$ is given by $D_G \times D_H$, where D_G and D_H are the minimum TD sets of $\overline{G}$ and $\overline{H}$, respectively.*

Proof. We show $D_G \times D_H$ is a dominating set of $G \times H$. For any vertex $(g, h) \in V(G \times H)$, if $g \notin D_G$, there exists $g' \in D_G$ with $gg' \in E(G)$. Similarly, if $h \notin D_H$, there exists $h' \in D_H$ with $hh' \in E(H)$. By the direct product edge definition, $(g,h) \sim (g',h')$ when $gg' \in E(G)$ and $hh' \in E(H)$, so $D_G \times D_H$ is a dominating set of $G \times H$.

Next, we prove $D_G \times D_H$ is a TD set of $\overline{G} \times \overline{H}$, i.e., for any $(g,h) \in D_G \times D_H$, there exists $(g',h') \in D_G \times D_H$ adjacent to it in $\overline{G} \times \overline{H}$. Since D_G is a TD set of $\overline{G}$ guaranteed by $\gamma_t(\overline{G}) = \overline{\gamma}(G)$, for any $g \in D_g$, there exists $g' \in D_G$ with $gg' \notin E(G)$. Similarly, for any $h \in D_H$, there exists $h' \in D_H$ with $hh' \notin E(H)$. Consider two cases.

Case 1. Take $g' \in D_G$ such that $gg' \notin E(G)$. For any $h \in D_H$, $(g,h) \sim (g',h)$ in $\overline{G} \times \overline{H}$.

Case 2. Take $h' \in D_H$ such that $hh' \notin E(H)$. For any $g \in D_G$, $(g,h) \sim (g,h')$ in $\overline{G} \times \overline{H}$.

Thus, every vertex in $D_G \times D_H$ has a neighbor within this set in $\overline{G} \times \overline{H}$, satisfying the total domination condition.
So, $D_G \times D_H$ is both a dominating set of $G \times H$ and a TD set of $\overline{G} \times \overline{H}$, hence an SDCTD set of $G \times H$. □

Corollary 5. *If hold the theorem 9's conditions, then* $|SDCTD(G \times H)| = \gamma_t(\overline{G}) \cdot \gamma_t(\overline{H}) = \overline{\gamma}(G) \cdot \overline{\gamma}(H)$.

Proof. Based on the above theorem, we only prove the minimum of the cardinality of the SDCTD set. Suppose there exists a smaller SDCTD set D'. Let $D'_G = \{g | \exists h, (g,h) \in D'\}$ and $D'_H = \{h | \exists g, (g,h) \in D'\}$ be the projection components onto G and H, respectively. They must satisfy D'_G is a TD set of $\overline{G}$, so $|D'_G| \geq \gamma_t(\overline{G})$. D'_H is a TD set of $\overline{H}$, so $|D'_H| \geq \gamma_t(\overline{H})$. Therefore, $|D'| \geq |D'_G| \cdot |D'_H|$, meaning $D_G \times D_H$ is the SDCTD set with minimum cardinality. □

Corollary 6. *If hold the theorem 9's conditions, then* $\overline{\gamma}(G \times H) \leq \overline{\gamma}(G) \cdot \overline{\gamma}(H)$.

Building on the above analysis, we characterize the graph classes for which the total domination number of the complement graph is equal to the SDCTD number, thereby leading to the following conclusions.

Theorem 10. *Let D be a minimum TD set of $\overline{G}$. Then $\gamma_t(\overline{G}) = \overline{\gamma}(G)$ if and only if for any $v \in (V(G) \setminus D), N_G(v) \cap D \neq \emptyset$.*

Proof. Let $\gamma_t(\overline{G}) = \overline{\gamma}(G) = k$. Since D is a minimum TD set of $\overline{G}$, we have $|D| = k$. Assume for contradiction that there exists $v_0 \in V(G) \setminus D$ such that $N_G(v_0) \cap D = \emptyset$, meaning v_0 is not adjacent to any vertex in D in G. By the definition of the complement graph, v_0 is adjacent to all vertices in D in $\overline{G}$, i.e., $N_{\overline{G}}(v_0) \supseteq D$. Although D is a TD set of $\overline{G}$, for any $u \in D$, there exists $u' \in D$ with $uu' \in E(\overline{G})$, the key contradiction arises $v_0 \notin D$ and $N_G(v_0) \cap D = \emptyset$, mean v_0 is not dominated by D in G, conflicting with D being an SDCTD set which requires domination in G. Therefore, the assumption is false, and no such v_0 exists, confirming D is a dominating set of G.

