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Upper bounds for $[1,2]$-domination number in trees

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Dedicated to Odile Favaron

Abstract: A set S of vertices is a $[1,2]$-set of a graph G if every vertex v not in S is adjacent to at least one but no more than two vertices in S. The minimum cardinality of a $[1,2]$-set is the $[1,2]$-domination number. In this paper, we present two upper bounds on the $[1,2]$-domination number of trees in terms of the order, number of support vertices and number of leaves. Furthermore, extremal trees reaching one of these two bounds are provided.

Keywords: $[1,2]$-set, $[1,2]$-domination number, trees.

AMS Subject classification: 05C69

1. Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Moreover, the *degree* $\deg_G(v)$ of v is $|N_G(v)|$. Let Δ denote the *maximum degrees* of vertices in G. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. Let $L(G)$ and $S(G)$ denote the set of leaves and support vertices of G, respectively, and let

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$\ell(G) = |L(G)|$ and $s(G) = |S(G)|$. A support vertex with two leaf neighbors or more is said to be *strong* while a support vertex with exactly one leaf neighbor is said to be *weak*. We also define one further set, namely the set of semi-support vertices $SS(G) = \{v \in V(G) - (S(G) \cup L(G)) : |N(v) \cap S(G)| \geq 1\}$.

A *tree* is an acyclic connected graph. A *star* is the graph $K_{1,t}$, with $t \geq 1$, where the vertex of degree t of the star is called the *center*. A *double star* $S_{r,s}$ is a tree obtained from two disjoint stars $K_{1,r}$ and $K_{1,s}$ by adding an edge joining their centers. As usual a *path* on n vertices is denoted by P_n.

A *dominating set* of a graph G is a subset S of vertices of G such that every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets of G.

In 2012, Caro et al. [2] introduced and studied k-fair dominating sets in graphs defined as follows. For an integer $k \geq 1$, a *k-fair dominating set* in G is a dominating set D such $|N(v) \cap D| = k$ for every vertex $v \in V - D$. The *k-fair domination number* $fd_k(G)$ is the minimum cardinality among all k-fair dominating sets of G. It is worth noting that, in particular, 1-fair dominating sets are perfect dominating sets introduced by Livingston and Stout [4].

In 2013, Chellali et al. [3] introduced and studied $[1, 2]$-sets of a graph G defined as follows. A subset $S \subseteq V$ in a graph G is a *$[1, 2]$-set* if for every vertex $v \in V - S$, $1 \leq |N(v) \cap S| \leq 2$, that is, every vertex in $V - S$ is adjacent to at least one, but not more than two vertices in S. The *$[1, 2]$-domination number* $\gamma_{[1,2]}(G)$ is the minimum cardinality among all $[1, 2]$-sets in G. It is worth noting that Cáceres et al [1] have studied this problem but in the general case which they called k-quasiperfect dominating sets.

It follows from the definition that for every graph G,

$$\gamma(G) \leq \gamma_{[1,2]}(G) \leq fd_1(G). \tag{1.1}$$

Caro et al. [2] showed that if T is a tree of order $n \geq 2$, then $fd_1(T) \leq \frac{n}{2}$, with equality if and only if T is the corona of some tree. By (1.1), it follows that for every tree T of order n, $\gamma_{[1,2]}(T) \leq \frac{n}{2}$.

Our aim in this paper is to improve the $\frac{n}{2}$-upper bound on the $[1, 2]$-domination number of trees by presenting two upper bounds in terms of the order, number of leaves and number of support vertices.

Before presenting these results, we need some further, but standard, notations and definitions. The *distance* between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G while the *diameter*, diam(G), of G is the maximum distance among all pairs of vertices in G. A *rooted tree* T is a tree with a distinguished special vertex r, called the root. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. Also, the *depth* of v, denoted depth(v), is the largest distance from v to a vertex descendant of v. The *maximal subtree* T_v at v is the subtree of T induced by v and the children and descendants of v.

2. First upper bound

In this section, we give an upper bound on the $[1,2]$-domination number of a tree. Moreover, we provide a characterization of all trees attaining this upper bound. For the purpose of characterizing extremal trees, we define the family $\mathcal{T}$ of trees $T = T_k$ can be obtained as follows. Let T_1 be a path P_2, and if $k \geq 2$, then T_{i+1} is recursively obtained from T_i by one of the following operations. Let one vertex of T_1 be considered a support vertex and the other a leaf, and thus $\ell(P_2) = 1$ and $s(P_2) = 1$.
Operation $\mathcal{O}_1$: Attach a path P_2 by joining one of its leaves to a vertex in $S(T_i) \cup SS(T_i)$.
Operation $\mathcal{O}_2$: Attach a path P_3 by joining one of its leaves to a leaf of T_i with $i \neq 1$.
It is worth noting that from the way a tree $T \in \mathcal{T}$ is constructed, no support vertex of T is strong. Also, every path P_{3t+1} for any integer $t \geq 2$ belongs to $\mathcal{T}$ since it can be obtained from T_1 by first applying Operation $\mathcal{O}_1$, and then $t-1$ times Operation $\mathcal{O}_2$.

