

On the global Italian domination of graphs

Guoliang Hao¹, Zhihong Xie^{2,*}, Yuqi Wu³, Seyed Mahmoud Sheikholeslami^{4,†}

¹School of Mathematics and Statistics, Heze University, Heze 274015, Shandong, China
*guoliang-hao@163.com

²School of Business, Heze University, Heze 274015, Shandong, China
*xiezh168@163.com

³School of Computer Information Engineering, Nanchang Institute of Technology,
Nanchang 330044, Jiangxi, P.R. China
wyq18870370890@163.com

⁴Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran
†s.m.sheikholeslami@azaruniv.ac.ir

Received: 24 August 2025; Accepted: 12 December 2025

Published Online: 16 December 2025

The authors would like to dedicate this paper to Dr. Odile Favaron, in recognition of her outstanding career in graph theory.

Abstract: Let H be a graph with vertex set V . An Italian dominating function (IDF) on H is a function from V to the set $\{0, 1, 2\}$ having the property that any vertex assigned 0 is adjacent to two vertices assigned 1 or one vertex assigned 2. The value $\sum_{x \in V} h(x)$ is called the weight of an IDF h on H . A global Italian dominating function (GIDF) on H is an IDF on H and its complement. The minimum weight of an IDF (resp., GIDF) on H is the Italian (resp., global Italian) domination number of H . In this paper, we establish several relations between the global Italian domination and Italian domination numbers. In particular, we determine the difference between these two parameters of cubic graphs.

Keywords: Italian domination, global Italian domination, cubic graph.

AMS Subject Classification: 05C69

1. Introduction

Let $H = (V(H), E(H))$ be a graph with vertex set $V(H)$ and edge set $E(H)$. The *open neighborhood* $N_H(u)$ (briefly $N(u)$) of a vertex $u \in V(H)$ is the set of its neighbors,

* *Corresponding Author*

† *Corresponding Author*

while its *closed neighborhood* $N_H[u]$ (briefly $N[u]$) is the set $\{u\} \cup N_H(u)$. We denote by $d(u) = d_H(u) = |N_H(u)|$ the *degree* of a vertex u in H and by $\Delta(H)$ (briefly Δ) the maximum degree among all vertices in H . For a vertex subset U of H , let $N_H(U)$ (briefly $N(U)$), $N_H[U]$ (briefly $N[U]$) and $H[U]$ be the sets $\bigcup_{x \in U} N_H(x)$, $\bigcup_{x \in U} N_H[x]$ and the subgraph induced by U , respectively. For subsets $U_1, U_2 \subseteq V(H)$, let $[U_1, U_2] = \{u_1u_2 \in E(H) : u_1 \in U_1 \text{ and } u_2 \in U_2\}$. For any vertices $u_1, u_2 \in V(H)$, the *distance* $d(u_1, u_2)$ between u_1 and u_2 is the length of a shortest (u_1, u_2) -path in H . The *diameter* $\text{diam}(H)$ of H is defined as $\text{diam}(H) = \max\{d(u_1, u_2) : u_1, u_2 \in V(H)\}$. The *complement* of H is denoted by \overline{H} , where $V(\overline{H}) = V(H)$ and $u_1u_2 \in E(\overline{H})$ if and only if $u_1u_2 \notin E(H)$. A *cubic graph* is a graph in which each vertex has degree three. For two subsets $U_1, U_2 \subseteq V(H)$, if $U_2 \subseteq N[U_1]$, then we say that U_1 dominates U_2 . For a vertex $u \in V(H)$ and a set $U \subseteq V(H)$, we say that u dominates U if $\{u\}$ dominates U .

A vertex subset S of a graph H with $N[S] = V(H)$ is called a *dominating set* (D-set) of H . The *domination number*, $\gamma(H)$ of H is the minimum cardinality of a D-set of H . In [5], a variant of the domination parameters, namely Italian domination, was introduced where the authors called it Roman $\{2\}$ -domination. A function h from the vertex set of a graph H to the set $\{0, 1, 2\}$ is called an *Italian dominating function* (IDF) on H if any vertex assigned 0 under h has two neighbors assigned 1 or one neighbor assigned 2. A *global Italian dominating function* (GIDF) on H , which is introduced in [6], is an IDF on both H and \overline{H} . Let $\omega(h)$ denote the *weight* $\sum_{x \in V(H)} h(x)$ of an IDF h on H . The minimum weight of an IDF (resp., GIDF) on H is called the *Italian domination number* (ID-number) $\gamma_I(H)$ (resp., *global Italian domination number* (GID-number) $\gamma_{gI}(H)$) of H . An IDF (resp., GIDF) on H having weight $\gamma_I(H)$ (resp., $\gamma_{gI}(H)$) is a $\gamma_I(H)$ -function (resp., $\gamma_{gI}(H)$ -function). For any $\gamma_I(H)$ -function h , we use W_i^h to denote the set $\{x \in V(H) : h(x) = i\}$, where $i \in \{0, 1, 2\}$, based on which we may use the notation (W_0^h, W_1^h, W_2^h) to denote the function h . For a sake of simplicity, we write (W_0, W_1, W_2) rather than (W_0^h, W_1^h, W_2^h) when the $\gamma_I(H)$ -function h is clear from the context. The concept on Italian domination was studied further in, for example, [1–4, 7–11].

In order to derive relations between GID-number and ID-number, we introduce a notation that is the key point for our following discussion. For arbitrary $\gamma_I(H)$ -function $h = (W_0, W_1, W_2)$, let

$$X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\}.$$

We will utilize the following two well-known results.

Theorem 1 ([5]). *For any graph H , $\gamma_I(H) \leq 2\gamma(H)$.*

Theorem 2 ([5]). *If H is a graph on n vertices, then $\gamma_I(H) \geq 2n/(\Delta + 2)$.*

2. Results for general graphs

Our aim in this section is to give some relations between GID-number and ID-number of general graphs.

It is known [6] that each graph H with diameter three or four satisfies $\gamma_{gI}(H) \leq \gamma_I(H) + 4$. We now make a slight improvement on this bound for graphs with diameter four.