Conversely, assume that D is a minimum TD set of $\overline{G}$, i.e., $|D| = \gamma_t(\overline{G})$. For any $u \in D$, there exists $u' \in D$ such that $uu' \in E(\overline{G})$. For any $v \in V(G)\setminus D$, $N_G(v) \cap D \neq \emptyset$ i.e., D is a dominating set of G. For any $v \in V(G)$, if $v \in D$, it is trivially dominated. If $v \notin S$, the premise ensures $N_G(v) \cap D \neq \emptyset$, then D is a dominating set of G.
Since D is a TD set of $\overline{G}$, its definition guarantees that for any $u \in D$, there is a neighbor $u' \in D$ in $\overline{G}$, satisfying the total domination condition for the complement.
Assume there exists a set $T \subset V(G)$ with $|T| < |S|$ that is an SDCTD set of G. Then T must be a TD set of $\overline{G}$, contradicting the minimality of D as the minimum TD set of $\overline{G}$. Hence, $\gamma_t(\overline{G}) = |D| = \overline{\gamma}(G)$. □

4. Domination number of $G \diamond H$

Theorem 11. *If $diam(H) = 2$, then $\gamma_t(\overline{H}) = 3$.*

Proof. We prove $\gamma_t(\overline{H}) \geq 3$ by contradiction. Suppose $\gamma_t(\overline{H}) \leq 2$, so there exists a TD set $D = \{u, v\}$.
In $\overline{H}$, every vertex must be adjacent to u or v. In H, if $u \sim v$, then $u \nsim v$ in $\overline{H}$. If such $w \in H$ exists which it does because $diam(H) = 2$, take w that is adjacent to both u and v. Then in $\overline{H}$, $w \nsim u$ and $w \nsim v$, meaning w is not dominated by D, a contradiction.
If there exist vertices w_1, w_2 in H such that $w_1 \sim u$, $w_1 \nsim v$ and $w_2 \sim v$, $w_2 \nsim u$, then the distance between w_1 and w_2 in H is at least 3, contradicting $diam(H) = 2$. Thus, the assumption fails, so $\gamma_t(\overline{H}) \geq 3$.
Next, we prove $\gamma_t(\overline{H}) \leq 3$ by constructing a TD set. Since $diam(H) = 2$ and H is not a complete graph, there exists a pair of non-adjacent vertices $u, v \in V(H)$, such that $u \sim w$ and $v \sim w$ in H. Construct $D = \{u, v, w\}$, which is a TD set for $\overline{H}$.
Since $u \nsim v$ in H, then $u \sim v$ in $\overline{H}$, so u and v dominate each other. Since $u \sim w$ and $v \sim w$ in H, then $u \nsim w$ and $v \nsim w$ in $\overline{H}$. For any vertex $x \in V(H)$, if $x \nsim u$ in H, then $x \sim u$ in $\overline{H}$, dominated by u. If $x \sim u$ in H, but $x \nsim v$ in H, then $x \sim v$ in $\overline{H}$, dominated by v. If x is adjacent to both u and v, consider $x \sim w$ in H. If $x \nsim w$ in H, then $x \sim w$ in $\overline{H}$, dominated by w. Since every vertex in $\overline{H}$ is adjacent to at least one vertex in D and since u and v dominate each other, $D = \{u, v, w\}$ is TD set for $\overline{H}$, so $\gamma_t(\overline{H}) \leq 3$. Combining both inequalities, $\gamma_t(\overline{H}) = 3$. □