Lemma 1. *If* $T \in \mathcal{T}$, *then* $\gamma(T) = \frac{n(T)+s(T)}{3}$.

Proof. We use induction on the number of operations k performed to construct T. The property is true for $T_1 = P_2$. Let $k \geq 2$, and suppose the property is true for all trees in $\mathcal{T}$ constructed with $k-1 \geq 0$ operations, and let $T' = T_{k-1}$. By the induction hypothesis on T', we have $\gamma(T') = \frac{n(T')+s(T')}{3}$. Let T be a tree obtained from T' using one of the operations $\mathcal{O}_1$ or $\mathcal{O}_2$.
Assume first that T was obtained from T' by Operation $\mathcal{O}_1$. Then $n(T) = n(T')+2$, and $s(T) = s(T') + 1$. Also, it is easy to see that $\gamma(T) = \gamma(T') + 1$. Using the fact that $\gamma(T') = \frac{n(T')+s(T')}{3}$, it follows that

$$\gamma(T) = \gamma(T') + 1 = \frac{n(T') + s(T')}{3} + 1 = \frac{n(T) + s(T)}{3}.$$

Assume now that T was obtained from T' by Operation $\mathcal{O}_2$. Clearly, $n(T) = n(T') + 3$, and since no support vertex of T' is strong $s(T) = s(T')$. As before, it is easy to see that $\gamma(T) = \gamma(T') + 1$. Now, since $\gamma(T') = \frac{n(T')+s(T')}{3}$, we obtain $\gamma(T) = \gamma(T') + 1 = \frac{n(T')+s(T')}{3} + 1 = \frac{n(T)+s(T)}{3}$. □

The following corollary is an immediate consequence of Lemma 1.

Corollary 1. *If* $T \in \mathcal{T}$, *then* $\gamma_{[1,2]}(T) \geq \frac{n(T)+s(T)}{3}$.

Now we are ready to prove the new upper bound.

Lemma 2. *For every tree* T *of order* $n \geq 2$, $\gamma_{[1,2]}(T) \leq \frac{n+s(T)}{3}$, *with equality only if* $T \in \mathcal{T}$.

Proof. We use the induction on the order n. If $n = 2$, then $T = P_2$, where $\gamma_{[1,2]}(P_2) = 1 = \frac{n+s(P_2)}{3}$ and P_2 belongs to $\mathcal{T}$. If $n = 3$, then $T = P_3$ and $\gamma_{[1,2]}(P_3) = 1 < \frac{n+s(P_3)}{3}$. This establishes the base case. Let $n \geq 4$ and assume that every tree T' of order $n' \geq 2$ with $n' < n$ satisfies $\gamma_{[1,2]}(T') \leq \frac{n+s(T')}{3}$, with equality only if $T' \in \mathcal{T}$. Let T be a tree of order n. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,n-1}$, and we have $\gamma_{[1,2]}(K_{1,n-1}) = 1 < \frac{n+1}{3}$. If $\mathrm{diam}(T) = 3$, then T is a double star $S_{r,t}$ and $\gamma_{[1,2]}(S_{r,t}) = 2 \leq \frac{n+2}{3}$ with equality if and only if $T = P_4$, where $P_4 \in \mathcal{T}$, since it is obtained from T_1 by using Operation $\mathcal{O}_1$. Moreover, if $\Delta = 2$, then $T = P_n$ and we have $\gamma_{[1,2]}(T) = \lceil \frac{n}{3} \rceil \leq \frac{n+2}{3}$, with equality if and only if $T = P_{3t+1}$ for some positive integer $t \geq 2$, an as noted above, $P_{3t+1} \in \mathcal{T}$ for any $t \geq 2$. Hence we may assume in the following that $\mathrm{diam}(T) \geq 4$ and $\Delta \geq 3$.

Let $P = v_1 v_2 \ldots v_k$ $(k \geq 5)$ be a longest path in T such that $\deg_T(v_2)$ is as large as possible. Root the tree T at the vertex v_k, and consider the following cases.

Case 1. $\deg_T(v_2) = 2$.