Theorem 3. *For any graph H with diameter four, $\gamma_{gI}(H) \leq \gamma_I(H) + 2$.*

Proof. Let $g = (W_0, W_1, W_2)$ be a $\gamma_I(H)$ -function and let x_1 and x_2 be vertices of H with $d(x_1, x_2) = 4$. If $x_1 \in W_2$ (the case $x_2 \in W_2$ is similar), then the function η given by $\eta(x_2) = 2$ and $\eta(z) = g(z)$ for any $z \in V(H) \setminus \{x_2\}$, is a GIDF on H with $\omega(\eta) \leq \omega(g) + 2$. If $x_1 \in W_0$ (the case $x_2 \in W_0$ is similar), then x_1 has a neighbor w_1 in W_2 or two neighbors w_2 and w_3 in W_1 . Clearly $d(x_2, w_i) \geq 3$ and so each vertex z of H is not adjacent to both w_i and x_2 for each i . Therefore, the function η given by $\eta(x_2) = 2$ and $\eta(z) = g(z)$ for each $z \in V(H) \setminus \{x_2\}$, is a GIDF on H with $\omega(\eta) \leq \omega(g) + 2$. Finally, assume that $x_1, x_2 \in W_1$. Then the function η given by $\eta(x_1) = \eta(x_2) = 2$ and $\eta(z) = g(z)$ for each $z \in V(H) \setminus \{x_1, x_2\}$, is a GIDF on H with $\omega(\eta) = \omega(g) + 2$. All in all, we deduce that $\gamma_{gI}(H) \leq \omega(\eta) \leq \omega(g) + 2 = \gamma_I(H) + 2$. \square

Theorem 4. *Let H be a graph. Then $\gamma_{gI}(H) \leq \gamma_I(H) + \gamma_I(\overline{H})$.*

Proof. Let $g = (W_0^g, W_1^g, W_2^g)$ be a $\gamma_I(H)$ -function and let $h = (W_0^h, W_1^h, W_2^h)$ be a $\gamma_I(\overline{H})$ -function. One can easily verify that the function η given by $\eta(x) = 0$ for any $x \in W_0^g \cap W_0^h$, $\eta(x) = 1$ for any $x \in (W_0^g \cap W_1^h) \cup (W_1^g \cap W_0^h) \cup (W_1^g \cap W_1^h)$ and $\eta(x) = 2$ for any $x \in W_2^g \cup W_2^h$, is a GIDF on H . This forces

$$\begin{aligned} \gamma_{gI}(H) &\leq \omega(\eta) \\ &= |(W_0^g \cap W_1^h) \cup (W_1^g \cap W_0^h) \cup (W_1^g \cap W_1^h)| + 2|W_2^g \cup W_2^h| \\ &\leq |W_0^g \cap W_1^h| + (|W_1^g \cap W_0^h| + |W_1^g \cap W_1^h|) + 2(|W_2^g| + |W_2^h|) \\ &\leq |W_1^h| + |W_1^g| + 2(|W_2^g| + |W_2^h|) \\ &= (|W_1^g| + 2|W_2^g|) + (|W_1^h| + 2|W_2^h|) \\ &= \gamma_I(H) + \gamma_I(\overline{H}), \end{aligned}$$

as desired. \square

Recall that $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\}$, where $h = (W_0, W_1, W_2)$ is a $\gamma_I(H)$ -function.

Lemma 1. *For any graph H with a $\gamma_I(H)$ -function $\eta = (W_0, W_1, W_2)$, if $\gamma_{gI}(H) \neq \gamma_I(H)$, then $X \neq \emptyset$.*

Proof. Note that η is $\gamma_I(H)$ -function. Thus if η is an IDF on \overline{H} , then η is a GIDF on H with $\gamma_{gI}(H) \leq \omega(\eta) = \gamma_I(H)$. Also, since $\gamma_{gI}(H) \geq \gamma_I(H)$, it follows that $\gamma_{gI}(H) = \gamma_I(H)$, a contradiction. Therefore η is not an IDF on \overline{H} , leading that there must exist $w \in W_0$ with $|W_1 \cap N_{\overline{H}}(w)| \leq 1$ and $W_2 \cap N_{\overline{H}}(w) = \emptyset$. This forces $|W_1 \setminus N_H(w)| \leq 1$ and $W_2 \subseteq N_H(w)$. Hence $w \in X$, that is, $X \neq \emptyset$. \square

Theorem 5. For any graph H with a $\gamma_I(H)$ -function $h = (W_0, W_1, W_2)$, if $\gamma_{gI}(H) = \gamma_I(H) + k$ ($k \geq 3$), then

- (a) Every subset of $V(H)$ that dominates X has cardinality at least $\lceil k/2 \rceil$ in \overline{H} .
- (b) X is a D-set of H and $|X| \geq \gamma_I(H)/2$.

Proof. Note that $\gamma_{gI}(H) = \gamma_I(H) + k$ ($k \geq 3$). Thus by Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let S be an arbitrary subset of $V(H)$ such that $X \subseteq N_{\overline{H}}[S]$. One can easily observe that the function η given by $\eta(z) = 2$ for every $z \in S$ and $\eta(z) = h(z)$ for each $z \in V(H) \setminus S$, is a GIDF on H and thus

$$\begin{aligned} \gamma_I(H) + k = \gamma_{gI}(H) &\leq \omega(\eta) = (\omega(h) - \sum_{z \in S} h(z)) + \sum_{z \in S} \eta(z) \\ &\leq \omega(h) + 2|S| = \gamma_I(H) + 2|S|, \end{aligned}$$

implying that $2|S| \geq k$. Moreover, since $|S|$ is an integer, we have $|S| \geq \lceil k/2 \rceil$, implying that (a) is true. Further, since $k \geq 3$, we have that the set S has at least $\lceil k/2 \rceil \geq 2$ vertices and so every vertex in $V(\overline{H}) \setminus X$ is not adjacent to all vertices of X in \overline{H} . As a result, $V(H) = N_H[X]$. This forces that X is a D-set of H and so by Theorem 1, $|X| \geq \gamma(H) \geq \gamma_I(H)/2$, that is, (b) holds. \square

Theorem 6. For any graph H with $\gamma_I(H) \geq 3$,

$$\gamma_{gI}(H) \leq \gamma_I(H) + 2 \left\lceil \frac{2\Delta - \gamma_I(H) + 2}{\gamma_I(H) - 2} \right\rceil + 2.$$

Proof. Since $\gamma_I(H) \geq 3$, we have

$$\lceil (2\Delta - \gamma_I(H) + 2)/(\gamma_I(H) - 2) \rceil = \lceil 2\Delta/(\gamma_I(H) - 2) \rceil - 1 \geq -1.$$

It is clear to observe that $\gamma_{gI}(H) \leq \gamma_I(H) + 2 \lceil (2\Delta - \gamma_I(H) + 2)/(\gamma_I(H) - 2) \rceil + 2$ if $\gamma_{gI}(H) = \gamma_I(H)$. Next, suppose that $\gamma_{gI}(H) \geq \gamma_I(H) + 1$. Let $h = (W_0, W_1, W_2)$ be a $\gamma_I(H)$ -function. By Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N_H(w)| \leq 1 \text{ and } W_2 \subseteq N_H(w)\} \neq \emptyset$.

