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Research Article

Independent location-domination number of graphs

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This paper is dedicated to Professor Odile Favaron
on the occasion of her 88th birthday.

Abstract: Let $G = (V(G), E(G))$ be a graph. A set $I \subseteq V(G)$ is independent if no two vertices of I are adjacent. A set $D \subseteq V(G)$ is dominating if every vertex $u \in V(G) \setminus D$ is adjacent to a vertex in D. A set $L \subseteq V(G)$ is an independent locating-dominating set (ILD-set) of G if L is independent and dominating with the additional property that $N(u) \cap L \neq N(v) \cap L$ for any pair of distinct $u, v \in V(G) \setminus L$. The independent location-domination number of a graph G is the minimum cardinality of an ILD-set of G and is denoted by $i_\ell(G)$. In this paper, we study the non-existence of ILD-sets of maximal outerplanar graphs and circulant graphs. In trees, we prove that $\frac{n+1}{3} \leq i_\ell(T) \leq n-1$ for every tree T of n vertices. We further prove that there exists a tree T with prescribed value $i_\ell(T)$ between these bounds.

Keywords: independence number; domination number; location-domination.

AMS Subject classification: 05C69

1. Introduction

Throughout this paper, we let a graph $G = (V(G), E(G))$ be finite and simple, no loops or multiple edges. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. For a vertex $x \in V(G)$, a *neighbour* of x in G is a vertex y such that $xy \in E(G)$. The

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neighbour set of x in G is the set of all the neighbours of x in G and is denoted by $N_G(x)$. For $A \subseteq V(G)$, we let $N_A(x) = N_G(x) \cap A$. The *closed neighbor set* of x in G is $N_G[x] = N_G(x) \cup \{x\}$. A vertex v is an *isolated vertex* of G if $N_G(v) = \emptyset$. A *complete graph*, a *path* and a *cycle* of order n are denoted by K_n, P_n and C_n, respectively. A *complete k-partite graph* having partite sets of $n_1, ..., n_k$ vertices is denoted by $K_{n_1,...,n_k}$. When $k = 2$, the graph K_{n_1,n_2} is called *complete bipartite.* Further a complete bipartite graph with $n_1 = 1$ or $n_2 = 1$ is called a *star.* A *tree* is a connected graph having no cycle as a subgraph. A vertex of degree one is called a *leaf* and a vertex which is adjacent to a leaf is called a *support vertex.* A set $I \subseteq V(G)$ is *independent* if no two vertices in I are adjacent. The length of a shortest cycle subgraph of G is called the *girth* of G and is denoted by $g(G)$. Since a tree is a graph that has no cycle as a subgraph, we say that the girth of a tree is infinite.

A graph G is called *planar* if there exists a drawing of G without crossing edges. An *outerplanar graph* is a planar graph whose all vertices lie on the infinite face. A *maximal outerplanar graph*, a *mop*, is an outerplanar graph such that no edges can be added without violating the outerplanarity. A *circulant graph* $C_n\langle a_1, \ldots, a_l\rangle$ of order n with the adjacency list $a_1, \ldots, a_l$ has the vertex set $\{x_0, x_1, \ldots, x_{n-1}\}$ and the edge set $\{x_i x_j : (i - j) \equiv (\pm a_k) \pmod{n}$ for some $1 \le k \le l\}$.

A set $I \subseteq V(G)$ is *independent* if every pair of vertices in I is non-adjacent. The maximum cardinality of an independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. A set $D \subseteq V(G)$ is *dominating* if every vertex in $V(G) \setminus D$ is adjacent to some vertex in D. A dominating set L is *locating* if $N_L(u) \neq N_L(v)$ for any pair of distinct vertices $u, v \in V(G) \setminus L$. The *location-domination number* of G is the minimum cardinality of a locating-dominating set, *an LD-set*, of G and is denoted by $\gamma_\ell(G)$.

The study of location-domination of graphs was initiated by Rall and Slater [16] and Slater [17, 18]. The aim was to introduce graph configurations that can provide fault detection in the whole network system. In a network system which consists of nodes and lines, an operation plan is to locate processors on some nodes to cover the rest and yet able to determine which nodes are broken. We label each of processors so that we can detect the nodes without processors. The question is:

Problem 1. What is the smallest number of nodes that are used to locate processors so that we can detect faulty processors in the whole systems?

To link with the entire system, a set L of nodes that possess the multiprocessors is a dominating set. The un-labeled nodes outside L can be specified individually as each of them is linked to different subsets of L. This is the concept of locating set. Location-domination does not only have widely applications (see [8, 15] for example) but also has a number of theoretical studies. In 2007, Blidia et al. [2] established the upper and lower bounds of $\gamma_\ell(G)$ when G is a tree having l leaves and s support vertices that:

Theorem 1. *If G is a tree of order $n \geq 3$, then*

$$\frac{n+l-s+1}{3} \leq \gamma_\ell(G) \leq \frac{n+l-s}{2}. \tag{1.1}$$

It is worth noting that the upper bound of Theorem 1 was improved to $\frac{n+l-s-sl}{2}$ by [3] where sl is the number of vertices that has only support vertices as the neighbors. Then, in 2010, Chellali et al. [7] extended this study by establishing an upper bound of $\gamma_\ell(G)$ when G has at most one cycle:

Theorem 2. *[7] If G is a connected graph of order $n \geq 2$ with at most one cycle, then*

$$\gamma_\ell(G) \leq \frac{n+l-s+1}{2}.$$

Chellali et al. [7] further established an interesting result, showing how to obtain a minimal LD-set from a maximum independent set of the graphs when the girth is given.

Theorem 3. *[7] If G is a graph with girth $g(G) \geq 5$, then every maximum independent set L is a minimal LD-set.*

An interesting extension of the study of LD-sets in graphs is when the sets are independent. This was introduced by Slater and Sewell in [19] as they pointed out when the detectors (vertices) in a monitoring set (LD-set) needs to be out of signal range (non-adjacent) to avoid signal interfere between the detectors. Hence, a set $L \subseteq V(G)$ is an *independent locating-dominating set*, or an *ILD-set*, if L is both locating-dominating and independent. The *independent location-domination number* of a graph G is the minimum cardinality of an ILD-set of G and is denoted by $i_\ell(G)$. Thus, we have by the definitions that

$$\gamma_\ell(G) \leq i_\ell(G) \leq \alpha(G). \tag{1.2}$$

In particular, Slater and Sewell [19] established the values of $i_\ell(G)$ of paths and cycles as shown in the following proposition. It is worth noting that the cycle C_n does not possess any ILD-set when $n \in \{3, 4\}$.

Proposition 1. *For a natural number n, we have that $i_\ell(P_n) = \lceil \frac{2n}{5} \rceil$. Further if $n \geq 5$, then we have that*

$$i_\ell(C_n) = i_\ell(P_n) = \left\lceil \frac{2n}{5} \right\rceil.$$

For more examples of the study on location-domination see [1, 4, 6, 9–11], and for some variants of location-domination see [5, 12–14].