Assume first that $\deg_T(v_3) \geq 3$, and consider the tree $T' = T - \{v_1, v_2\}$. Then $n(T') = n(T) - 2 \geq 4$ and $s(T') = s(T) - 1$. We note that v_3 is either a support vertex or adjacent to a support vertex in T as well as in T', and thus $v_3 \in S(T') \cup SS(T')$. Now, since any $[1, 2]$-set of T' can be extended to a $[1, 2]$-set of T by adding to it vertex v_1, we have $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. It follows that

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 1 \leq \frac{n(T') + s(T')}{3} + 1 \\ &= \frac{(n(T) - 2) + (s(T) - 1)}{3} + 1 = \frac{n(T) + s(T)}{3}.\end{aligned}$$

Further if $\gamma_{[1,2]}(T) = \frac{n+s(T)}{3}$, then we have equality throughout this inequality chain. In particular, $\gamma_{[1,2]}(T') = \frac{n(T')+s(T')}{3}$, and thus by the induction hypothesis on T', $T' \in \mathcal{T}$. It follows that $T \in \mathcal{T}$, since it is obtained from T' by using Operation $\mathcal{O}_1$.

Next, assume that $\deg_T(v_3) = 2$, and consider the tree $T' = T - T_{v_3}$. Since $k \geq 5$, we have $n(T') \geq 2$. Note that if $n(T') = 2$, then T is a path P_5, where $\gamma_{[1,2]}(P_5) = 2 < \frac{n+2}{3}$. Hence we assume that $n(T') = n(T) - 3 \geq 3$, and thus $s(T') \leq s(T)$. As before, any $[1, 2]$-set of T' can be extended to a $[1, 2]$-set of T by adding to it vertex v_2, leading to $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. It follows that

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 1 \leq \frac{n(T') + s(T')}{3} + 1 \\ &= \frac{(n(T) - 3) + s(T)}{3} + 1 = \frac{n(T) + s(T)}{3}.\end{aligned}$$

Further if $\gamma_{[1,2]}(T) = \frac{n+s(T)}{3}$, then we have equality throughout this inequality chain. In particular, $\gamma_{[1,2]}(T') = \frac{n(T')+s(T')}{3}$ and $s(T) = s(T')$, meaning that v_5 is a support vertex in T' having v_4 as a unique leaf neighbor. By the induction hypothesis on T', $T' \in \mathcal{T}$. It follows that $T \in \mathcal{T}$, since it is obtained from T' by using Operation $\mathcal{O}_2$.

Case 2. $\deg_T(v_2) \geq 5$.
Let $T' = T - v_1$. Then $n(T') = n(T) - 1$ and $s(T') = s(T)$. Since v_2 has at least three leaf neighbors in T', v_2 belongs to every $[1,2]$-set of T', and such a set remains a $[1,2]$-set of T. Thus $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T')$, and using the induction hypothesis on T' we obtain

$$\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') \leq \frac{(n(T)-1)+s(T)}{3} < \frac{n(T)+s(T)}{3}.$$

Case 3. $\deg_T(v_2) = 4$.
Let $v_1 = x_1, x_2, x_3$ denote the three leaf neighbors of v_2. Assume first that v_3 has another child w of degree four with leaf neighbors w_1, w_2, w_3. Consider the tree T' obtained from T by removing the leaves x_1, x_2, x_3, w_1, w_2 and w_3. Clearly, $n(T') = n(T) - 6$ and $s(T') \leq s(T) - 1$. Now, if D is a minimum $[1,2]$-set of T', then either $v_3 \in D$ and so $D \cup \{v_2, w\}$ is a $[1,2]$-set of T or $v_3 \notin D$, and so $v_2, w \in D$ and clearly D remains a $[1,2]$-set of T. In either case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 2$. Using the induction hypothesis on T' it follows that

$$\begin{aligned}\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 2 &\leq \frac{n(T')+s(T')}{3} + 2\\ &\leq \frac{(n(T)-6)+(s(T)-1)}{3} + 2 < \frac{n(T)+s(T)}{3}.\end{aligned}$$

Next assume that v_3 has another child w of degree 3 with leaf neighbors w_1 and w_2. Consider the tree T' obtained from T by removing the vertices x_1, x_2, x_3, w, w_1 and w_2. Clearly $n(T') = n(T) - 6$ and $s(T') \leq s(T) - 1$. If D is a minimum $[1,2]$-set of T', then as before, either $v_3 \in D$ and so $D \cup \{v_2, w\}$ is a $[1,2]$-set of T or $v_3 \notin D$, so $v_2 \in D$ and $D \cup \{w_1, w_2\}$ is a $[1,2]$-set of T. In either case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 2$. Applying the induction hypothesis on T' we get

$$\begin{aligned}\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 2 &\leq \frac{n(T')+s(T')}{3} + 2\\ &\leq \frac{(n(T)-6)+(s(T)-1)}{3} + 2 < \frac{n(T)+s(T)}{3}.\end{aligned}$$