$N_H(w)\} \neq \emptyset$. Let w be a vertex of X and $A = N_H(w) \cap X$. Note that $X \subseteq W_0$, $|W_1 \setminus N_H(w)| \leq 1$ and $W_2 \subseteq N_H(w)$. Thus

$$\begin{aligned} |A| &= |N_H(w) \cap X| \\ &\leq |N_H(w) \cap W_0| \\ &= |N_H(w)| - |N_H(w) \cap W_1| - |N_H(w) \cap W_2| \\ &= d_H(w) - (|W_1| - |W_1 \setminus N_H(w)|) - |W_2| \\ &\leq \Delta - (|W_1| - 1) - |W_2| \\ &= \Delta - (|W_1| + |W_2|) + 1. \end{aligned}$$

Moreover, since $W_2 \cup W_1$ is a D-set of H , this forces $\gamma(H) \leq |W_1| + |W_2|$ and hence

$$|A| \leq \Delta - (|W_1| + |W_2|) + 1 \leq \Delta - \gamma(H) + 1. \tag{2.1}$$

Now suppose that $A = N_H(w) \cap X = \emptyset$. Then the function η given by $\eta(w) = 2$ and $\eta(z) = h(z)$ for each $z \in V(H) \setminus \{w\}$, is a GIDF on H , implying that

$$\gamma_{gI}(H) \leq \omega(\eta) = (\omega(h) - h(w)) + \eta(w) = \gamma_I(H) + 2. \tag{2.2}$$

In addition, since $\gamma_I(H) \geq 3$ and $\Delta - \gamma(H) + 1 \geq |A| = 0$ by (2.1), we get $\lceil (\Delta - \gamma(H) + 1) / (\gamma_I(H) - 2) \rceil \geq 0$. Thus by Theorem 1 and (2.2),

$$\begin{aligned} \gamma_{gI}(H) &\leq \gamma_I(H) + 2 \\ &\leq \gamma_I(H) + 2 \lceil (\Delta - \gamma(H) + 1) / (\gamma_I(H) - 2) \rceil + 2 \\ &\leq \gamma_I(H) + 2 \lceil (2\Delta - 2\gamma(H) + 2) / (\gamma_I(H) - 2) \rceil + 2 \\ &\leq \gamma_I(H) + 2 \lceil (2\Delta - \gamma_I(H) + 2) / (\gamma_I(H) - 2) \rceil + 2. \end{aligned}$$

Next suppose that $A = N_H(w) \cap X \neq \emptyset$. By Theorem 1, $\gamma(H) \geq \gamma_I(H)/2 \geq 3/2$. This forces $\gamma(H) \geq 2$. We now choose k disjoint subsets A_1, A_2, \dots, A_k of the set A as follows:

- (a) If $|A| \leq \gamma(H) - 1$, then set $k = 1$ and $A_k = A$, and if $|A| > \gamma(H) - 1$, then let $B_1 = A$.
- (b) If $|B_i| > \gamma(H) - 1$, then let $A_i \subseteq B_i$ with $|A_i| = \gamma(H) - 1$ and let $B_{i+1} = B_i \setminus A_i$.
- (c) If $|B_{i+1}| \leq \gamma(H) - 1$, then set $k = i + 1$ and $A_k = B_{i+1}$. Otherwise, increment i and return to Step (b).

It is not complicated to check that $A = A_1 \cup A_2 \cup \dots \cup A_k$. Since $|A_j| \leq \gamma(H) - 1$ for each $j \in \{1, 2, \dots, k\}$, we have that A_j is not a D-set of H and thus there must

exist some $a_j \in V(H) \setminus A_j$ with $A_j \cap N_H(a_j) = \emptyset$, implying that $A_j \subseteq N_{\overline{H}}(a_j)$. Let $S = \bigcup_{j=1}^k \{a_j\}$. Then

$$|S| \leq k = \lceil |A| / (\gamma(H) - 1) \rceil. \tag{2.3}$$

Observe that the function η given by $\eta(z) = 2$ for each $z \in \{w\} \cup S$ and $\eta(z) = h(z)$ for all other vertices z of H , is a GIDF on H . Thus by (2.1), (2.3) and Theorem 1,

$$\begin{aligned} \gamma_{gI}(H) &\leq \omega(\eta) \\ &= \left(\omega(h) - \sum_{z \in \{w\} \cup S} h(z) \right) + \sum_{z \in \{w\} \cup S} \eta(z) \\ &\leq \omega(h) + 2|S| + 2 \\ &\leq \gamma_I(H) + 2 \lceil |A| / (\gamma(H) - 1) \rceil + 2 \\ &= \gamma_I(H) + 2 \lceil 2|A| / (\gamma_I(H) - 2) \rceil + 2 \\ &\leq \gamma_I(H) + 2 \lceil 2(\Delta - \gamma(H) + 1) / (\gamma_I(H) - 2) \rceil + 2 \\ &\leq \gamma_I(H) + 2 \lceil (2\Delta - \gamma_I(H) + 2) / (\gamma_I(H) - 2) \rceil + 2, \end{aligned}$$

as desired. □

Theorem 7. *If H is a graph on $n \geq \Delta^2 + 3$ vertices, then $\gamma_{gI}(H) = \gamma_I(H)$.*

Proof. By contradiction, suppose that $\gamma_{gI}(H) \neq \gamma_I(H)$. Let $h = (W_0, W_1, W_2)$ be an IDF on H with weight $\gamma_I(H)$. By Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let $w \in X$. Then $|W_1 \setminus N(w)| \leq 1$ and $W_2 \subseteq N(w)$. First, assume that $|W_1 \setminus N(w)| = 0$. One can check that $W_2 \cup W_1 \subseteq N(w)$. Thus $|W_2 \cup W_1| \leq \Delta$ and any vertex in $W_2 \cup W_1$ has at most $\Delta - 1$ neighbors in $W_0 \setminus N[w]$. Moreover, since $W_2 \cup W_1$ is a D-set of H , this forces that $W_2 \cup W_1$ dominates $W_0 \setminus N[w]$, implying that $|W_0 \setminus N[w]| \leq (\Delta - 1)|W_2 \cup W_1| \leq (\Delta - 1)\Delta$. Note that $|(W_2 \cup W_1) \setminus N[w]| = 0$ since $W_2 \cup W_1 \subseteq N(w)$. Therefore,

$$\begin{aligned} n &= |N[w]| + |V(H) \setminus N[w]| \\ &= |N[w]| + (|(W_2 \cup W_1) \setminus N[w]| + |W_0 \setminus N[w]|) \\ &= |N[w]| + |W_0 \setminus N[w]| \\ &\leq (\Delta + 1) + (\Delta - 1)\Delta \\ &= \Delta^2 + 1, \end{aligned}$$

a contradiction.