In this paper, in Section 2, we study the existence of ILD-sets of complete graphs and complete multi-partite grahs. In Section 3, apart from complete graphs, we show that there are infinitely many circulant graphs that do not possess ILD-sets as well as there are infinitely many circulant graphs that possess an ILD-set. In Section 4, we establish the sharp upper bound of $i_\ell(G)$ when G is a maximal outerplanar graph (mop) and show that there are infinitely many mops that do not possess ILD-set. In section 5, we obtain both a sharp upper bound and a sharp lower bound of $i_\ell(G)$ when G is a tree and further show that there exists a tree T having $i_\ell(T) = k$ for any k between the bounds.

2. Independent locating-dominating sets of common classes of graphs

In this section, we study independent location-dominations of common classes of graphs which are complete graphs and complete multipartite graphs. In complete graphs, we prove that the graphs have an ILD-set only when the order is less than 3. We let G be a graph that has vertices u, v and w such that $N[v] = N[u] = N[w]$. We let L be an ILD-set of G. Because the set L is independent, it follows that $|\{u, v, w\} \cap L| < 2$. Renaming vertices if necessary, we let $u, v \notin L$. Thus, $N_L(v) = N_L(u)$, contradicting L is locating. Hence, if there exist three vertices $u, v, w \in V(G)$ with $N[v] = N[u] = N[w]$, then graph G does not admit an ILD-set.

Proposition 2. *For a natural number n, every K_n has an ILD-set if and only if $n \leq 2$. Moreover, $i_\ell(K_n) = 1$ for $n \in \{1, 2\}$.*

Proof. Clearly, the complete graph K_n when $n \in \{1, 2\}$ possesses an ILD-set on a singleton vertex. Hence, it suffices to show that K_n does not have an ILD-set when $n \geq 3$. Assume that $n \geq 3$. Thus there are three vertices $u, v, w \in V(G)$ with $N[v] = N[u] = N[w]$. Therefore, the complete graph K_n does not possess an ILD-set when $n \geq 3$. This completes the proof. □

Next, we establish the existence of ILD-sets of complete k-partite graphs when $k \geq 2$. We begin with the result when $k = 2$, the bipartite graphs. We next prove that the complete bipartite graph $K_{n,m}$ when $n, m \geq 2$ does not possess an ILD-set.

Proposition 3. *For natural numbers n, m, we have that $K_{n,m}$ has ILD-sets if and only if $n = 1$ or $m = 1$.*

Proof. Let the complete bipartite graph $K_{n,m}$ have the bipartition $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. Suppose to the contrary that $K_{n,m}$ has L as ILD-set when $n, m \geq 2$. Without loss of generality, we let $x_1 \in L$. Because L is independent,

$y_1, \ldots, y_m \notin L$. Because L is a dominating set, it follows that $x_2, \ldots, x_n \in L$. Hence $L = \{x_1, \ldots, x_n\}$. Therefore, $N_L(y_1) = \{x_1, \ldots, x_n\} = N_L(y_2)$, contradicting L is a locating set. Thus, if $K_{n,m}$ has an ILD-set, then $n = 1$ or $m = 1$.

Conversely, without loss of generality, we let $n = 1$. Clearly, $\{y_1, \ldots, y_m\}$ is an ILD-set. This completes the proof. □

In the next theorem, we show that the complete multi-partite graphs do not possess any ILD-set if the number of partite sets is greater than two.

Theorem 4. *For natural numbers $k \geq 3$ and $n_1, \ldots, n_k \geq 1$, the graph $K_{n_1,\ldots,n_k}$ does not have ILD-sets.*

Proof. Suppose to the contrary that $K_{n_1,\ldots,n_k}$ has L as an ILD-set. Without loss of generality, we let $x_1 \in V_1$ be a vertex in L. Hence, $L \cap V_i = \emptyset$ for all $i \in \{2, \ldots, k\}$ because L is independent. If $|V_1| = 1$, then $L = \{x_1\}$. Therefore, $N_L(y) = \{x_1\} = N_L(z)$ for some $y \in V_2$ and $z \in V_3$, contradicting L is locating. Hence, $|V_1| \geq 2$. To dominate $x_2, \ldots, x_n$, we have that $\{x_2, \ldots, x_{n_1}\} \subseteq L$, implying that $L = \{x_1, \ldots, x_{n_1}\}$. Thus, $N_L(y) = \{x_1, \ldots, x_{n_1}\} = N_L(z)$, contradicting L is locating set. □

By Proposition 3 and Theorem 4, we have that:

Corollary 1. *The complete k-partite graph $K_{n_1,\ldots,n_k}$ has an ILD-set if and only if $k = 2$ and $\min\{n_1, n_2\} = 1$.*

3. Circulant Graphs

In this section, we first establish the existence of circulant graphs that do not possess any ILD-set.

Theorem 5. *For each $n \geq 3$, there are at least two circulant graphs $C_n\langle a_1, \ldots, a_l\rangle$ of order n which do not have an ILD-set. That is, the family of circulant graphs of order n that do not possess an ILD-set is non-singleton.*

Proof. Clearly, the complete graph K_n is the circulant graph $C_n\langle 1, 2, \ldots, n-1\rangle$. By Proposition 2, it suffices to establish one more class of circulant graphs which do not have an ILD-set.

When n is 3 and 4, the graph $C_n\langle 1\rangle$ is the cycle C_n as respectively shown in Figures 1 (A) and (B) in which do not have ILD-sets. We next show that $C_n\langle 1, 2\rangle$ does not have ILD-set for all $n \geq 5$. We first consider the case when $5 \leq n \leq 9$. We first assume that $n = 5$. Thus, $C_5\langle 1, 2\rangle$ is a complete graph K_5 as shown in Figure 1(C). By Proposition 2, the graph $C_5\langle 1, 2\rangle$ does not have an ILD-set.