Corollary 7. *If both graphs G and H have diameter 2, then $\gamma(G \diamond H) \leq \min\{\gamma(G) + \gamma(H) - 1, 5\}$.*

Proof. By Theorem 8 in the [1], the upper bound for the domination number of a modular product graph is given by $\gamma(G \diamond H) \leq \min\{\gamma(G) + \gamma(H) - 1, \gamma_t(\overline{G}) + \gamma_t(\overline{H}) - 1\}$. By the above theorem 11, when G and H have diameter 2, their complement graphs satisfy $\gamma_t(\overline{G}) = \gamma_t(\overline{H}) = 3$. Substituting these values yields $\gamma_t(\overline{G}) + \gamma_t(\overline{H}) - 1 = 3 + 3 - 1 = 5$. Thus, $\gamma(G \diamond H) \leq \min\{\gamma(G) + \gamma(H) - 1, 5\}$. □

For graph classes with low tree-width and hierarchical decompositions, we employ a hierarchical construction method based on tree decomposition: by averaging the cardinality of the SDCTD set across layers, we construct a snake-like cross-dominating set, thereby deriving an upper bound for the domination number of modular product graphs. First, we present the definition of tree decomposition and that of the hierarchical partition of the SDCTD set D_G with respect to tree decomposition levels.

Conduct a BFS on the tree decomposition T_G of graph G, dividing the nodes into L_G layers $\{L_1^G, L_2^G, \cdots, L_{l_G}^G\}$. Each layer L_i^G corresponds to a vertex subset $V_i^G \subseteq V(G)$, satisfying the connectivity condition of the tree decomposition. Similarly, define the hierarchical layers of as $\{L_1^H, L_2^H, \cdots, L_{l_H}^H\}$.

The corresponding SDCTD set D_G is partitioned into l_G subsets $D_G^1, D_G^2, \cdots, D_G^{l_G}$ according to the tree decomposition levels, satisfying $\bigcup_{i=1}^{l_G} D_G^i = D_G$ and $\sum_{i=1}^{l_G} |D_G^i| = \overline{\gamma}(G)$. Analogously, partition D_H into l_H subsets $D_H^1, D_H^2, \cdots, D_H^{l_H}$.
Additionally, we have obtained the following results, which lay a foundation for the subsequent discussion.

Observation 12 ([1]). A vertex (g, h) is dominated by (g', h') in $G \diamond H$ if and only if either $g \in N_G[g']$ and $h \in N_H[h']$ or $g \notin N_G[g']$ and $h \notin N_H[h']$.

Theorem 13. *If the tree decompositions of G and H have l_G and l_H layers respectively, and SDCTD sets are constructed via hierarchical tree decomposition, then $\gamma(G \diamond H) \leq \min\{\frac{\overline{\gamma}(G)}{l_G}, \frac{\overline{\gamma}(H)}{l_H}\} \cdot (l_G + l_H - 1)$. When $l_G = l_H = l$, we have $\gamma(G \diamond H) \leq \min\{\overline{\gamma}(G), \overline{\gamma}(H)\} \cdot \frac{2l-1}{l}$.*