Second, assume that $|W_1 \setminus N(w)| = 1$. Note that $W_2 \subseteq N(w)$. Thus

$$\begin{aligned} |W_1| + |W_2| &= |W_1 \cup W_2| \\ &= |(W_1 \cup W_2) \cap N(w)| + |(W_1 \cup W_2) \setminus N(w)| \end{aligned}$$

$$\begin{aligned}
 &= |(W_1 \cup W_2) \cap N(w)| + |W_1 \setminus N(w)| \\
 &\leq |N(w)| + 1 \\
 &\leq \Delta + 1.
 \end{aligned} \tag{2.4}$$

Now let $W_1 \setminus N(w) = \{v\}$. Moreover, since $w \in X$, this forces that w is adjacent to any vertex belonging to $W_2 \cup (W_1 \setminus \{v\})$ in H and so each vertex of $W_2 \cup (W_1 \setminus \{v\})$ has at most $\Delta - 1$ neighbors in $W_0 \setminus N[w]$. Thus by (2.4),

$$\begin{aligned}
 |(W_0 \setminus N[w]) \cap N((W_1 \setminus \{v\}) \cup W_2)| &\leq (\Delta - 1)|(W_1 \setminus \{v\}) \cup W_2| \\
 &= (\Delta - 1)(|W_2| + |W_1| - 1) \\
 &\leq \Delta(\Delta - 1).
 \end{aligned} \tag{2.5}$$

Since h is an IDF on H , every vertex of W_0 is adjacent to one vertex in W_2 or two vertices in W_1 . Hence every neighbor of v in W_0 is adjacent to some vertex in $(W_1 \setminus \{v\}) \cup W_2$, implying that $N(v) \cap W_0 \subseteq N((W_1 \setminus \{v\}) \cup W_2) \cap W_0$. Thus $(W_0 \setminus N[w]) \cap N(W_1 \cup W_2) = (W_0 \setminus N[w]) \cap N((W_1 \setminus \{v\}) \cup W_2)$. Furthermore, since $W_1 \cup W_2$ is a D-set of H , this forces $W_0 \setminus N[w] \subseteq N(W_1 \cup W_2)$. Therefore, by (2.5),

$$\begin{aligned}
 |W_0 \setminus N[w]| &= |(W_0 \setminus N[w]) \cap N(W_1 \cup W_2)| \\
 &= |(W_0 \setminus N[w]) \cap N((W_1 \setminus \{v\}) \cup W_2)| \\
 &\leq \Delta(\Delta - 1) \\
 &= \Delta^2 - \Delta.
 \end{aligned} \tag{2.6}$$

Since $W_2 \subseteq N(w)$ and $W_1 \setminus N(w) = \{v\}$, it follows from (2.6) that

$$\begin{aligned}
 n &= |N[w]| + |V(H) \setminus N[w]| \\
 &= |N[w]| + (|W_2 \setminus N[w]| + |W_1 \setminus N[w]| + |W_0 \setminus N[w]|) \\
 &= |N[w]| + |W_1 \setminus N[w]| + |W_0 \setminus N[w]| \\
 &\leq (\Delta + 1) + 1 + (\Delta^2 - \Delta) \\
 &= \Delta^2 + 2,
 \end{aligned}$$

a contradiction, and this complete our proof. □

Next we demonstrate that the condition $n \geq \Delta^2 + 3$ in Theorem 7 is optimal. To show the optimality, we consider a graph H obtained from $\Delta \geq 4$ copies of stars $K_{1, \Delta-1}$, say $S_1, S_2, \dots, S_\Delta$ centred at $x_1, x_2, \dots, x_\Delta$ respectively, by adding a new vertex x and the edge xx_i for each $i \in \{1, 2, \dots, \Delta\}$, and attaching a pendant edge at a unique leaf in S_1 . One can easily see that H has $\Delta^2 + 2$ vertices and has a unique $\gamma_I(H)$ -function which is not a GIDF on H . Thus $\gamma_{gI}(H) \neq \gamma_I(H)$. In fact, we have $\gamma_{gI}(H) = \gamma_I(H) + 1$.

3. Results for cubic graphs

Our aim in the section is to derive the difference between GID -number and ID -number for cubic graphs.

Lemma 2. *If H is a cubic graph on n vertices with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$, then $n \leq 10$. In particular, $n \in \{4, 6, 8, 10\}$.*

Proof. By Theorem 7, $n \leq \Delta^2 + 2 = 11$. Moreover, since H is cubic, it follows from Euler's handshaking lemma that $2|E(H)| = \sum_{z \in V(H)} d(z) = 3n$. This forces that n is even and so $n \in \{4, 6, 8, 10\}$. □

Lemma 3. *Let H be a graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H . Then $\gamma_I(H) \leq d(w) + |W_2| + 1$, where $w \in X$.*

Proof. Since $w \in X$, we have $|W_1 \setminus N(w)| \leq 1$ and $W_2 \subseteq N(w)$. Therefore

$$\begin{aligned} d(w) &= |N(w) \cap W_0| + |N(w) \cap W_1| + |N(w) \cap W_2| \\ &\geq |N(w) \cap W_1| + |N(w) \cap W_2| \\ &= (|W_1| - |W_1 \setminus N(w)|) + |W_2| \\ &\geq (|W_1| - 1) + |W_2| \\ &= \gamma_I(H) - |W_2| - 1, \end{aligned}$$

as desired. □

Lemma 4. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$. Then $\gamma_I(H) \leq 7$.*

Proof. Let $h = (W_0, W_1, W_2)$ be a minimum IDF on H . Note that $\gamma_{gI}(H) \geq \gamma_I(H) + 1$. Thus by Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let w_0 be a vertex of X . Clearly $W_2 \subseteq N(w_0)$. Thus by Lemma 3, $\gamma_I(H) \leq d(w_0) + |W_2| + 1 \leq 3 + |N(w_0)| + 1 = 7$. □

Lemma 5. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H . If $\gamma_I(H) = 7$, then $|W_2| = 3$ and $|W_1| = 1$.*