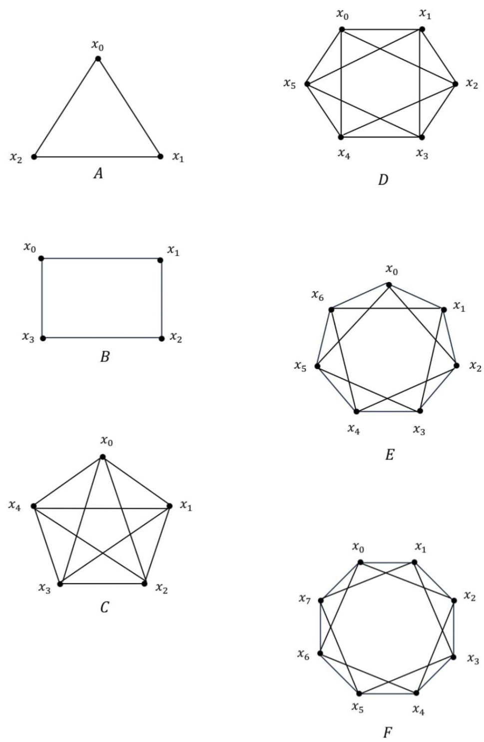

Figure 1. **Circulant graphs of orders $3, \ldots, 8$ that do not possess ILD-sets.**

Throughout the proof, we suppose to the contrary that the circulant graph $C\langle 1, 2\rangle$ has L as an ILD-set. We next assume that $n = 6$. The circulant graph $C_6\langle 1, 2\rangle$ is illustrated in Figure 1(D). Without loss of of generality let $x_5 \in L$. Because L is independent, $x_0, x_1, x_3, x_4 \notin L$. Thus, to dominate x_2, we have that $x_2 \in L$. So, we have that $L = \{x_2, x_5\}$. This implies that $N_L(x_0) = \{x_5, x_2\} = N_L(x_4)$, contradicting L is locating. Hence, $C_n\langle 1, 2\rangle$ does not have ILD-sets.

We consider the case when $n = 7$. We next show that the circulant graph of order 7 that is shown Figure 1(E) does not have any ILD-set. Suppose to the contrary that the graph has L as ILD-set. Suppose without loss of generality that $x_0 \in L$. Because L is independent, $x_1, x_2, x_6, x_5 \notin L$. Hence, to dominate x_3, x_4, we have that $x_3 \in L$ or $x_4 \in L$. If $x_3 \in L$, then $L = \{x_0, x_3\}$. Hence, $N_L(x_1) = \{x_0, x_3\} = N_L(x_2)$ contradicting L is locating. If $x_4 \in L$, then $L = \{x_0, x_4\}$. Hence, $N_L(x_2) = \{x_0, x_4\} = N_L(x_6)$ contradicting L is locating. Thus, $C_7\langle 1, 2\rangle$ does not have an ILD-set.

We now consider the case when $n = 8$. The circulant graph $C_8\langle 1, 2\rangle$ is shown in Figure 1(F). Suppose to the contrary that $C_8\langle 1, 2\rangle$ has an ILD-set. We let without loss of generality that $x_0 \in L$. Because L is independent, $x_1, x_2, x_7, x_6 \notin L$. Hence, to dominate x_3, x_4 and x_5, we have that $\{x_3, x_4, x_5\} \cap L \neq \emptyset$. If $x_3 \in L$, then

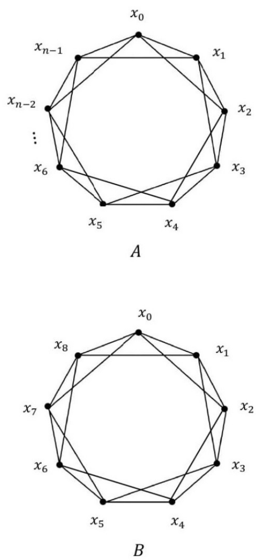

Figure 2. **The circulant graphs $C_n\langle 1,2\rangle$ when $n \in \{9, 10\}$.**

$L = \{x_0, x_3\}$. So, $N_L(x_1) = \{x_0, x_3\} = N_L(x_2)$ contradicting L is locating. If $x_4 \in L$, then $L = \{x_0, x_4\}$. Thus, $N_L(x_1) = \{x_0\} = N_L(x_7)$ contradicting L is locating. If $x_5 \in L$, then $L = \{x_0, x_5\}$. Hence, $N_L(x_6) = \{x_0, x_5\} = N_L(x_7)$ contradicting L is locating. Therefore, when $n = 8$, the graph $C_8\langle 1,2\rangle$ does not have an ILD-set.

We now consider the case when $n = 9$. The circulant graph $C_9\langle 1,2\rangle$ is shown in Figure 2(A). Suppose to the contrary that $C_9\langle 1,2\rangle$ has an ILD-set. Without loss of generality let $x_0 \in L$. Because L is independent, $x_1, x_2, x_7, x_8 \notin L$. Hence, to dominate x_3, x_4, x_5 and x_6, we have that $\{x_3, x_4, x_5, x_6\} \cap L \neq \emptyset$. If $x_3 \in L$ and $x_6 \in L$, then $L = \{x_0, x_3, x_6\}$. So, $N_L(x_1) = \{x_0, x_3\} = N_L(x_2)$ contradicting L is locating. If $x_4 \in L$, then $L = \{x_0, x_4\}$. So, $N_L(x_1) = \{x_0\} = N_L(x_8)$ contradicting L is locating. If $x_5 \in L$, then $L = \{x_0, x_5\}$. So, $N_L(x_6) = \{x_5\} = N_L(x_4)$ contradicting L is locating. Therefore, when $n = 9$, the graph $C_9\langle 1,2\rangle$ does not have an ILD-set.

Thus, we assume that $n \geq 10$. An example of the graph $C_{10}\langle 1,2\rangle$ is shown in Figure 2(B). Suppose to the contrary that the graph $C_n\langle 1,2\rangle$ has an ILD-set. Without loss of generality let $x_1 \in L$. Thus $x_2, x_3, x_0, x_{n-1} \notin L$ because L is independent. If $x_4 \in L$, then $x_5, x_6 \notin L$. Thus $N_L(x_2) = \{x_1, x_4\} = N_L(x_3)$ contradicting L is locating. Thus $x_4 \notin L$. If $x_5 \notin L$, then $N_L(x_2) = \{x_1\} = N_L(x_3)$ contradicting L is locating. So $x_5 \in L$ implying $x_6, x_7 \notin L$. If $x_8 \in L$, then $x_9, x_{10} \notin L$. Hence $N_L(x_7) = \{x_5, x_8\} = N_L(x_6)$ contradicting L is locating. Hence, $x_8 \notin L$. We have that $N_L(x_4) = \{x_5\} = N_L(x_6)$ contradicting L is locating. So, when $n \geq 10$, $C_n\langle 1,2\rangle$ does not have an ILD-set. This completes the proof. □

Although we prove that there are infinitely many circulant graphs that do not possess ILD-sets, we also have infinitely many circulant graphs that contain ILD-sets too. In particular, we have at least two circulant graphs of order n that admit ILD-sets when n is a composite number qr where $r \geq 5$. It is obvious that the circulant grah $C_n\langle n\rangle$ is the union of n isolated vertices which admits an ILD-set. We then turn our attention to circulant graphs which are isolate-free (graphs who do not contain an isolated vertex).

Theorem 6. *For a natural number $n \geq 5$, the family of isolate-free circulant graphs of order n that contain an ILD-set is non-empty. Further, if $n = qr$ for natural numbers $r \geq 5, q \geq 1$, the family of isolate-free circulant graphs of order n is non-singleton.*

Proof. The first obvious example is, when $n \geq 5$, the graph $C_n\langle 1\rangle$ is a cycle C_n which contains ILD-sets because of Proposition 1. This proves the first statement of the theorem.