Proof. First, construct the hierarchical partition of the SDCTD sets. For graph G, by the Pigeonhole Principle, there exists at least one hierarchical partition such that the size of each SDCTD subset satisfies $|\ D_G^i\ | \leq \lceil \frac{\overline{\gamma}(G)}{l_G} \rceil$. Similarly, $|\ D_H^j\ | \leq \lceil \frac{\overline{\gamma}(H)}{l_H} \rceil$. Let $k_G = \lceil \frac{\overline{\gamma}(G)}{l_G} \rceil$ and $k_H = \lceil \frac{\overline{\gamma}(H)}{l_H} \rceil$, so $k_G \geq \frac{\overline{\gamma}(G)}{l_G}$ and $k_H \geq \frac{\overline{\gamma}(H)}{l_H}$.
Next, construct the snake-shaped set D, defined as the union of inter-layer cross vertex pairs $D = \bigcup_{\substack{1 \leq i \leq l_G \\ 1 \leq j \leq l_H \\ i+j \leq l_G + l_H - 1}} \left(D_G^i \times D_H^j \right)$.
For any vertex $(g, h) \in V(G \diamond H)$, let g be in layer i_0 of the tree decomposition, i.e., $g \in V_{i_0}^G$ and h in layer j_0, i.e., $h \in V_{j_0}^H$. Define $i' = \min\{i_0, l_G\}$, $j' = \min\{j_0, l_H\}$, and take $i = i'$, $j = j'$, so that $i + j \leq i_0 + j_0 \leq l_G + l_H$.
If $i + j \leq l_G + l_H - 1$, then (i, j) is within the construction scope of D. Since D_G is a dominating set of G, as D_G^i covers the neighborhood of layer i, there exists $g' \in D_G^i$ dominating g; Similarly, $h' \in D_H^j$ dominates h.
If $i + j = l_G + l_H$, then $i = l_G$ and $j = l_H$. By the layer connectivity of the tree decomposition, g is adjacent to a vertex in $D_G^{l_G}$, and h is adjacent to a vertex in $D_H^{l_H}$. Further, $(l_G, l_H - 1)$ or $(l_G - 1, l_H)$ falls within $D's$ construction scope, dominating (g, h) via co-direct edges.

By Observation 12, (g,h) is dominated by (g',h') if and only if $g \in N_G[g']$ and $h \in N_H[h']$ or $g \notin N_G[g']$ and $h \notin N_H[h']$. Since D_G and D_H are SDCTD sets, if $g \in N_G[g']$ and $h \in N_H[h']$, domination holds directly. If $g \notin N_G[g']$, then g' is adjacent to g in $\overline{G}$, and similarly for h' in $\overline{H}$, ensuring (g,h) is adjacent to (g',h') via co-direct edges.

The number of (i,j) pairs with $i+j \leq l_G + l_H - 1$ is $l_G + l_H - 1$, and each pair satisfies $\mid D_G^i \times D_H^j \mid \leq k_G \cdot k_H$. Let $k = \min\{k_G, k_H\}$, then $|D| \leq k \cdot (l_G + l_H - 1)$. Substituting $k_G \geq \frac{\overline{\gamma}(G)}{l_G}$ and $k_H \geq \frac{\overline{\gamma}(H)}{l_H}$, we get $\mid D \mid \leq \min\{\frac{\overline{\gamma}(G)}{l_G}, \frac{\overline{\gamma}(H)}{l_H}\} \cdot (l_G + l_H - 1)$. When $l_G = l_H = l$, $\mid D \mid \leq \min\{\frac{\overline{\gamma}(G)}{l_G}, \frac{\overline{\gamma}(H)}{l_H}\} \cdot (2l-1) \leq \min\{\overline{\gamma}(G), \overline{\gamma}(H)\} \cdot \frac{2l-1}{l}$. □

Subsequently, we have resolved the problem of determining the upper bound for the domination number of the modular product of two regular graphs.

Theorem 14. *Let G and H be k-regular graphs. Then $\gamma(G \diamond H) \leq 2k+1$.*

Proof. Let G and H be k-regular graphs. Define a dominating set as follows: choose $g_0 \in V(G)$, $h_0 \in V(H)$, and let $D = \{(g_0,h_0)\} \cup (N_G(g_0) \times \{h_0\}) \cup (\{g_0\} \times N_H(h_0))$, where $N_G(g_0), N_H(h_0)$ are neighborhoods of size k. Thus, $|D| = 2k+1$. For any vertex $(g,h) \in V(G \diamond H)$, we prove that (g,h) is dominated by at least one vertex in D. We consider three cases.