Proof. By Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let w_0 be a vertex of X . By Lemma 3, $|W_2| \geq \gamma_I(H) - d(w_0) - 1 = 3$. Moreover, since $|W_2| = (\gamma_I(H) - |W_1|)/2 \leq 7/2$ and $|W_2|$ is an integer, we have $|W_2| = 3$ and so $|W_1| = \gamma_I(H) - 2|W_2| = 1$. □

Lemma 6. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$. Then $\gamma_I(H) \leq 6$.*

Proof. By Lemma 2, H has order $n \in \{4, 6, 8, 10\}$. Let $v \in V(H)$ and $N_H(v) = \{v_1, v_2, v_3\}$. If $n \in \{4, 6, 8\}$, then the function η given by $\eta(v) = 2$, $\eta(v_i) = 0$ for each $i \in \{1, 2, 3\}$ and $\eta(z) = 1$ for all other vertices z of H , is an IDF on H , leading that $\gamma_I(H) \leq 2 + (n - 4) \leq 6$. Now let $n = 10$ and let $Y = V(H) \setminus \{v, v_1, v_2, v_3\} = \{u_1, u_2, \dots, u_6\}$. Noting that H is cubic, we have $|[Y, N_H(v)]| \leq 6$ and $|[\{v\}, N_H(v)]| = 3$, leading that

$$\sum_{i=1}^6 d_{H[Y]}(u_i) \geq \sum_{z \in V(H)} d_H(z) - 2|[Y, N_H(v)]| - 2|[\{v\}, N_H(v)]| \geq 30 - 12 - 6 = 12.$$

This forces $\Delta(H[Y]) \geq 2$. Wlog, assume that u_1 is adjacent to u_2 and u_3 in H . One can check that the mapping η given by $\eta(u_1) = \eta(v) = 2$, $\eta(z) = 0$ for each $z \in \{v_1, v_2, v_3, u_2, u_3\}$ and $\eta(z) = 1$ for all other vertices z of H , is an IDF on H , leading that $\gamma_I(H) \leq 7$. If $\gamma_I(H) = 7$, then η is a $\gamma_I(H)$ -function with $W_2^\eta = \{u_1, v\}$, a contradiction to Lemma 5. Thus $\gamma_I(H) \leq 6$. \square

By applying a similar approach as described in the proof of Lemma 5, we can get the next two results.

Lemma 7. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H . If $\gamma_I(H) = 6$, then $|W_2| = 3$ and $|W_1| = 0$, or $|W_2| = |W_1| = 2$.*

Lemma 8. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H . If $\gamma_I(H) = 5$, then $|W_2| = 2$ and $|W_1| = 1$, or $|W_2| = 1$ and $|W_1| = 3$.*

Lemma 9. *Let H be a cubic graph on n vertices with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H . Then*

$$n \leq \begin{cases} 1 + 3|W_2| + |W_1|, & \text{if } |W_1| \leq 1, \\ 1 + 3|W_2| + 2|W_1|, & \text{if } |W_1| \geq 2. \end{cases}$$

Proof. By Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let v_0 be a vertex of X , $W_{01} = \{z \in W_0 \setminus \{v_0\} : |N(z) \cap W_2| \geq 1\}$ and let $W_{02} = \{z \in W_0 \setminus \{v_0\} : |N(z) \cap W_1| \geq 2\}$. Clearly $W_2 \subseteq N(v_0)$. Moreover, since H is cubic, we have that any vertex of W_2 is adjacent to at most two vertices of $W_0 \setminus \{v_0\}$ in H and hence $|W_{01}| \leq 2|W_2|$. Furthermore, by the definition of $\gamma_I(H)$ -function, $W_0 \setminus \{v_0\} = W_{01} \cup W_{02}$. Hence

$$|W_0| = 1 + |W_0 \setminus \{v_0\}| = 1 + |W_{01} \cup W_{02}| \leq 1 + |W_{01}| + |W_{02}| \leq 1 + 2|W_2| + |W_{02}|. \tag{3.1}$$

If $|W_1| \leq 1$, then clearly $|W_{02}| = 0$ and hence by (3.1),

$$n = |W_0| + |W_1| + |W_2| \leq (1 + 2|W_2| + |W_{02}|) + |W_1| + |W_2| = 1 + |W_1| + 3|W_2|,$$

as desired. We next assume that $|W_1| \geq 2$. Since $v_0 \in X$, we have $|W_1 \setminus N(v_0)| \leq 1$. Moreover, since $W_{02} = \{z \in W_0 \setminus \{v_0\} : |N(z) \cap W_1| \geq 2\}$ and H is cubic, we obtain

$$\begin{aligned} |W_{02}| &\leq \frac{1}{2} |[W_{02}, W_1]| \\ &\leq \frac{1}{2} \left(\sum_{z \in W_1} d(z) - |W_1 \cap N(v_0)| \right) \\ &= \frac{1}{2} (3|W_1| - (|W_1| - |W_1 \setminus N(v_0)|)) \\ &\leq \frac{1}{2} (2|W_1| + 1) = |W_1| + \frac{1}{2}. \end{aligned}$$

Noting that $|W_{02}|$ and $|W_1|$ are integers, we obtain $|W_{02}| \leq |W_1|$. Thus by (3.1),

$$\begin{aligned} n &= |W_0| + |W_1| + |W_2| \leq (1 + 2|W_2| + |W_{02}|) + |W_1| + |W_2| \\ &\leq (1 + 2|W_2| + |W_1|) + |W_1| + |W_2| = 1 + 3|W_2| + 2|W_1|, \end{aligned}$$

which completes our proof. □

Proposition 1. *Let H be a cubic graph on 10 vertices. Then $\gamma_{gI}(H) = \gamma_I(H)$.*

Proof. Suppose that $\gamma_{gI}(H) \neq \gamma_I(H)$ and let $h = (W_0, W_1, W_2)$ be a $\gamma_I(H)$ -function. By Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let v_0 be a vertex of X . Clearly $v_0 \in W_0$, $|W_1 \setminus N(v_0)| \leq 1$ and $W_2 \subseteq N(v_0)$. Since $n = 10$ and $\Delta = 3$, it follows from Theorem 2 that $\gamma_I(H) \geq 2n/(\Delta+2) = 4$. On the other hand, by Lemma 6, $\gamma_I(H) \leq 6$. Therefore $\gamma_I(H) \in \{4, 5, 6\}$. Let $V(H) \setminus \{v_0\} = \{v_i : 1 \leq i \leq 9\}$.

Case 1. $\gamma_I(H) = 4$.