Now we assume that $n = qr$ for natural numbers $r \geq 5, q \geq 1$. It can be checked that the circulant graph $C_{qr}\langle q\rangle$ is the union of q cycles of length $r \geq 5$. By Proposition 1, the graphs $C_{qr}\langle q\rangle$ admit an ILD-set. This proves the theorem. □

In fact, from Theorem 6, we see that every graphs in the proof has girth at least 5, and contains an ILD-set by Theorem 3. It would be interesting to ask if there exists a circulant graph of girth at most 4 that contains an ILD-set. The answer is yes.

For $t \geq 2$, we let G_1 be the circulant graph $C_{6t}\langle 1, 3\rangle$ containing vertices $x_0, \ldots, x_{6t-1}$. It can be checked that any two consecutive vertices x_i and x_{i+1} are adjacent (the subscripts are taken modulo n). However, x_i and x_{i+1} are not adjacent to any vertex in common. Further, x_i and x_{i+3} are adjacent but x_i and x_{i+3} are not adjacent to any vertex in common because $n \geq 12$. Thus, $g(G_1) \geq 4$. We see that $x_i, x_{i+1}, x_{i+4}, x_{i+3}, x_i$ is a cycle of length 4. Thus, $g(G_1) = 4$. We see that the set $\{x_0, x_2, \ldots, x_{n-2}\}$ is and ILD-set of G_1. Therefore, we have that:

Corollary 2. *For $t \geq 2$, there exists a connected circulant graph G of order $n = 6t$ having girth $g(G) = 4$ that contains an ILD-set.*

We can further conclude the Corollary 2 in general as follows.

Corollary 3. *For $t \geq 2$, there exists a connected graph G of order $n = 6t$ having girth $g(G) = 4$ that contains an ILD-set.*

4. Maximal Outerplanar Graphs

In this section, we study ILD-set in maximal outerplanar graphs. We first establish the upper bound of $i_\ell(G)$ in terms of the order. For the sake of convenience, we may call a maximal outerplanar graph shortly a *mop*.

Theorem 7. *Let G be a maximal outerplanar graph of order $n \geq 5$. Then, $i_\ell(G) \leq \lfloor \frac{n}{2} \rfloor$ and this bound is sharp.*

Proof. Let G be a maximal outerplanar graph of order n. It is well-known that G has a Hamiltonian cycle. Thus, the independence number of G is at most $\frac{n}{2}$. Since an ILD-set is independent, we have that $i_\ell(G) \leq \lfloor \frac{n}{2} \rfloor$ and this proves the upper bound. We now establish the construction of maximal outerplanar graphs satisfying the upper bound. We first assume that n is even. For an integer $k \geq 3$, we let $n = 2k$. Then, we let $x_1, \ldots, x_k$ be vertices and G' be a maximal outerplanar graph of order k. We can let $C = y_1 \ldots y_k y_1$ be a Hamiltonian cycle of G'. Hence, the graph G_k is obtained from $x_1, \ldots, x_k$ and G' by joining x_i to the vertices y_i and y_{i+1} for all $1 \leq i \leq k-1$ and joining x_k to y_k and y_1. It suffices to show that $i_\ell(G_k) = k$. Let L be an i_ℓ-set of G_k. We next show that $\{x_1, \ldots, x_k\} \subseteq L$. If $y_i \in L$ for some $2 \leq i \leq k-1$, then $y_{i-1}, y_{i+1}, x_{i-1}, x_{i+1} \notin L$. Hence, $N_L(x_{i-1}) = N_L(x_i) = \{y_i\}$, contradicting the set L is locating. So, $y_i \notin L$ for all $2 \leq i \leq k-1$. Similarly, $y_1, y_k \notin L$. To dominate $x_1, \ldots, x_k$, we have that $\{x_i : 1 \leq i \leq k\} \subseteq L$. Thus, $|L| = i_\ell(G_k) \geq k$. Clearly, the set $\{x_i : 1 \leq i \leq k\}$ is an ILD-set of G_k. By the minimality of $i_\ell(G_k)$, we have $i_\ell(G_k) \leq k$ implying that $i_\ell(G_k) = k$.

When n is odd, we let $n = 2k+1$. Then, we let $x_1, \ldots, x_k$ be vertices and G' be a maximal outerplanar graph of order $k+1$. Then, we let $C = y_1 \ldots y_{k+1} y_1$ be a Hamiltonian cycle of G'. Hence, the graph G_k is obtained from $x_1, \ldots, x_k$ and G' by joining x_i to the vertices y_i and y_{i+1} for all $1 \leq i \leq k$. We let L be an i_ℓ-set of G_k. By the similar arguments as the case when n is even, we can prove that $y_2, \ldots, y_k \notin L$. Thus, $x_2, \ldots, x_{k-1} \in L$. Further, to dominate x_1 and x_k, we have that $L \cap \{x_1, y_1\} \neq \emptyset$ and $L \cap \{x_k, y_{k+1}\} \neq \emptyset$. Therefore, $|L| = i_\ell(G_k) \geq k$. Similarly, the set $\{x_i : 1 \leq i \leq k\}$ is an ILD-set of G_k. By the minimality of $i_\ell(G_k)$, we have $i_\ell(G_k) \leq k$ implying that $i_\ell(G_k) = k$. This completes the proof. □

Theorem 8. *For $n \in \{3, 4, \ldots\} \setminus \{5\}$, there exists a mop G of order n such that G does not possess an ILD-set.*

Proof. We first show that every mop of order 3 and 4 does not have an ILD-set. There is only one mop of order 3 which is C_3 (Figure 3(A)). By Proposition 1, the mop of order 3 does not possess any ILD-set.

We consider the case when $n = 4$. The only mop of order 4 is a diamond, see Figure 3(B), where we can let the vertices be $x_1, \ldots, x_4$ in which x_1 and x_3 are the vertices of degree three where x_2 and x_4 are the vertices of degree two. Suppose to the contrary that G has an ILD-set L. If $x_1 \in L$, then $x_2, x_3, x_4 \notin L$ because L is independent. Hence $N_L(x_2) = \{x_1\} = N_L(x_3)$, a contradiction. Therefore $x_1 \notin L$. Similarly $x_3 \notin L$. So without loss of generality we let $x_2 \in L$.Thus $x_4 \in L, x_1 \notin L$ and $x_3 \notin L$ because L is independent dominating set. So $N_L(x_1) = \{x_2, x_4\} = N_L(x_3)$ when $n = 4$, G does not have an ILD-set.