Consequences of the aforementioned situations, we can assume that every child of v_3 besides v_2 is either a leaf or a support vertex of degree 2. Suppose that $\deg_T(v_3) = 2$, and consider the tree $T' = T - T_{v_3}$. Clearly, $n(T') = n(T) - 5 \geq 2$, since $k \geq 5$. If $n(T') = 2$, then one can easily see that $\gamma_{[1,2]}(T) = 2 < \frac{n(T)+s(T)}{3}$. Hence let $n(T') \geq 3$, and so $s(T') \leq s(T)$. Since any $[1,2]$-set of T' can be extended to a $[1,2]$-set of T by adding to it v_2, we have that $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. By the induction hypothesis we have

$$\begin{aligned}\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1 &\leq \frac{n(T')+s(T')}{3} + 1\\ &\leq \frac{(n(T)-5)+s(T)}{3} + 1 < \frac{n(T)+s(T)}{3}.\end{aligned}$$

Thus, in the following, we assume that $\deg_T(v_3) \geq 3$. Suppose first that v_3 is a support vertex, and consider the tree $T' = T - \{x_1, x_2, x_3\}$. Clearly $n(T') = n(T) - 3 \geq 5$ and $s(T') = s(T) - 1$. If D is a $[1,2]$-set of T', then $v_3 \notin D$ and so D remains a $[1,2]$-set of T, or $v_3 \in D$, and so $D \cup \{v_2\}$ is a $[1,2]$-set of T. In either case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T')+1$. Using the induction hypothesis on T', it follows that $\gamma_{[1,2]}(T) < \frac{n(T)+s(T)}{3}$. Suppose now that v_3 is not a support vertex. Let $y_1, \ldots, y_t$ denote the children of v_3 with degree 2 and let y_i' be the leaf neighbor of y_i. Note that $t \geq 1$, since $\deg_T(v_3) \geq 3$. In this case, consider the tree T' obtained from T by removing v_2 and its three leaf neighbors. Clearly $n(T') = n(T) - 4 \geq 5$ and $s(T') = s(T) - 1$. As before, if v_3 belongs to some minimum $[1,2]$-set of T', then $D \cup \{v_2\}$ is a $[1,2]$-set of T leading to $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. Thus assume that v_3 is in no minimum $[1,2]$-set of T', and let D be a minimum $[1,2]$-set of T'. In this case, if $v_4 \in D$, then we can further assume that every $y_i' \in D$, so no $y_i \in D$, and thus $D \cup \{v_2\}$ is a $[1,2]$-set of T. Hence let $v_4 \notin D$. We can further assume that y_1 is the only child of v_3 in T' that belongs to D, and so again $D \cup \{v_2\}$ is a $[1,2]$-set of T. In all case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. Now, using the induction on T', it follows that

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 1 \leq \frac{n(T') + s(T')}{3} + 1 \\ &= \frac{(n(T) - 4) + (s(T) - 1)}{3} + 1 < \frac{n(T) + s(T)}{3}.\end{aligned}$$

Case 4. $\deg_T(v_2) = 3$.
Let x_1 and x_2 be the leaf neighbors of v_2, where $v_1 = x_1$. As in Case 2, one can see that $\gamma_{[1,2]}(T) < \frac{n(T)+s(T)}{3}$ when $\deg_T(v_3) = 2$. Henceforth, we assume that $\deg_T(v_3) \geq 3$. Moreover, in the remainder of the proof of this case, set D will always be referred to as a minimum $[1,2]$-set of the tree T' that is under consideration.
Assume first that $v_3 \in S(T)$, and consider the tree $T' = T - \{x_1, x_2\}$. Clearly $n(T') = n(T) - 2 \geq 5$ and $s(T') = s(T) - 1$. Observe that v_3 is a strong support vertex in T', and so $T' \notin \mathcal{T}$. Hence $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$. Now if $v_2 \in D$, then D remains a $[1,2]$-set of T, while if $v_2 \notin D$, then $v_3 \in D$ and so $D \cup \{v_2\}$ is a $[1,2]$-set of T. In either case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. It follows that

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 1 < \frac{n(T') + s(T')}{3} + 1 \\ &= \frac{(n(T) - 2) + (s(T) - 1)}{3} + 1 = \frac{n(T) + s(T)}{3}.\end{aligned}$$