Noting that $2|W_2| + |W_1| = \gamma_I(H) = 4$, we have that $|W_1| = 0$ and $|W_2| = 2$, or $|W_1| = 2$ and $|W_2| = 1$, or $|W_1| = 4$ and $|W_2| = 0$. By Lemma 9, if $|W_1| = 0$ and $|W_2| = 2$, then $n \leq 1 + 3|W_2| + |W_1| = 7$; if $|W_1| = 2$ and $|W_2| = 1$, then $n \leq 1 + 3|W_2| + 2|W_1| = 8$ and if $|W_1| = 4$ and $|W_2| = 0$, then $n \leq 1 + 3|W_2| + 2|W_1| = 9$. In each case, we have a contradiction to the assumption $n = 10$.

Case 2. $\gamma_I(H) = 5$.

By Lemma 8, we have two possibilities $|W_1| = 1$ and $|W_2| = 2$, or $|W_1| = 3$ and $|W_2| = 1$. If $|W_1| = 1$ and $|W_2| = 2$, then by Lemma 9, $n \leq 1 + 3|W_2| + |W_1| = 8$, a contradiction. Therefore $|W_1| = 3$ and $|W_2| = 1$. Let $W_2 = \{v_1\}$ and let $W_1 = \{v_2, v_3, v_4\}$. Then $W_0 \setminus \{v_0\} = \{v_i : 5 \leq i \leq 9\}$. Since $d(v_0) = 3$, $|W_1 \setminus N(v_0)| \leq 1$ and $\{v_1\} = W_2 \subseteq N(v_0)$, we have

$$|N(v_0) \cap W_1| = 2. \tag{3.2}$$

Therefore we can assume that $N(v_0) = \{v_1, v_2, v_3\}$. Furthermore, since $d(v_1) = 3$, we have $|N(v_1) \cap (W_0 \setminus \{v_0\})| \leq 2$.

First, suppose that $|N(v_1) \cap (W_0 \setminus \{v_0\})| \leq 1$. Let $v_6, v_7, v_8, v_9 \notin N(v_1) \cap (W_0 \setminus \{v_0\})$. It is evident from the definition of $\gamma_I(H)$ -function that $|N(v_i) \cap W_1| \geq 2$ for each $i \in \{6, 7, 8, 9\}$. Thus by (3.2),

$$\begin{aligned} d(v_2) + d(v_3) + d(v_4) &\geq |[v_0], W_1| + |[v_6, v_7, v_8, v_9], W_1| \\ &\geq |N(v_0) \cap W_1| + 2|\{v_6, v_7, v_8, v_9\}| \\ &= 10, \end{aligned}$$

a contradiction.

Second, suppose that $|N(v_1) \cap (W_0 \setminus \{v_0\})| = 2$. Let $N(v_1) \cap (W_0 \setminus \{v_0\}) = \{v_5, v_6\}$. Note that $|N(v_i) \cap \{v_2, v_3, v_4\}| = |N(v_i) \cap W_1| \geq 2$ for each $i \in \{7, 8, 9\}$. Moreover, since H is cubic and $N(v_0) = \{v_1, v_2, v_3\}$, the function η given by $\eta(v_i) = 1$ for each $i \in \{2, 3, 4, 5, 6, \}$ and $\eta(v_i) = 0$ for each $i \in \{0, 1, 7, 8, 9, \}$ is a GIDF on H , leading that $\gamma_{gI}(H) \leq 5$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 5$, implying that $\gamma_{gI}(H) = 5 = \gamma_I(H)$, a contradiction.

Case 3. $\gamma_I(H) = 6$.

It follows from Lemma 7 that $|W_1| = 0$ and $|W_2| = 3$, or $|W_1| = |W_2| = 2$. First, assume that $|W_1| = 0$ and $|W_2| = 3$. Let $W_2 = \{v_1, v_2, v_3\}$. It is clear that $W_0 \setminus \{v_0\} = \{v_4, v_5, \dots, v_9\}$.

Claim. $N(W_2) \cap (W_0 \setminus \{v_0\}) = W_0 \setminus \{v_0\}$ and any vertex in W_2 is adjacent to exactly two vertices of $W_0 \setminus \{v_0\}$.

Proof of Claim. By the definition of $\gamma_I(H)$ -function, any vertex of $W_0 \setminus \{v_0\}$ has at least one neighbor in W_2 , implying that $(W_0 \setminus \{v_0\}) \cap N(W_2) = W_0 \setminus \{v_0\}$ and $|[W_0 \setminus \{v_0\}, W_2]| \geq |W_0 \setminus \{v_0\}| = 6$. Also, since H is cubic and $W_2 \subseteq N(v_0)$, we have $|[W_2, W_0 \setminus \{v_0\}]| \leq 2|W_2| = 6$. Thus $|[W_2, W_0 \setminus \{v_0\}]| = 6$. This forces that any vertex in W_2 is adjacent to exactly two vertices in $W_0 \setminus \{v_0\}$. ■

By Claim, we let $N(v_1) \cap (W_0 \setminus \{v_0\}) = \{v_4, v_5\}$, $N(v_2) \cap (W_0 \setminus \{v_0\}) = \{v_6, v_7\}$ and let $N(v_3) \cap (W_0 \setminus \{v_0\}) = \{v_8, v_9\}$. One can verify that the function η given by $\eta(v_1) = 2$, $\eta(v_i) = 1$ for each $i \in \{6, 7, 8, 9\}$ and $\eta(v_i) = 0$ for each $i \in \{0, 2, 3, 4, 5\}$, is a GIDF on H , leading that $\gamma_{gI}(H) \leq \omega(\eta) = 6$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 6$, this forces $\gamma_{gI}(H) = 6 = \gamma_I(H)$, a contradiction.

Second, assume that $|W_1| = |W_2| = 2$. Let $W_2 = \{v_1, v_2\}$ and let $W_1 = \{v_3, v_4\}$. Let $U = (W_0 \setminus \{v_0\}) \cap N(v_3) \cap N(v_4)$. If $U = \emptyset$, then by the definition of $\gamma_I(H)$ -function, each vertex of $W_0 \setminus \{v_0\}$ has a neighbor in W_2 . Furthermore, since $W_2 \subseteq N(v_0)$, we obtain $d(v_1) + d(v_2) \geq 7$, a contradiction. Suppose next that $U \neq \emptyset$. Since $|W_1 \setminus N(v_0)| \leq 1$ and $W_2 \subseteq N(v_0)$, we have $|N(v_0) \cap W_1| = 1$ (noting that H is cubic). Let $N(v_0) \setminus \{v_1, v_2\} = \{v_3\}$. Thus v_3 has at most two neighbors in $W_0 \setminus \{v_0\}$, implying that $|U| \in \{1, 2\}$.