Case 1. $g = g_0$ or $h = h_0$. If $(g,h) = (g_0,h_0)$, it belongs to D and it is obviously dominated. If $(g,h) = (g_0,h)(h \neq h_0)$, since H is a k-regular graph, h and h_0 are either adjacent or non-adjacent. If $h \in N_H(h_0)$, then $(g_0,h) \in D$ is directly dominated. If $h \notin N_H(h_0)$, then (g_0,h) is adjacent to $(g_0,h_0) \in D$ via a Cartesian edge and the non-adjacency of h and h_0 does not affect the existence of the Cartesian edge. And since $(g_0,h_0) \in D$, (g_0,h) is dominated. If $(g,h) = (g,h_0)(g \neq g_0)$, by symmetry, when g is adjacent to g_0,$(g,h_0) \in D$. Otherwise, it is adjacent to (g_0,h_0) through a Cartesian edge and is dominated.

Case 2. $g \in N_G(g_0)$ and $h \in N_H(h_0)$. According to the edge definition of the modular product graph, (g,h) is adjacent to at least one vertex in D. By the definition of the Cartesian edge, (g,h) adjacent to $(g,h_0) \in D$, or (g,h) adjacent to $(g_0,h) \in D$. Thus, g,h is dominated by D.

Case 3. $g \notin N_G[g_0]$ and $h \notin N_H[h_0]$. Since $g \notin N_G[g_0]$, then $g \neq g_0$ and $g \nsim g_0$; Similarly, $h \neq h_0$ and $h \nsim h_0$. By the definition of co-direct edges in modular product graphs, when $g \nsim g_0$ and $h \nsim h_0$, (g,h) is adjacent to $(g_0,h_0) \in D$. Since $(g_0,h_0) \in D$, (g,h) is dominated.

Through the analysis of the above three cases, any vertex $(g,h) \in V(G \diamond H)$ is covered by the dominating set D, and $|D| = 2k+1$. Therefore, for any k-regular graphs G and H, we conclude $\gamma(G \diamond H) \leq 2k+1$. □

Proposition 1 ([1]). *Let G and H be two graphs such that they contains two adjacent vertices g_1, g_2 and h_1, h_2, respectively, such that $N_G(g_1) \cap N_G(g_2) = \emptyset$ and $N_H(h_1) \cap N_H(h_2) = \emptyset$. If $N_G(g_1) \cup N_G(g_2) \neq V(G)$ and $N_H(h_1) \cup N_H(h_2) \neq V(H)$, then $\gamma(G \diamond H) \leq 5$.*

Problem 1 ([1]). Find a pair of graphs G and H satisfying conditions of Proposition 1 for which $\gamma(G \diamond H) = 5$.

Based on the conditions established in Proposition 1, we characterized the graph pairs (G, H) satisfy $\gamma(G \diamond H) = 5$, thereby resolving Problem 1 in [1].

Theorem 15. *Under Proposition 1's conditions, $\gamma(G \diamond H) = 5$, if and only if there exist $u \in V(G) \setminus (N_G[g_1] \cup N_G[g_2])$ and $v \in V(H) \setminus (N_H[h_1] \cup N_H[h_2])$ such that (u, v) evades domination by any 4-vertex subset.*

Proof. Assume $\gamma(G \diamond H) = 5$. By definition, this means no 4-vertex subset of $V(G \diamond H)$ can dominate the graph. Thus, there must exist at least one vertex $(u, v) \in V(G \diamond H)$ that is not dominated by any 4-vertex set. Under the conditions of Proposition 1 in the [1], G and H have adjacent vertex pairs (g_1, g_2) and (h_1, h_2) with $N_G(g_1) \cap N_G(g_2) = \emptyset$, $N_H(h_1) \cap N_H(h_2) = \emptyset$, and $N_G(g_1) \cup N_G(g_2) \neq V(G)$, $N_H(h_1) \cup N_H(h_2) \neq V(H)$. Let $u \in V(G) \setminus (N_G(g_1) \cup N_G(g_2))$ and $v \in V(H) \setminus (N_H(h_1) \cup N_H(h_2))$. Such u and v exist by the problem 1's premise.