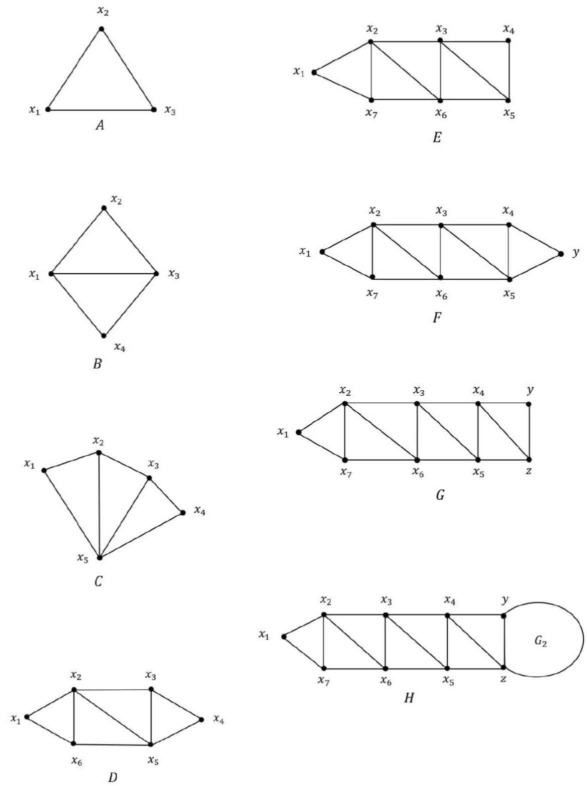

Figure 3. **The mops of different orders.**

We now consider the case when $n = 5$. It can be checked that there is exactly one mop of order 5 which is shown in Figure 3(C). In this case, we let $L = \{x_1, x_4\}$. Hence $N_L(x_0) = \{x_1, x_4\} \neq N_L(x_2) = \{x_1\} \neq N_L(x_3) = \{x_4\}$. So, when $n = 5$, we have $\{x_1, x_4\}$ as ILD-set.

When $n = 6$, we next show that the mop of order 6 that is shown in Figure 3(D) does not have any ILD-set. Suppose to the contrary that this graph G has L as ILD-set. We first consider the case when $x_1 \in L$. Because L is an independent set, it follows that $x_2, x_6 \notin L$. If $x_3 \in L$, then $L = \{x_1, x_3\}$. Hence, $N_L(x_5) = \{x_3\} = N_L(x_4)$ contradicting the set L is locating. If $x_5 \in L$, then $L = \{x_1, x_5\}$. So, $N_L(x_3) = \{x_5\} = N_L(x_4)$ contradicting the set L is locating. If $x_4 \in L$, then $L = \{x_1, x_4\}$. Thus, $N_L(x_3) = \{x_4\} = N_L(x_5)$ contradicting L the set is locating. Then, $x_1 \notin L$. To dominate x_1, we have $x_2 \in L$ or $x_6 \in L$. We consider the case when $x_2 \in L$. Thus, $x_1, x_3, x_5, x_6 \notin L$. So, $N_L(x_1) = \{x_2\} = N_L(x_6)$ contradicting L is locating. Therefore, $x_6 \in L$. Thus, $L = \{x_3, x_6\}$ or $\{x_4, x_6\}$. If $L = \{x_3, x_6\}$, then $N_L(x_2) = \{x_3, x_6\} = N_L(x_5)$. If $L = \{x_4, x_6\}$, then $N_L(x_1) = \{x_6\} = N_L(x_2)$. The both cases contradict L is a locating set. Thus, when $n = 6$, there is a mop which does not have an ILD-set.

When $n \geq 7$, let G_1 be a mop as shown in Figure 3(E). When $n = 8$, let G be a mop that is obtained from G_1 and a vertex y by joining y to x_4 and x_5 as shown in Figure 3(F). Further, when $n = 9$, let G be a mop that is obtained from G_1 and the vertices y and z by joining z to x_4, x_5 and y to x_4 and z as shown in Figure 3(G). Finally, when $n \geq 10$, let G be a mop which can be partitioned into mops G_1 and G_2 by a diagonal y, z as shown in Figure 3(H). Let G_2 be subgraph of G such that $V(G_2) = V(G) - \{x_1, \ldots, x_7\}$.

Observe that all our mops of orders $n = 7, 8, 9$ and $n \geq 10$ have G_1 as an induced subgraph.

We next show that G does not have ILD-sets. Suppose to the contrary that G has L as an ILD-set. Observe that $L \cap \{x_1, x_2, x_7\} \neq \emptyset$. We next prove the following claim

Claim A: $L \cap \{x_1, \ldots, x_7\} = \{x_1, x_3\}$.

Proof of Claim A. Suppose first that x_2 or $x_7 \in L$. If $x_2 \in L$, then $x_6 \notin L$ because L is independent. Thus $N_L(x_1) = \{x_2\} = N_L(x_7)$ contradicting L is locating set. So we assume that $x_7 \in L$. In this case $x_6 \notin L$ because L is independent. Thus $x_3 \in L$, otherwise $N_L(x_1) = N_L(x_2)$. Clearly $x_2, x_6 \notin L$. Because $N_L(x_2) \neq N_L(x_6)$, if follows that $x_5 \in L$. But $x_3, x_5 \in E(G_1)$ contradicting L is independent. Therefore $x_2 \notin L$ and $x_7 \notin L$.

Thus $x_1 \in L$. If $x_6 \in L$, then $x_3 \notin L$ because L is independent. So $N_L(x_2) = \{x_1, x_6\} = N_L(x_7)$ contradicting L is locating. Hence, $x_6 \notin L$ and $x_3 \in L$. So, by independence of L, we have that $L \cap \{x_1, \ldots, x_7\} = \{x_1, x_3\}$ completing the proof of the claim.

When $n = 8$. By Claim A, we have that $L \cap \{x_1, ..., x_7\} = \{x_1, x_3\}$. Thus $y \in L$ implying $N_L(x_4) = \{x_3, y\} = N_L(x_5)$, contradicting L is locating. Hence, when $n = 8$, the graph G does not have an ILD-set.

Finally, we suppose that $n \geq 9$. By Claim A, $L \cap \{x_1, \ldots, x_7\} = \{x_1, x_3\}$. If $z \in L$, then $y \notin L$ because L is independent. Thus $N_L(x_4) = \{x_3, z\} = N_L(x_5)$, a contradiction. So $z \notin L$ and $y \in L$. Therefore $L = \{x_1, x_3, y\}$. We see that $N_L(x_6) = \{x_3\} = N_L(x_5)$ contradicting is locating. Thus G does not have ILD- sets. This completes the proof. □

5. Independent locating-dominating sets of trees

In this section, we prove that $\frac{n+1}{3} \leq i_\ell(T) \leq n - 1$ for every tree T of order $n \geq 2$. Further, we establish constructions of trees, showing the existence of trees T with a prescribed value $i_\ell(T)$ from $\lceil \frac{n+1}{3} \rceil$ to $n - 1$ when the order is given to be n. It is worth noting that the girth of a tree is infinite. By Theorem 3, every tree possess an ILD-set.