Next, we can assume that $v_3 \notin S(T)$. Let $w_1 = v_2, w_2, \ldots, w_p$ ($p \geq 2$) be the children of v_3. If $\deg_T(w_i) = 2$ for some $i \geq 2$ and w_i' is the leaf neighbor of w_i, then let $T' = T - \{w_i, w_i'\}$. Clearly $n(T') = n(T) - 2 \geq 6$ and $s(T') = s(T) - 1$. As before, since T' has a strong support vertex, namely $v_2 = w_1$, $T' \notin \mathcal{T}$ and thus $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$. Now since $D \cup \{w_i'\}$ is a $[1,2]$-set of T, we obtain as above $\gamma_{[1,2]}(T) < \frac{n(T)+s(T)}{3}$. Accordingly, taking into account the previous cases (1, 2 and

3), we assume that $\deg_T(w_i) = 3$ for each $i \in \{1, \ldots, p\}$. In this case, let w_i^1 and w_i^2 be the leaf neighbors of w_i, for every i. To complete the proof, we consider the following subcases.

Subcase 4.1. $p \geq 5$.
Consider the tree $T' = T - \{w_1^1, w_2^1, w_i^1, w_i^2 \mid 3 \leq i \leq p\}$. Then $n(T') = n - 2(p-1)$ and $s(T') = s(T) - p + 3$. Also since v_3 is a support vertex with at least three leaf neighbors in T', $T' \notin \mathcal{T}$ and thus $\gamma_{[1,2]}(T')) < \frac{n(T')+s(T')}{3}$. It also implies that $v_3 \in D$, and $v_2 = w_1$ and $w_2 \in D$. Hence $D \cup \{w_i \mid 3 \leq i \leq p\}$ is a $[1,2]$-set of T, leading to $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + (p-2)$. It follows that

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + (p-2)\\ &< \frac{n(T) - 2(p-1) + (s(T) - p + 3)}{3} + (p-2)\\ &< \frac{n(T) + s(T)}{3}.\end{aligned}$$

Subcase 4.2. $p = 3$.
Consider the tree $T' = T - \{w_1^1, w_2^1, w_3^1, w_3^2\}$. Note that $n(T') = n(T) - 4$ and $s(T') = s(T)$. Now, if $v_3 \in D$, then let $v_2 = w_1, w_2 \in D$ and obviously $D \cup \{w_3\}$ is a $[1,2]$-set of T. Also, if $v_3 \notin D$, then $w_3 \in D$ and we can further assume that $v_4 \notin D$ (for otherwise replacing w_3 in D by v_3 leads us back to the previous situation). In this case we may assume $w_1 = v_2$ and $w_2^2 \in D$, implying that $D \cup \{w_2^1\}$ is a $[1,2]$-set of T. Hence in either case $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. Using the induction hypothesis on T' we obtain

$$\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1 = \frac{n(T) - 4 + s(T)}{3} + 1 < \frac{n(T) + s(T)}{3}.$$

Subcase 4.3. $p = 4$.
Consider the tree $T' = T - \{w_1^1, w_2^1, w_3^1, w_3^2, w_4^1, w_4^2\}$. Then $n(T') = n(T) - 6$ and $s(T') = s(T) - 1$. Now, if $v_3 \in D$, then let $v_2 = w_1, w_2 \in D$ and obviously $D \cup \{w_3, w_4\}$ is a $[1,2]$-set of T. Also, if $v_3 \notin D$, then $w_3, w_4 \in D$ and thus w_1^2 and $w_2^2 \in D$. In this case $D \cup \{w_1^2, w_2^1\}$ is a $[1,2]$-set of T. Hence in either case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 2$. By induction on T' we obtain

$$\begin{aligned}\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 2 &\leq \frac{n(T') + s(T')}{3} + 2\\ &= \frac{n(T) - 6 + (s(T) - 1)}{3} + 2 < \frac{n(T) + s(T)}{3}.\end{aligned}$$

Subcase 4.4. $p = 2$.
Thus $\deg_T(v_3) = 3$. Further situations need to be examined.

(a) v_4 has a child z_3 with depth 2.