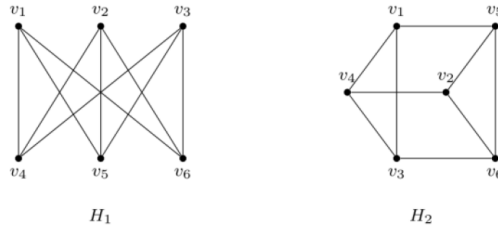


Figure 1. Two non-isomorphic cubic graphs H_1 and H_2 of order 6.

Now suppose that $|U| = 1$. Let $U = \{v_9\}$. Clearly $W_0 \setminus (\{v_0\} \cup U) = \{v_5, v_6, v_7, v_8\}$. We deduce from the method analogous to the proof of Claim that $N(W_2) \cap (W_0 \setminus (\{v_0\} \cup U)) = W_0 \setminus (\{v_0\} \cup U)$ and any vertex in W_2 is adjacent to exactly two vertices of $W_0 \setminus (\{v_0\} \cup U)$. We now let $N(v_1) \cap (W_0 \setminus (\{v_0\} \cup U)) = \{v_5, v_6\}$ and $N(v_2) \cap (W_0 \setminus (\{v_0\} \cup U)) = \{v_7, v_8\}$. Observe that the function η given by $\eta(v_1) = 2$, $\eta(v_i) = 1$ for each $i \in \{3, 4, 7, 8\}$ and $\eta(v_i) = 0$ for each $i \in \{0, 2, 5, 6, 9\}$, is a GIDF on H , leading that $\gamma_{gI}(H) \leq \omega(\eta) = 6$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 6$, this forces $\gamma_{gI}(H) = 6 = \gamma_I(H)$, a contradiction.

We next suppose that $|U| = 2$. Let $U = \{v_8, v_9\}$. Then $W_0 \setminus (\{v_0\} \cup U) = \{v_5, v_6, v_7\}$. We conclude from the method similar to the proof of Claim that $N(W_2) \cap (W_0 \setminus (\{v_0\} \cup U)) = W_0 \setminus (\{v_0\} \cup U)$ and one vertex in W_2 is adjacent to exactly two vertices in $W_0 \setminus (\{v_0\} \cup U)$ and the other is adjacent to exactly one or two vertices in $W_0 \setminus (\{v_0\} \cup U)$. We now let $N(v_1) \cap (W_0 \setminus (\{v_0\} \cup U)) = \{v_5, v_6\}$ and $v_7 \in N(v_2) \cap (W_0 \setminus (\{v_0\} \cup U))$. Observe that the function η given by $\eta(v_1) = 2$, $\eta(v_i) = 1$ for each $i \in \{2, 7, 8, 9\}$ and $\eta(v_i) = 0$ for each $i \in \{0, 3, 4, 5, 6\}$, is a GIDF on H , leading that $\gamma_{gI}(H) \leq \omega(\eta) = 6$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 6$, this forces $\gamma_{gI}(H) = 6 = \gamma_I(H)$, a contradiction. This concludes the proof. \square

Theorem 8. For any cubic graph H on n vertices,

$$\gamma_{gI}(H) - \gamma_I(H) = \begin{cases} 2, & \text{if } n = 4, \\ 1, & \text{if } n = 6, \\ 0, & \text{if } n \notin \{4, 6\}. \end{cases}$$

Proof. Noting that H is cubic, we obtain n is even. If $n = 4$, then $H = K_4$ and clearly $\gamma_{gI}(H) - \gamma_I(H) = 4 - 2 = 2$. If $n = 6$, then $H \in \{H_1, H_2\}$ (see Figure 1). Then the mapping g given by $g(v_i) = 0$ for each $i \in \{1, 2, 3\}$ and $g(v_i) = 1$ for each $i \in \{4, 5, 6\}$ is a $\gamma_I(H)$ -function. Furthermore, the mapping h given by $h(v_1) = h(v_4) = 0$ and $h(v_i) = 1$ for each $i \in \{2, 3, 5, 6\}$ is a $\gamma_{gI}(H)$ -function. Thus $\gamma_{gI}(H) - \gamma_I(H) = \omega(h) - \omega(g) = 4 - 3 = 1$. If $n \geq 10$, then by Lemma 2 and Proposition 1, $\gamma_{gI}(H) = \gamma_I(H)$. Suppose next that $n = 8$. Suppose that $f = (W_0, W_1, W_2)$ be a $\gamma_I(H)$ -function.

By Theorem 2 and the fact that $\gamma_I(H)$ is an integer, we obtain $\gamma_{gI}(H) \geq \gamma_I(H) \geq 4$. Thus it suffices to prove that $\gamma_{gI}(H) \leq 4$. Let $v_1v_2 \in E(H)$. Since $n = 8$ and $d(v_1) = d(v_2) = 3$, there must exist two vertices, say v_3 and v_4 , in H with $v_3, v_4 \notin N(v_1) \cup N(v_2)$. Let $V(H) \setminus \{v_i \mid 1 \leq i \leq 4\} = \{v_i \mid 5 \leq i \leq 8\}$.

Case 1. $v_3v_4 \in E(H)$.

Note that $v_1v_2, v_3v_4 \in E(H)$ and $v_3, v_4 \notin N(v_1) \cup N(v_2)$. Moreover, since H is cubic, we have that each vertex of $\{v_1, v_2, v_3, v_4\}$ has exactly two neighbors in $\{v_5, v_6, v_7, v_8\}$. Thus the mapping h given by $h(v_i) = 0$ for each $i \in \{1, 2, 3, 4\}$ and $h(v_i) = 1$ for each $i \in \{5, 6, 7, 8\}$, is a GIDF on H , implying that $\gamma_{gI}(h) \leq 4$.

Case 2. $v_3v_4 \notin E(H)$.