For a tree T of order $n \geq 3$, we let L be an i_ℓ-set of T. Since the set L is independent and T has order $n \geq 3$, it follows that L cannot contain all vertices of T. Hence,

$i_\ell(T) \le n-1$. Further, we have by (4) and Theorem 1 that $i_\ell(T) \ge \gamma_\ell(T) \ge \frac{n+l-s+1}{3}$ where s is the number of support vertices and l is the number of leaves of T. Since $s \le l$, we have the following corollary that:

Corollary 4. *Let T be a tree of order $n \ge 3$. Then*

$$\frac{n+1}{3} \le i_\ell(T) \le n-1.$$

For the sharpness of the lower bound, we will give the constructions as follows.

When $n = 3k-1$ for $k \ge 2$, we let $P_3^i = c_i a_i b_i$ for $1 \le i \le k-1$ be $k-1$ copies of a path of order 3 and let $P_2 = c_k a_k$ be a path of order 2. The tree T_1 of order $n = 3k-1$ is constructed from $P_3^1, \dots, P_3^{k-1}, P_2$ by joining b_i to a_{i+1} for all $1 \le i \le k-1$. Clearly, $\{a_1, \dots, a_k\}$ is an ILD-set. Thus, $i_\ell(T_1) \le k = \frac{n+1}{3}$. Let L_1 be an ILD-set of T_1. To dominate c_i, we have that $\{c_i, a_i\} \cap L_1 \ne \emptyset$ for all $1 \le i \le k$. Therefore, $i_\ell(T_1) \ge k$, implying that $i_\ell(T_1) = k = \frac{n+1}{3}$.

When $n = 3k-2$, we let T_2 be the tree which is constructed from T_1 by removing the vertex c_k. Clearly, $\{a_1, \dots, a_k\}$ is an ILD-set. Thus, $i_\ell(T_2) \le k = \lceil \frac{n+1}{3} \rceil$. Let L_2 be an ILD-set of T_2. To dominate c_i and a_k, we have that $\{c_i, a_i\} \cap L_2 \ne \emptyset$ for all $1 \le i \le k-1$ and $\{b_{k-1}, a_k\} \cap L_2 \ne \emptyset$. Therefore, $i_\ell(T_2) \ge k$, implying that $i_\ell(T_2) = k = \lceil \frac{n+1}{3} \rceil$.

When $n = 3k+3$ for $k \ge 3$, we let $P_3^i = c_i a_i b_i$ for $1 \le i \le k$ be k copies of a path of order 3 and let x, y, z be three vertices. The tree T_3 of order $n = 3k+3$ is constructed from $P_3^1, \dots, P_3^k, x, y, z$ by joining b_i to a_{i+1} for all $1 \le i \le k-1$ and joining x to c_1, joining y to c_k and joining z to y. Clearly, $\{x, a_1, \dots, a_k, y\}$ is an ILD-set. Thus, $i_\ell(T_1) \le k+2 = \lceil \frac{n+1}{3} \rceil$. Let L_3 be an ILD-set of T_3. To dominate c_i, z and b_k, we have that $\{c_i, a_i\} \cap L_3 \ne \emptyset$ for all $2 \le i \le k-1$ and $\{y, z\} \cap L_3 \ne \emptyset$ and $\{a_k, b_k\} \cap L_3 \ne \emptyset$. Further, it can be seen that $|\{x, c_1, a_1, b_1\} \cap L_3| \ge 2$. Therefore, $i_\ell(T_3) \ge k+2$, implying that $i_\ell(T_2) = k+2 = \lceil \frac{n+1}{3} \rceil$.

For the realisability of T of the order between $\frac{n+2}{3}$ and $n-1$, we distinguish 3 cases according to the remainder of when n is divided by 3.

The Class $\mathcal{G}_1(\ell)$

For a natural number $k \ge 3$ and $n = 3k+1$, we let the tree T in the class $\mathcal{G}_1(\ell)$ be defined as follows: when $\ell = \lceil \frac{n+2}{3} \rceil = k+1$, we let $P_3^i = c_i a_i b_i$ for $1 \le i \le k$ be k copies of a path of order 3 and let x be a vertex. The tree T in the class $\mathcal{G}_1(k+1)$ is obtained from $P_3^1, P_3^2, \dots, P_3^k$ and x by joining b_j to a_{j+1} for all $1 \le j \le k-1$ and joining b_k to x.

When $\ell = k+2t$ for any $1 \le t \le k-1$, we let $P_3^i = c_i a_i b_i$ for $1 \le i \le k-t$ be $k-t$ copies of a path of order 3 and let $x, x_1, \dots, x_{3t}$ be vertices. The tree T in the class $\mathcal{G}_1(k+2t)$ is obtained from $P_3^1, \dots, P_3^{k-t}, x, x_1, \dots, x_{3t}$ by joining b_j to a_{j+1} for $1 \le j \le k-t-1$ and joining $x_1, \dots, x_{3t}$ to b_{k-t} and x to x_1.

When $\ell = k+2p+1$ for any $1 \le p \le k-1$, we let $P_3^i = c_i a_i b_i$ for $1 \le i \le k-p$ be $k-p$ copies of a path of order 3 and let $x_1, x_2, \dots, x_{3p+1}$. The tree T of the class

$\mathcal{G}_1(k+2p+1)$ is obtained from $P_3^1, \ldots, P_3^{k-p}, x_1, \ldots, x_{3p+1}$ by joining b_j to a_{j+1} for $1 \le j \le k-p-1$ and joining $x_1, \ldots, x_{3p+1}$ to b_{k-p}.

Lemma 1. *If $G \in \mathcal{G}_1(\ell)$, then $i_\ell(G) = \ell$.*

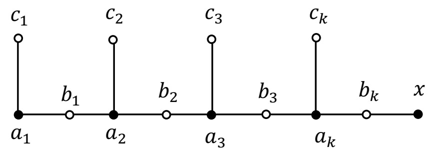

Figure 4. **The graph G when $\ell = k+1$.**

Proof. We first consider the case when $\ell = \lceil \frac{n+2}{3} \rceil = k+1$. To dominate $\{c_1, \ldots, c_k, x\}$, $L \cap \{a_i, c_i\} \ne \emptyset$ for all $1 \le i \le k$ and $L \cap \{b_k, x\} \ne \emptyset$. Thus, $i_\ell(G) = |L| \ge k+1$. Clearly $\{a_1, a_2, \ldots, a_k, x\}$ is an independent locating-dominating set of G. Thus, $i_\ell(G) \le |\{a_1, a_2, \ldots, a_k, x\}| = k+1$. Therefore, $i_\ell(G) = k+1 = \ell$. Let L be a smallest ILD-set.