Let $v_4z_3z_2z_1$ be a path in T. If z_2 is a strong support vertex, then let $T' = T - T_{v_3}$. Clearly $n(T') = n(T) - 7$ and $s(T') = s(T) - 2$. Also, since $D \cup \{w_1, w_2^1, w_2^2\}$ is a [1, 2]-set of T, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 3$. Moreover, since T' has strong support vertex, $T' \notin \mathcal{T}$, and by induction on T', $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$. Therefore, $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. Next, assume that $\deg_T(z_2) = 2$. If $\deg_T(z_3) \geq 3$, then let $T' = T - T_{z_2}$. Clearly $n(T') = n(T) - 2$ and $s(T') = s(T) - 1$. Also, $D \cup \{z_1\}$ is a [1, 2]-set of T, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. As before, $T' \notin \mathcal{T}$, and thus by induction on T', $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$. Therefore, $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. Finally assume that $\deg_T(z_3) = 2$, and consider the tree $T' = T - T_{z_3}$. Then $n(T') = n(T) - 3$, $s(T') = s(T) - 1$, and $D \cup \{z_2\}$ is a [1, 2]-set of T. It follows that $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$, and using the induction on T', we get $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

(b) v_4 has a child z_2 with depth 1.

Let $v_4z_2z_1$ be a path in T. If z_2 is a strong support vertex, then let $T' = T - T_{v_3}$. Clearly $n(T') = n(T) - 7 \geq 5$ and $s(T') = s(T) - 2$. Also, since $D \cup \{w_1, w_2^1, w_2^2\}$ is a [1, 2]-set of T, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 3$. Moreover, since T' has strong support vertex, $T' \notin \mathcal{T}$, and by induction on T', $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$. Therefore, $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. Next, assume that $\deg_T(z_2) = 2$. Let $T' = T - T_{z_2}$. Clearly $n(T') = n(T) - 2$ and $s(T') = s(T) - 1$. Also, $D \cup \{z_1\}$ is a [1, 2]-set of T, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. As before, $T' \notin \mathcal{T}$, and thus by induction on T', $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$. Therefore, $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

(c) v_4 is a strong support vertex.

Consider the tree $T' = T - T_{v_3}$, and apply a similar argument to that used for Item (a). Hence $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

(d) v_4 is a weak support vertex.

According to Items (a), (b) and (c), $\deg_T(z_3) = 3$. Let w be the unique leaf neighbor of v_4 and consider the tree $T' = T - \{w_1^1, w_2^1, w\}$. Then $n(T') = n(T) - 3$ and $s(T') = s(T) - 1$. Now, if $v_3 \in D$, then we may assume that $w_1, w_2 \in D$ and thus $(D - \{v_3\}) \cup \{w\}$ is a [1, 2]-set of T. Also, if $v_3, v_4 \notin D$, then we may assume that $w_1, w_2 \in D$ and so $D \cup \{w\}$ is a [1, 2]-set of T. Finally, if $v_3 \notin D$ and $v_4 \in D$, then we may suppose that $w_1, w_2^2 \in D$ and thus $D \cup \{w_2^1\}$ is a [1, 2]-set of T. In either case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. Using the induction on T', we obtain $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

According to Items (a), (b),(c) and (d), we can assume in the following that $\deg_T(v_4) = 2$.

(e) v_5 has a child z_4 with depth one, two or three or v_5 is a strong support vertex.

Consider the tree $T' = T - T_{v_4}$. Then $n(T') = n(T) - 8$ and $s(T') = s(T) - 2$. Also, since $D \cup \{v_3, w_1, w_2\}$ is a $[1,2]$-set of T, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 3$. Using the induction on T', we obtain $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

(f) v_5 is a weak support vertex.

Let $T' = T - T_{v_3}$, and note that v_5 is a strong support vertex in T'. Thus, $T' \notin \mathcal{T}$, and so by induction on T', $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$. Now, since $D \cup \{w_1, w_2^1, w_2^2\}$ is a $[1,2]$-set of T, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 3$. Using the fact that $n(T') = n(T) - 7$ and $s(T') = s(T) - 2$, we obtain as above $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

(g) $\deg_T(v_5) = 2$.

We first assume that $\deg_T(v_6) \geq 3$, where we briefly provide hints showing in this case that $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. If v_6 is a support vertex or it has a descendant different from w_1 and w_2 that is a strong support vertex, then consider the tree $T' = T - T_{v_4}$. We note here that if v_6 is a support vertex in T, then it is a strong support vertex in T'. Clearly, $n(T') = n(T) - 8$, $s(T') = s(T) - 2$ and $D \cup \{v_3, w_1, w_2\}$ is a $[1,2]$-set of T. Since $T' \notin \mathcal{T}$, $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$ and we obtain $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. Next, if there is a descendant of v_6 which is a support vertex x of degree 2 whose parent has degree at least 3, then let $T' = T - \{x, x'\}$, where x' is the leaf neighbor of x. Then $n(T') = n(T) - 2$, $s(T') = s(T) - 1$ and $D \cup \{x'\}$ is a $[1,2]$-set of T. As before, $T' \notin \mathcal{T}$, $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$ and we obtain $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. Form these previous situations, we conclude that v_6 belongs to pendant paths of length 3, 4 or 5. For one of these pendant paths, consider the furthest leaf from v_6, say x_1, and let $x_1x_2x_3$ be a (sub)path, where x_3 is not necessarily adjacent to v_6. Consider the tree $T' = T - T_{x_3}$. Then $n(T') = n(T) - 3$, $s(T') \leq s(T)$ and $D \cup \{x_2\}$ is a $[1,2]$-set of T. As before, $T' \notin \mathcal{T}$, $\gamma_{[1,2]}(T') < \frac{n(T')+s(T')}{3}$ and we obtain $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