First, suppose that $N(v_1) \cap N(v_2) \neq \emptyset$. Now let $v_8 \in N(v_1) \cap N(v_2)$. Since $v_1v_2 \in E(H)$, we obtain $\{v_1, v_2, v_8\} \subseteq N[v_1] \cap N[v_2]$. Moreover, since H is cubic, we obtain

$$|N[v_1] \cup N[v_2]| = |N[v_1]| + |N[v_2]| - |N[v_1] \cap N[v_2]| \leq 4 + 4 - |\{v_1, v_2, v_8\}| = 5.$$

Note that $n = 8$ and $v_3, v_4 \notin N[v_1] \cup N[v_2]$. Thus there must exist some vertex, say v_5 , in $\{v_5, v_6, v_7\}$ with $v_5 \notin N[v_1] \cup N[v_2]$. If v_3, v_4 and v_5 are pairwise nonadjacent vertices in H , then since each of v_3, v_4 and v_5 has degree three, we have $v_3, v_4, v_5 \in \bigcap_{i=6}^8 N(v_i)$. Further, since $v_8 \in N(v_1) \cap N(v_2)$, we have $\{v_i : 1 \leq i \leq 5\} \subseteq N(v_8)$ and hence $d(v_8) \geq 5$, a contradiction. Noting that $v_3v_4 \notin E(H)$, we have either $v_3v_5 \in E(H)$ or $v_4v_5 \in E(H)$. We may presume that $v_3v_5 \in E(H)$. Moreover, since H is cubic, we obtain:

(i) For any $i \in \{1, 2\}$, $v_{3-i}, v_8 \in N(v_i)$ and $|N(v_i) \cap \{v_6, v_7\}| = 1$.

(ii) $|N(v_3) \cap \{v_6, v_7, v_8\}| = 2$ and $|N(v_5) \cap \{v_4, v_6, v_7, v_8\}| = 2$.

Thus the mapping h given by $h(v_i) = 0$ for each $i \in \{1, 2, 3, 5\}$ and $h(v_i) = 1$ for each $i \in \{4, 6, 7, 8\}$, is a GIDF on H , leading that $\gamma_{gI}(H) \leq 4$.

Second, suppose that $N(v_1) \cap N(v_2) = \emptyset$. Since $d(v_1) = d(v_2) = 3$, $v_1v_2 \in E(H)$ and $v_3, v_4 \notin N(v_1) \cup N(v_2)$, we may presume that $N(v_1) \setminus \{v_2\} = \{v_5, v_6\}$ and $N(v_2) \setminus \{v_1\} = \{v_7, v_8\}$. Moreover, since $d(v_3) = 3$ and $v_1, v_2, v_4 \notin N(v_3)$, we may presume that $N(v_3) = \{v_5, v_6, v_7\}$. Noting that $v_1, v_2, v_3 \notin N(v_4)$, we have $N(v_4) \subseteq \{v_i : 5 \leq i \leq 8\}$. If $v_8 \notin N(v_4)$, then since $d(v_4) = 3$, we have $N(v_4) = \{v_5, v_6, v_7\}$ and so $d(v_8) = |N(v_8)| = |\{v_2\}| = 1$, a contradiction. Therefore $v_8 \in N(v_4)$, implying that $N(v_4) \setminus \{v_8\} \subseteq \{v_5, v_6, v_7\}$. Recall that H is cubic. If $N(v_4) \setminus \{v_8\} = \{v_i, v_7\}$ for some $i \in \{5, 6\}$, then clearly $v_{11-i}v_8 \in E(H)$ and so the mapping h given by $h(v_j) = 1$ for each $j \in \{2, 3, 8, i\}$ and $h(v_j) = 0$ for each $j \notin \{2, 3, 8, i\}$, is a GIDF on H , implying that $\gamma_{gI}(H) \leq 4$, and if $N(v_4) \setminus \{v_8\} = \{v_5, v_6\}$, then $v_7v_8 \in E(H)$ and hence the mapping h given by $h(v_j) = 1$ for each $j \in \{2, 3, 4, 6\}$ and $h(v_j) = 0$ for each $j \notin \{2, 3, 4, 6\}$, is a GIDF on H , leading that $\gamma_{gI}(H) \leq 4$, which completes our proof. □

Acknowledgements: This study was supported by the National Natural Science Foundation of China (12061007) and the Doctor Fund of Heze University (XY23BS12, XY23BS48).

Conflict of Interest: The fourth author is an editor of Communications in Combinatorics and Optimization.

Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

- [1] H. Abdollahzadeh Ahangar, M. Chellali, M. Hajjari, and S.M. Sheikholeslami, *Further progress on the total Roman $\{2\}$ -domination number of graphs*, Bull. Iranian Math. Soc. **48** (2022), no. 3, 1111–1119.
<https://doi.org/10.1007/s41980-021-00565-z>.
- [2] H. Abdollahzadeh Ahangar, M. Chellali, S.M. Sheikholeslami, and J.C. Valenzuela-Tripodoro, *Total Roman $\{2\}$ -dominating functions in graphs*, Discuss. Math. Graph Theory **42** (2024), no. 3, 937–958.
<https://doi.org/10.7151/dmgt.2316>.
- [3] S. Banerjee, M.A. Henning, and D. Pradhan, *Perfect Italian domination in cographs*, Appl. Math. Comput. **391** (2021), 125703.
<https://doi.org/10.1016/j.amc.2020.125703>.
- [4] A. Cabrera-Martínez, A. Conchado Peiró, and J.M. Rueda-Vázquez, *Further results on the total Italian domination number of trees*, AIMS Math. **8** (2023), no. 5, 10654–10664.
<https://doi.org/10.3934/math.2023540>.
- [5] M. Chellali, T.W. Haynes, S.T. Hedetniemi, and A.A. McRae, *Roman $\{2\}$ -domination*, Discrete Appl. Math. **204** (2016), 22–28.
<https://doi.org/10.1016/j.dam.2015.11.013>.
- [6] G. Hao, K. Hu, S. Wei, and Z. Xu, *Global Italian domination in graphs*, Quaest. Math. **42** (2019), no. 8, 1101–1115.
<https://doi.org/10.2989/16073606.2018.1506831>.
- [7] Z. Liang, L. Wu, and J. Yang, *The Italian domination numbers of some generalized Sierpiński networks*, Discrete Math. Algorithms Appl. **16** (2024), no. 4, 2350045.
<https://doi.org/10.1142/S1793830923500453>.
- [8] J. Lyle, *Regular graphs with large Italian domatic number*, Commun. Comb. Optim. **7** (2022), no. 2, 257–271.
<https://doi.org/10.22049/CCO.2021.27092.1194>.
- [9] K. Paul and A. Pandey, *Perfect Italian domination on some generalizations of cographs*, Comp. Appl. Math. **43** (2024), no. 6, Article number: 390.
<https://doi.org/10.1007/s40314-024-02901-5>.

-
- [10] D. Pradhan, S. Banerjee, and J.B. Liu, *Perfect Italian domination in graphs: Complexity and algorithms*, *Discrete Appl. Math.* **319** (2022), 271–295.
<https://doi.org/10.1016/j.dam.2021.08.020>.
- [11] L. Volkmann, *Remarks on the restrained Italian domination number in graphs*, *Commun. Comb. Optim.* **8** (2023), no. 1, 183–191.
<https://doi.org/10.22049/CCO.2021.27471.1269>.