For (u, v) to not be dominated by any 4-vertex set, consider the control conditions in $G \diamond H$. (u, v) is dominated by (d_g, d_h) if and only if $u \in N_G[d_g]$ and $v \in N_H[d_h]$, or $u \notin N_G[d_g]$ and $v \notin N_H[d_h]$. Since $u \notin N_G[g_1] \cup N_G[g_2]$ and $v \notin N_H[h_1] \cup N_H[h_2]$, u and v cannot be dominated by vertices whose projections to G or H are within $\{g_1, g_2\}$ or $\{h_1, h_2\}$ via the first condition. The second condition requires $d_g \notin N_G[u]$ and $d_h \notin N_H[v]$, but with only 4 vertices, the projections cannot cover all cases, leaving (u, v) undominated.

Conversely, suppose there exist $u \in V(G) \setminus (N_G(g_1) \cup N_G(g_2))$ and $v \in V(H) \setminus (N_H(h_1) \cup N_H(h_2))$ such that (u, v) is not dominated by any 4-vertex set. By Proposition 1 [1], $\gamma(G \diamond H) \leq 5$.

To show $\gamma(G \diamond H) \geq 5$, we assume for contradiction that $\gamma(G \diamond H) \leq 4$, i.e., a 4-vertex set D dominates $G \diamond H$. For D to dominate (u, v), there must exist $(d_g, d_h) \in D$ such that $u \in N_G[d_g]$ and $v \in N_H[d_h]$, or $u \notin N_G[d_g]$ and $v \notin N_H[d_h]$. However, $u \notin N_G[g_1] \cup N_G[g_2]$ implies $u \notin N_G[d_g]$ for $d_g \in \{g_1, g_2\}$, and since $N_G[g_1] \cup N_G[g_2] \neq V(G)$, u may be an isolated vertex or far from g_1, g_2, making $N_G[u] = \emptyset$ or minimal. Similarly, $v \notin N_H[h_1] \cup N_H[h_2]$ implies $v \notin N_H[d_h]$ for $d_h \in \{h_1, h_2\}$, with $N_H[v] = \emptyset$ or minimal. If $d_g \notin \{g_1, g_2\}$ and $d_h \notin \{h_1, h_2\}$, then $u \notin N_G[d_g]$ and $v \notin N_H[d_h]$. With only 4 vertices, D cannot include enough vertices to satisfy $u \notin N_G[d_g]$ and $v \notin N_H[d_h]$ for all cases, contradicting the assumption that D dominates (u, v).
Thus, we have $\gamma(G \diamond H) \geq 5$, and combining with Proposition 1's upper bound, so $\gamma(G \diamond H) = 5$. □

For instance, in Figure 2, the modular product graph of the cycle graph C_{13} and the complete graph K_3 is constructed. These red vertices form a minimum dominating set $D = \{(0, 0), (1, 0), (4, 0), (7, 0), (10, 0)\}$. Consequently, it follows that $\gamma(C_{13} \diamond K_3) = 5$.

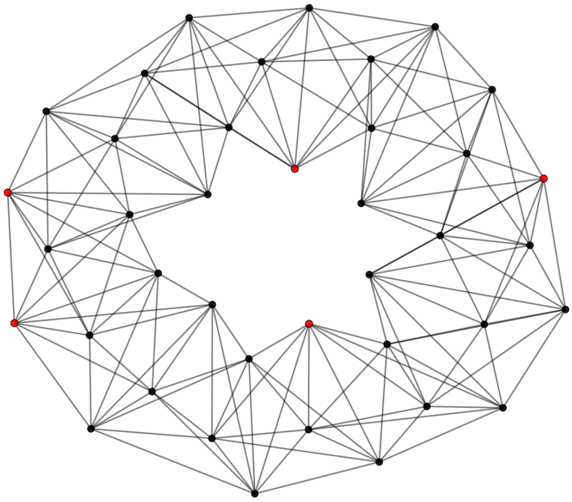


Figure 2. Illustration of the modular product graph $C_{13} \diamond K_3$. Vertices in the dominating set are highlighted in red.

