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Short Note

A note on distance-fall colorings

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Dedicated to Odile Favaron

Abstract: We say a proper coloring of a graph is distance-k fall if every vertex is within distance k of at least one vertex of every color. We show that if G is a connected graph of order at least 3 that is 3-colorable, then it has a distance-2 fall 3-coloring. Further, for every integer $k \geq 2$, if T is a tree of order at least k, then T has a k-coloring such that every vertex is within distance $k-1$ of every color. This proves an old conjecture of Beineke and Henning that every tree of order n has an independent distance-d-dominating set of size at most $n/(d+1)$.

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1. Introduction

A *fall coloring* is a proper coloring of the vertices of a graph such that every vertex sees every color. That is, for each vertex v its closed neighborhood $N[v]$ contains all colors. Not all graphs have a fall coloring. The simplest example is the 5-cycle. In this paper we consider proper colorings where every vertex is "near" to every color. We say a coloring is *distance-k fall* if every vertex is within distance k of at least one vertex of every color. For example, for a graph of diameter 2, every proper coloring is distance-2 fall; and every odd cycle has a 3-coloring that is distance-2 fall.

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The terminology "*fall coloring*" was introduced in the 2000 paper by Dunbar et al. [5]. But the concept is older, having been studied before as partitioning a graph into independent dominating sets; see the references of [5]. Recall that a set S of vertices is *independent* if no two elements of S are joined by an edge. A set S is *dominating* if every vertex is either in S or adjacent to at least one vertex of S. An *independent dominating* set is one that is both independent and dominating. More generally, a set S is *distance-k dominating* if every vertex is within distance k of at least one vertex of S. Thus a distance-k fall coloring is a partition of the vertex set into independent distance-k dominating sets. The parameter *independent distance-k domination*, that is, the minimum size of an independent distance-k dominating set, was originally studied by Beineke and Henning [1].

2. Distance-Fall Colorings and Chromatic Number

It is immediate that if a (connected nontrivial) graph is bipartite, then the bipartite coloring is a fall coloring. So the next question to consider is 3-colorable graphs. For a coloring, we say a vertex is *d-good* if every color appears within distance d of the vertex; otherwise the vertex is *d-bad*. The following theorem shows that if the graph is 3-colorable then there is a proper coloring with 3 colors such that every vertex is 2-good.

Theorem 1. *If G is a connected graph of order at least 3 that is 3-colorable, then G has a distance-2 fall 3-coloring.*

Proof. Suppose to the contrary, that G does not admit a distance-2 fall 3-coloring. Take a proper 3-coloring of G that minimizes the number of 2-bad vertices. Without loss of generality, choose a 2-bad vertex $v \in V(G)$. Then $N(v)$ is monochromatic.
If there exists a vertex $w_1 \in N(v)$ with $\deg(w_1) = 1$, then by recoloring w_1 with the missing color in $N[v]$, we obtain a proper 3-coloring of G with strictly fewer 2-bad vertices, contradicting the minimality of the original coloring.
Therefore, assume $\deg(w) \geq 2$ for all $w \in N(v)$. Since v is 2-bad, we have $N(v) \cap N(w) = \emptyset$ and $N(w)$ is monochromatic for every $w \in N(v)$. That is, v and all vertices in its second neighborhood share the same color.
Now recolor v with the missing color in $N[v]$. Vertex v then becomes 2-good, and every neighbor of v remains 2-good. Moreover, no previously 2-good vertex becomes 2-bad under this recoloring. Thus, the total number of 2-bad vertices decreases, contradicting the minimality of the original coloring.
Therefore, there exists a proper 3-coloring of G with zero 2-bad vertices. □

It is perhaps interesting to note that an independent distance-2 dominating set is related to an independent isolating set. An isolating (or vertex-edge dominating) set S can be viewed as a 2-dominating set S with the added condition that if vertex v is at distance 2 from S, then all its neighbors are at distance 1 from S. An independent isolating set is an isolating set that is also independent. See for example [2, 3, 6]. It is easy to see that Theorem 1 does not extend to a partition into three independent isolating sets (for example C_5). However, in [3] it was shown that it almost does: specifically, if G is 3-colorable there exist three sets A, B, and C of vertices, such that each is an independent isolating set, their union is all vertices, and such that A and B overlap in at most one vertex, while C is disjoint from $A \cup B$.