In the following we can assume that $\deg_T(v_6) \leq 2$. If $\deg_T(v_6) = 1$, then $D = \{v_5, w_1, w_2\}$ is a $[1,2]$-set of T and clearly $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. Thus assume that $\deg_T(v_6) = 2$. If $\deg_T(v_7) = 1$, then $D = \{v_6, v_3, w_1, w_2\}$ is a $[1,2]$-set of T and again $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$. Finally suppose that $\deg_T(v_7) \geq 2$, and let $T' = T - T_{v_6}$. Then $n(T') = n(T) - 10$, $s(T') \leq s(T) - 1$ and $D \cup \{v_5, w_1, w_2\}$ is a $[1,2]$-set of T. Applying the induction on T', we obtain $\gamma_{[1,2]}(T) < \frac{n+s(T)}{3}$.

This completes the proof. □

According to Corollary 1 and Lemma 2, the main result of this section follows.

Theorem 1. *Let T be a tree of order $n(T) \geq 3$. Then $\gamma_{[1,2]}(T) = \frac{n(T)+s(T)}{3}$ if and only if $T \in \mathcal{T}$.*

3. Second upper bound

In this section, we present a second upper bound on the [1, 2]-domination number of a tree in terms of the order, number of leaves and the number of support vertices. Recall that for a path P_2, it is assumed that $\ell(P_2) = 1$ and $s(P_2) = 1$.

Theorem 2. *If T is a tree of order $n \geq 2$, then*

$$\gamma_{[1,2]}(T) \leq \frac{2n(T) + 3s(T) - 2\ell(T) - 1}{4}.$$

Proof. The proof is by induction on n. If $n \leq 5$, then it is easy to verify that $\gamma_{[1,2]}(T) \leq \frac{2n(T)+3s(T)-2\ell(T)-1}{4}$, thus establishing the base case. Let $n \geq 6$ and assume that each tree T' with n' such that $2 \leq n' < n$ satisfies $\gamma_{[1,2]}(T') \leq \frac{2n(T')+3s(T')-2\ell(T')-1}{4}$. Let T be a tree of order n. If $\mathrm{diam}(T) \leq 3$, then the result is clearly trivial. Hence we assume that $\mathrm{diam}(T) \geq 4$. Let $v_1 v_2 \dots v_k$ ($k \geq 5$) be a diametral path in T and root T at v_k. For any support vertex $v \in S(T)$, let L_v denote the set of leaf neighbors of v. Also, along this proof, set D will be referred to as a minimum [1, 2]-set of the tree T' that is under consideration.

Assume first that $v_3 \in S(T)$, and consider the tree $T' = T - L_{v_2}$. Then $n(T') = n(T) - |L_{v_2}|$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - |L_{v_2}| + 1$. By the induction hypothesis on T', $\gamma_{[1,2]}(T') \leq \frac{2(n(T)-|L_{v_2}|)+3(s(T)-1)-2(\ell(T)-|L_{v_2}|+1)-1}{4}$. Moreover, since $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$, it follows that

$$\begin{aligned}
\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 1 \\
&\leq \frac{2(n(T) - |L_{v_2}|) + 3(s(T) - 1) - 2(\ell(T) - |L_{v_2}| + 1) - 1}{4} + 1 \\
&\leq \frac{2n(T) + 3s(T) - 2\ell(T) - 1}{4}.
\end{aligned}$$

In the following, we assume that $v_3 \notin S(T)$. We consider the following cases.

Case 1. $\deg_T(v_3) = 2$.

Let $T' = T - T_{v_3}$ and note that $n(T') = n(T) - |L_{v_2}| - 2 \geq 2$. If $n(T') = 2$, then $\{v_2, v_4\}$ is a minimum [1, 2]-set of T, and the result is valid. Hence let $n(T') \geq 3$. Thus $s(T') \leq s(T)$ and $\ell(T') \geq \ell(T) - |L_{v_2}|$. Also, since $D \cup \{v_2\}$ is a [1, 2]-set of T, we have $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 1$. Using the induction on T', we obtain

$$\begin{aligned}
\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 1 \\
&\leq \frac{2(n(T) - |L_{v_2}| - 2) + 3s(T) - 2(\ell(T) - |L_{v_2}|) - 1}{4} + 1 \\
&= \frac{2n(T) + 3s(T) - 2\ell(T) - 1}{4}.
\end{aligned}$$

Case 2. $\deg_T(v_3) = 3$.