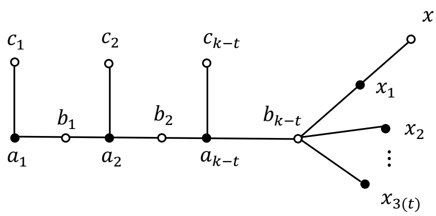

Figure 5. **The graph G when $\ell = k+2t$.**

We then consider the case when $\ell = k+2t$ for $1 \le t \le k-1$. To dominate $\{c_1, \ldots, c_{k-t}\}$, we have $L \cap \{a_i, c_i\} \ne \emptyset$ for all $1 \le i \le k-t$. If $b_{k-t} \in L$, then $\{x_1, \ldots, x_{3t}\} \cap L = \emptyset$ because L is independent. This implies $N_L(x_2) = N_L(x_3) = \{b_{k-t}\}$ contradicting L is locating. Thus, $b_{k-t} \notin L$. So, $\{x_1, x\} \cap L \ne \emptyset$ and $\{x_2, \ldots, x_{3t}\} \subseteq L$. Hence, $i_\ell(G) = |L| \ge (k-t) + 1 + 3t - 1 = k+2t$. Observe that $\{a_1, \ldots, a_{k-t}, x_1, \ldots, x_{3t}\}$ is an ILD-set. So $i_\ell(G) \le k-t+3t = k+2t$. Hence, $i_\ell(G) = k+2t = \ell$.

Finally we consider when $\ell = k+2p+1$ for $1 \le p \le k-1$. By similar argument as the case when $\ell = k+2t$, we have $i_\ell(G) = |L| \ge k-p+3p+1 = k+2p+1$. Moreover, we have $\{a_1, \ldots, a_{k-p}, x_1, \ldots, x_{3p+1}\}$ is an ILD-set. Thus $i_\ell(G) \le k-p+3p+1 = k+2p+1$. This implies $i_\ell(G) = k+2p+1 = \ell$. □

The Class $\mathcal{G}_0(\ell)$

For a natural number $k \ge 3$ and $n = 3k$, we let the tree T in the class $\mathcal{G}_0(\ell)$ be defined as follows: when $\ell = \lceil \frac{n+2}{3} \rceil = \lceil \frac{3k+2}{3} \rceil = k+1$, we let $P_3^i = c_i a_i b_i$ for

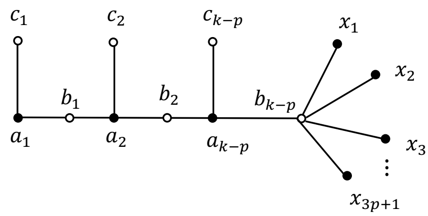

Figure 6. **The graph G when $\ell = k + 2p + 1$.**

$1 \leq i \leq k-1$ be $k-1$ copies of a path of order 3 and let x, x_1, x_2 be three vertices. The tree T in the class $\mathcal{G}_0(k+1)$ is obtained from $P_3^1, P_3^2, \ldots, P_3^{k-1}, x, x_1, x_2$ by joining b_j to a_{j+1} for all $1 \leq j \leq k-2$ and joining b_{k-1} to x_1 and joining x_2 to b_{k-1} and x.

When $\ell = k + 2t$ for any $1 \leq t \leq k-1$, we let $P_3^i = c_i a_i b_i$ for $1 \leq i \leq k-t$ be $k-t$ copies of a path of order 3 and let $x_1, \ldots, x_{3t}$ be vertices. The tree T in the class $\mathcal{G}_0(k+2t)$ is obtained from $P_3^1, P_3^2, \ldots, P_3^{k-t}$ and $\{x_1, \ldots, x_{3t}\}$ by joining b_j to a_{j+1} for all $1 \leq j \leq k-t-1$ and joining $x_1, \ldots, x_{3t}$ to b_{k-t}.

When $\ell = k + 2p + 1$ for any $1 \leq p \leq k-2$, we let $P_3^i = c_i a_i b_i$ for $1 \leq i \leq k-p$ be $k-p$ copies of a path of order 3 and let $x_1, \ldots, x_{3p}$ be vertices. The tree T in the class $\mathcal{G}_0(k+2p+1)$ is obtained from $P_3^1, P_3^2, \ldots, P_3^{k-p}$ and $x_1, \ldots, x_{3p}$ by joining b_j to a_{j+1} for all $1 \leq j \leq k-p-1$, joining a_{k-p} to $x_2, \ldots, x_{3p}$ and joining x_1 to b_{k-p-1}.

Lemma 2. *If $G \in \mathcal{G}_0(\ell)$, then $i_\ell(G) = \ell$.*

Proof. Let L be a smallest ILD-set. We first consider the case when $\ell = \lceil \frac{n+2}{3} \rceil = \lceil \frac{3k+2}{3} \rceil = k+1$. To dominate $\{c_1, \ldots, c_{k-1}, x_1, x_2\}$, we have $L \cap \{a_i, c_i\} \neq \emptyset$ for all $1 \leq i \leq k-1$, $L \cap \{x_1, b_{k-1}\}$ and $L \cap \{x_2, x\} \neq \emptyset$. Thus, $i_\ell(G) = |L| \geq k - 1 + 1 + 1 = k+1$. We see that $\{a_1, \ldots, a_{k-1}, x_1, x_2\}$ is an ILD-set. Therefore $i_\ell(G) \leq |\{a_1, a_2, \ldots, a_{k-1}, x_1, x_2\}| = k+1$. Hence $i_\ell(G) = k+1 = \ell$.

We next consider the case when $\ell = k + 2t$ for all $1 \leq t \leq k-1$. To dominate $\{c_1, \ldots, c_{k-t}, x_1, \ldots, x_{3t}\}$, we have $L \cap \{a_i, c_i\} \neq \emptyset$ for all $1 \leq i \leq k-t$. Similarly $b_{k-t} \notin L$, implying that $\{x_1, \ldots, x_{3t}\} \subseteq L$. Thus $i_\ell(G) = |L| \geq k - t + 3t = k + 2t$. We see that $\{a_1, \ldots, a_{k-t}, x_1, \ldots, x_{3t}\}$ is an ILD-set. By minimality of $i_\ell(G)$, we have $i_\ell(G) \leq k - t + 3t = k + 2t$. So, $i_\ell(G) = k + 2t$.