Lemma 1 ([1]). *Let G and H be arbitrary graphs, then $\gamma(G \diamond H) \geq \max\{\overline{\gamma}(G), \overline{\gamma}(H)\}$.*

Lemma 2. *Let G be an arbitrary graph, then $\gamma(G \diamond G) \geq \overline{\gamma}(G)$. The necessary and sufficient condition for the equality to hold is that there exists $S \subseteq V(G)$ such that S is both a dominating set of G and a TD set of $\overline{G}$, and $\mid S \mid = \overline{\gamma}(G)$; For any $(u, v) \in V(G \diamond G)$, there exists $(s, t) \in S \times S$ such that $u \in N_G[s] \cup \{s\}$ and $v \in N_G[t] \cup \{t\}$, or $u \notin N_G[s]$ and $v \notin N_G[t]$.*

Proof. Let D be an arbitrary dominating set of $G \diamond G$, and define its projection onto G as $P_G(D) = \{u \in V(G) \mid \exists v \in V(G), (u, v) \in D\}$. For any vertex $(u, v) \in V(G \diamond G)$, there exists $(u', v') \in D$ such that $u = u'$ and $v \in N_G[v']$, or $v = v'$ and $u \in N_G[u']$, or $u \notin N_G[u']$ and $v \notin N_G[v']$.

As a dominating set of G, for any $u \in V(G)$, take $V \in V(G)$, then (u, v) is dominated by D. If dominated via a Cartesian edge or a direct edge, there exists $u' \in P_G(D)$ such that $u \in N_G[u']$ or $v \in N_G[v']$. In particular, when v is arbitrary, $P_G(D)$ must cover the neighborhoods of all vertices in G, meaning $P_G(D)$ is a dominating set of G. As a TD set of $\overline{G}$, for any $u \in V(G)$, take $v \in V(G)$ such that $v \notin N_G[u]$, then (u, v) is dominated by D. If dominated via a co-direct edge, there exists $(u', v') \in D$ such that $u \notin N_G[u']$ and $v \notin N_G[v']$. Here, $u \notin N_G[u']$ implies u' is adjacent to u in $\overline{G}$, so $P_G(D)$ is a TD set of $\overline{G}$.

By the definition of $\overline{\gamma}(G)$, it is the minimal size of a set satisfying both conditions above. Since $P_G(D)$ is both a dominating set of G and a TD set of $\overline{G}$, we have $|P_G(D)| \geq \overline{\gamma}(G)$. Moreover, as $|D| \geq |P_G(D)|$, it follows that $\gamma(G \diamond G) = \min\{|D|\} \geq \min\{|P_G(D)|\} \geq \overline{\gamma}(G)$. □

In Corollary 15 of the Reference [1], it is proved that if $diam(G) \geq 3$, then $\gamma(G \diamond G) \leq \gamma(G) + 2$. We further investigate by strengthening and optimizing the conditions, establishing that $\gamma(G \diamond G) \geq \gamma(G) + 2$, leading to the following theorem:

Theorem 16. *If $diam(G) \geq 3$, $\overline{\gamma}(G) \geq \gamma(G) + 1$, and contains no 2-ECD set of G, then $\gamma(G \diamond G) = \gamma(G) + 2$.*

Proof. We only need to prove $\gamma(G \diamond G) \geq \gamma(G) + 2$. Assume for contradiction that there exists a dominating set S of $G \diamond G$ with $|S| = \gamma(G) + 1$. Let $P_G(S)$ denote the projection onto G, so $|P_G(S)| \leq \gamma(G) + 1$. We first show $P_G(S)$ is a dominating set of G and a TD set of $\bar{G}$. By lemma 2, $P_G(S)$ dominates G, so for any $a \in V(G)$, there exists $s \in P_G(S)$ with $a \in N_G[s] \cup \{s\}$. Similarly, for $b \in V(G)$ with $d_G(a, b) = 3$, there exists $t \in P_G(S)$ with $b \in N_G[t] \cup \{t\}$. Since $d_G(a, b) = 3$, if $a \in N_G[s]$, then $d_G(s, b) \geq 2$, so $b \notin N_G[s]$. Symmetrically, if $b \in N_G[t]$, then $a \notin N_G[t]$.