Let $N_T(v_3) - \{v_4\} = \{v_2 = w_1, w_2\}$. Assume first that $\deg_T(v_4) \geq 3$, and consider the tree $T' = T - T_{v_3}$. Then $n(T') = n(T) - |L_{w_1}| - |L_{w_2}| - 3 \geq 3$, $s(T') = s(T) - 2$ and $\ell(T') = \ell(T) - |L_{w_1}| - |L_{w_2}|$. Now if $v_4 \in D$, then $D \cup \{v_3, w_1, w_2\}$ is a $[1,2]$-set of T, while if $v_4 \notin D$, then $D \cup \{w_1, w_2\}$ is a $[1,2]$-set of T. Hence in either case, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 3$. Using the induction on T', it follows that

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 3\\ &\leq \frac{2(n(T) - |L_{w_1}| - |L_{w_2}| - 3) + 3(s(T) - 2) - 2(\ell(T) - |L_{w_1}| - |L_{w_2}|) - 1}{4} + 3\\ &= \frac{2n(T) + 3s(T) - 2\ell(T) - 1}{4}.\end{aligned}$$

Assume now that $\deg_T(v_4) = 2$, and consider the tree $T' = T - T_{v_4}$. Note that $n(T') = n(T) - |L_{w_1}| - |L_{w_2}| - 4 \geq 1$. If $n(T') \in \{1, 2\}$, then $\{w_1, w_2, v_5\}$ is a minimum $[1,2]$-set of T and in either case, the result is valid. Thus, let $n(T') \geq 3$. Then $s(T) - 2 \leq s(T') \leq s(T) - 1$ and $\ell(T) - |L_{w_1}| - |L_{w_2}| \leq \ell(T') \leq \ell(T) - |L_{w_1}| - |L_{w_2}| + 1$. Since $D \cup \{v_3, w_1, w_2\}$ is a $[1,2]$-set of T, $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + 3$. It follows from the induction on T' that

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + 3\\ &\leq \frac{2(n(T) - |L_{w_1}| - |L_{w_2}| - 4) + 3s(T') - 2(\ell(T')) - 1}{4} + 3.\end{aligned}$$

Now, if $s(T') = s(T) - 2$, then one can check that $\gamma_{[1,2]}(T) \leq \frac{2n(T)+3s(T)-2\ell(T)-1}{4}$. Hence let $s(T') = s(T) - 1$. Then v_6 becomes a new support vertex in T' with leaf neighbor v_5 leading to $\ell(T') = \ell(T) - |L_{w_1}| - |L_{w_2}| + 1$. After substitution and a simple calculation, we obtain $\gamma_{[1,2]}(T) < \frac{2n(T)+3s(T)-2\ell(T)-1}{4}$.

Case 3. $\deg_T(v_3) \geq 4$.

Let $N_T(v_3) - \{v_4\} = \{v_2 = w_1, w_2, \ldots, w_q\}$ and consider $T' = T - \cup_{i=1}^{q} L_{w_i}$. Then $n(T') = n(T) - \sum_{i=1}^{q} |L_{w_i}|$, $s(T') = s(T) - q + 1$ and $\ell(T') = \ell(T) - \sum_{i=1}^{q} |L_{w_i}| + q$. Since $q \geq 3$, v_3 has at least three leaf neighbor in T' and so $v_3 \in D$. Then $D \cup \{w_1, w_2, \ldots, w_q\}$ is a $[1,2]$-set of T, and thus $\gamma_{[1,2]}(T) \leq \gamma_{[1,2]}(T') + q$. Using the induction on T' we obtain

$$\begin{aligned}\gamma_{[1,2]}(T) &\leq \gamma_{[1,2]}(T') + q\\ &\leq \frac{2(n(T) - \sum_{i=1}^{q} |L_{w_i}|) + 3(s(T) - q + 1) - 2(\ell(T) - \sum_{i=1}^{q} |L_{w_i}| + q) - 1}{4} + q\\ &\leq \frac{2n(T) + 3s(T) - 2\ell(T) - 1}{4},\end{aligned}$$

and the proof is complete. □

It is worth noting that the bound of Theorem 2 is better than the one of Theorem 1 for all trees T whose order $n \leq 3\ell(T) + 1 - \frac{5}{2}s(T)$.

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