It is, however, unclear if Theorem 1 generalizes to larger number of colors. The next step would be to determine whether every connected 4-colorable graph of order at least 4 has a 4-coloring such that every vertex is within distance three of all colors. This is best possible because of path–complete graph, meaning the graph that consists of a path and a complete graph joined by an edge. However, in the case of trees, the above theorem does generalize:

Theorem 2. *For integer $k \geq 2$, if T is a tree of order at least k, then T has a k-coloring such that every vertex is within distance $k-1$ of every color.*

Proof. Let T be a tree of diameter d. If $d < k$, then any coloring that uses each color at least once automatically has the desired property.
So, assume $d \geq k$. Let $P := v_1, v_2, \ldots, v_d$ be a diametral path in T. Define a coloring $c : V(P) \to \{0, 1, 2, \ldots, k-1\}$ on the vertices of P such that $c(v_i) = i \pmod{k}$.
Now, let $e = uv$ be a central edge of P, and let T_u and T_v be the components of $T - e$ containing u and v, respectively. If $T \setminus P$ denotes the forest induced by the vertices of T not contained in P, then color every vertex of $T \setminus P$ such that, in each of the components T_u and T_v, any two vertices at the same distance from u or v receive the same color.
We claim that this coloring has the desired property. If a vertex w lies within distance $(k-1)/2$ of the edge e, then w is within distance $k-1$ of each color on P. Otherwise, the unique path from w to P enters P at some vertex and continues along P via e; the first k vertices on this path all have distinct colors. Hence the theorem follows. $\square$

Theorem 1 is equivalent to saying that the vertices of a 3-colorable graph can be partitioned into three (disjoint) independent distance-2 dominating sets. And thus the independent distance-2 domination number of a 3-colorable graph is at most $n/3$, where n is the order. This generalizes Theorem 2 of [1], which proved the bound for trees. Furthermore, Theorem 2 shows that the independent distance-k domination number of a tree is at most $n/(k+1)$, which establishes the Conjecture at the end of [1].

We note that Theorem 2 and thus the conjecture in [1] was recently also proved by Bujtás et al. [4]. Indeed, they showed that Theorem 2 generalizes to bipartite graphs.

There is a version of Theorem 1 for general graphs with a slightly weaker distance condition. A *partial coloring* means a proper coloring where only some of the vertices are colored.

Theorem 3. *If G is a connected graph of order at least 3, then G has a partial 3-coloring such that every vertex is within distance 3 of every color.*

Proof. Consider any partial 3-coloring that maximizes the number of colored vertices. Note that any uncolored vertex v has neighbors of every color, since it is not possible to color v. Thus, out of all partial 3-colorings with the maximum number of colored vertices, take the one with the minimum number of 2-bad vertices and let v be such a vertex.

We know v is colored. Furthermore, all of its neighbors are colored. If some vertex w at distance 2 from v is uncolored, then v is 3-good since $N(w)$ contains all colors. So assume every vertex within distance two of v is colored. Then, as in the proof of Theorem 1, one can re-color either v or a neighbor of v so that v and its neighbors are 2-good. Note that if this recoloring makes an uncolored vertex bad, then we have a contradiction of the original requirement that the maximum number of vertices were colored. So every vertex that is 2-bad is 3-good; or in other words, every vertex is 3-good. □

This strengthens the first result in [1] for the case that $k = 3$.

We conclude with a brief comment about graph operations. Kaul and Mitillos [7] showed that if a graph G has a fall k-coloring and a graph H has a k-coloring, then the cartesian product $G \Box H$ has a fall k-coloring. The same idea works here: if G has a distance-d fall k-coloring f_G and H has a k-coloring f_H, then the cartesian product $G \Box H$ has a distance-d fall k-coloring. As per usual in the cartesian product, consider both colorings as assigning integers in the range 1 to k and take the coloring of vertex (g, h) in the product to be the sum $f_G(g) + f_H(h)$ modulo k.

We note further that if G has a distance-d_G fall k_G-coloring and H has a distance-d_H fall k_H-coloring, then the cartesian product $G \Box H$ has a distance-$(d_G + d_H)$ fall $k_G k_H$-coloring. Simply take the coloring of vertex (g, h) in the product to be the ordered pair $(f_G(g), f_H(h))$.

Similar results can be shown for other products.

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