In $\overline{G}$, $s' \in P_G(S)$, s.t. $a \notin N_G[s']$, and $\exists\ t \in P_G(S)$, s.t. $b \notin N_G[t']$. Due to the absence of 2-ECD sets, consider two cases:

Case 1. If $s' = t'$, since $d_G(a, b) = 3$, $N_G[s']$ cannot cover neighborhoods of both a and b, so $P_G(S)$ fails to dominate vertices x, y on the path $a - x - y - b$, a contradiction.

Case 2. If $s' \neq t'$, then $P_G(S)$ requires at least $\gamma(G)$ vertices to dominate G, plus distinct s' and t', totaling $\gamma(G) + 2$ vertices contradicting $|P_G(S)| \leq \gamma(G) + 1$. The assumption $|S| = \gamma(G) + 1$ is invalid, proving $\gamma(G \diamond G) \geq \gamma(G) + 2$. Thus, the equality $\gamma(G \diamond G) = \gamma(G) + 2$ holds. □

5. Conclusions and further work

This paper primarily investigates the relationships, equalities, and inequalities among SDCTD sets, ECD sets, and PD sets for vertex subsets of three types of product graphs. It further explores the connections between SDCTD sets and PD sets in regular graphs and bipartite graphs. In the preceding section, we performed a hierarchical partition of SDCTD sets via tree decomposition, deriving new upper bounds for the domination number of modular product graphs and that of the modular product of two regular graphs thereby resolving Open Problems 33 and 35 in [1]. Our subsequent research will focus on generalizing the modular product $G \diamond H$ to the n-fold modular product $G_1 \diamond G_2 \diamond \cdots \diamond G_n$, with the aim of investigating the domination number $\gamma\left(\prod_{i=1}^{n} G_i\right)$.

First, a n-dimensional snake set is defined as a sequence of vertex subsets in $V(G_1) \times \cdots \times V(G_n)$, where each subset modifies only one coordinate at a time, and the projection onto the vertex set of each factor graph G_i forms a dominating set or a TD set. When adding a new factor graph G_k, the snake set projection domination method superimposes a 1-dimensional dominating set onto the $(k-1)$-dimensional snake set, requiring only $\gamma(G_k) - 1$ additional vertices per dimension.

Conjecture 17. For the n-fold modular product graph, we have

$$\gamma(\prod_{i=1}^{n} G_i) \leq \sum_{i=1}^{n} \gamma\left(G_i\right) - (n-1).$$

For any factor graph G_i, the domination number must take the minimum of $\gamma(G_i)$ and $\gamma_t(\overline{G}_j)$ of the other $n-1$ factor complements $\overline{G}_j$, then take the maximum over all factors to satisfy cross-dimensional constraints.

Conjecture 18. For any graphs G_i, $i = 1, 2, \cdots, n$, we have

$$\gamma(\prod_{i=1}^{n} G_i) \geq \max_{1\leq i\leq n} \left\{ \min \left\{ \gamma(G_i), \min_{j\neq i} \gamma_t \left(\overline{G_j}\right) \right\} \right\}.$$

In special graph classes, such as ECD graph, due to the closed-neighborhood partition property of ECD sets of ECD graphs, which projects directly to multi-dimensional modular products.

Conjecture 19. If G_i is an ECD graph, then

$$\gamma(\prod_{i=1}^{n} G_i) \leq \min_{1\leq i\leq n} \gamma(G_i).$$

Another compelling case when the factor is a complete graph K_t, the modular product does not change the domination number of the modular product of other factors. Thus, the following conjecture is proposed.

Conjecture 20. If $G_k = K_t$, then

$$\gamma(\prod_{i=1}^{n} G_i) = \gamma \left(\prod_{\substack{i=1 \\ i\neq k}}^{n} G_i \right).$$

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