Finally we consider the case when $\ell = k + 2p + 1$ for all $1 \leq p \leq k-2$. To dominate $\{c_1, \ldots, c_{k-p}, x_1, \ldots, x_{3p}\}$, we have $L \cap \{a_i, c_i\} \neq \emptyset$ for all $1 \leq i \leq k-p-1$. Similarly, $\{b_{k-p-1}, x_1\} \cap L \neq \emptyset$. If $a_{k-p} \in L$, then $\{c_{k-p}, b_{k-p}, x_2, \ldots, x_{3p}\} \cap L = \emptyset$ because L is independent. Thus, $N_L(x_2) = N_L(x_3) = \{a_{k-p}\}$ contradicting L is locating. Hence, $a_{k-p} \notin L$, implying that $\{c_{k-p}, x_2, \ldots, x_{3p}, b_{k-p}\} \subseteq L$. Thus $|L| \geq (k-p-1) + 1 + 3p + 1 = k + 2p + 1$. Clearly $\{a_1, a_2, \ldots, a_{k-p-1}, c_{k-p}, b_{k-p}, x_1, \ldots, x_{3p}\}$ is an ILD-set. By the minimality of $i_\ell(G)$, we have $i_\ell(G) \leq (k-p-1) + 1 + 1 + 3p = k + 2p + 1$. □

The Class $\mathcal{G}_2(\ell)$

For a natural number $k \geq 3$ and $n = 3k+2$, we let T in the class $\mathcal{G}_2(\ell)$ be defined as follows: when $\ell = \lceil \frac{n+2}{3} \rceil = \lceil \frac{3k+4}{3} \rceil = k+2$, we let $P_3^i = c_i a_i b_i$ for $1 \leq i \leq k$ be k copies of a path of order 3 and let x_1, x_2 be two vertices. The tree T in the class $\mathcal{G}_2(k+2)$ is obtained from $P_3^1, P_3^2, \ldots, P_3^k$ and x_1, x_2 by joining b_j to a_{j+1} for all $1 \leq j \leq k-1$ and joining b_k to x_1, x_2.

When $\ell = k+2t+3$ for any $0 \leq t \leq k-2$, we let $P_3^i = c_i a_i b_i$ for $1 \leq i \leq k-t$ be $k-t$ copies of a path of order 3 and let $x_1, \ldots, x_{3t+2}$ be vertices. The tree T in the class $\mathcal{G}_2(k+2t+3)$ is obtained from $P_3^1, P_3^2, \ldots, P_3^{k-t}$ and $x_1, \ldots, x_{3t+2}$ by joining b_j to a_{j+1} for all $0 \leq j \leq k-t-1$, joining a_{k-t} to $x_2, \ldots, x_{3t+2}$ and joining x_1 to b_{k-t-1}.

When $\ell = k+2p+2$ for any $1 \leq p \leq k-1$, we let $P_3^i = c_i a_i b_i$ for $1 \leq i \leq k-p$ be $k-p$ copies of a path of order 3 and let $x_1, \ldots, x_{3p+1}$ and x_{3p+2} be vertices. The tree T in the class $\mathcal{G}_2(k+2p+2)$ is obtained from $P_3^1, P_3^2, \ldots, P_3^{k-p}, x_1, \ldots, x_{3p+1}, x_{3p+2}$ in the two such following ways. When $p \leq k-2$, the tree T is obtained by joining b_j to a_{j+1} for all $1 \leq j \leq k-p-1$ and joining a_{k-p} to $x_2, \ldots, x_{3p+1}$, joining b_{k-p} to x_{3p+2} and joining x_1 to b_{k-p-1}. When $p = k-1$, we have that $k-p = 1$. That is, there is only one $P_3^1 = c_1 a_1 b_1$. The tree T is obtained by joining a_1 to $x_1, \ldots, x_{3p+1}$, joining b_1 to x_{3p+2}.

Lemma 3. *If $G \in \mathcal{G}_2(\ell)$, then $i_\ell(G) = \ell$.*

Proof. Let L be a smallest ILD-set. We first consider the case when $\ell = \lceil \frac{n+2}{3} \rceil = \lceil \frac{3k+4}{3} \rceil = k+2$. To dominate $\{c_1, \ldots, c_k, x_1, x_2\}$, we have $L \cap \{a_i, c_i\} \neq \emptyset$ for all $1 \leq i \leq k$, $L \cap \{b_k, x_1, x_2\} \neq \emptyset$. If $b_k \in L$, then $x_1, x_2 \notin L$, implying $N_L(x_1) = N_L(x_2) = \{b_k\}$. This contradicts L is locating. So $b_k \notin L$ and $x_1, x_2 \in L$. Thus, $i_\ell(G) = |L| \geq k+2$. Clearly $\{a_1, \ldots, a_k, x_1, x_2\}$ is an ILD-set. Therefore by the minimality of $i_\ell(G) \leq |\{a_1, a_2, \ldots, a_k, x_1, x_2\}| = k+2$. Hence $i_\ell(G) = k+2 = \ell$.

We next consider the case when $\ell = k+2t+3$ for all $0 \leq t \leq k-2$. To dominate $\{c_1, \ldots, c_{k-t-1}, b_{k-t-1}, x_1\}$, we have $L \cap \{a_i, c_i\} \neq \emptyset$ for all $1 \leq i \leq k-t-1$ and $L \cap \{x_1, b_{k-t-1}\} \neq \emptyset$. Similarly $a_{k-t} \notin L$, implying that $\{b_{k-t}, c_{k-t}, x_2, \ldots, x_{3t+2}\} \subseteq L$. Thus $i_\ell(G) = |L| \geq (k-t-1) + 1 + 3t + 3 = k+2t+3$. We see that $\{a_1, \ldots, a_{k-t-1}, x_1, \ldots, x_{3t+2}, c_{k-t}, b_{k-t}\}$ is an ILD-set. So, by the minimality of $i_\ell(G) \leq k-t-1+3t+4 = k+2t+3$. Hence, $i_\ell(G) = k+2t+3 = \ell$.

Finally we consider the case where $\ell = k+2p+2$ for all $1 \leq p \leq k-1$. We first consider the case when $p = k-1$. Thus $k-p = 1$ implying that $\ell = 3p+3$. Clearly, the graph G is a star with one edge subdivided which $c_1, x_1, \ldots, x_{3p+2}$ are the vertices of degree one. Hence, it can be checked that $i_\ell(G) = 3p+3 = \ell$. We now consider the case when $p \leq k-2$. Recall that x_1 is adjacent to b_{k-p-1} in this case. To dominate $\{c_1, \ldots, c_{k-p-1}, b_{k-p-1}, x_1, b_{k-p}, x_{3p+2}\}$, we have $L \cap \{a_i, c_i\} \neq \emptyset$ for all $1 \leq i \leq k-p-1$ and $L \cap \{b_{k-p-1}, x_1\} \neq \emptyset$ and $L \cap \{b_{k-p}, x_{3p+2}\} \neq \emptyset$. If $a_{k-p} \in L$, then $\{c_{k-p}, b_{k-p}, x_2, \ldots, x_{3p+1}\} \cap L = \emptyset$ because L is independent. Thus, $N_L(x_2) = N_L(x_3) = \{a_{k-p}\}$ contradicting L is locating. Hence, $a_{k-p} \notin L$, which implies that $\{c_{k-p}, x_2, \ldots, x_{3p+1}\} \subseteq L$. Thus, $|L| \geq k-p-1+1+1+3p+1 = k+2p+2$. Clearly $\{a_1, a_2, \ldots, a_{k-p-1}, c_{k-p}, b_{k-p}, x_1, \ldots, x_{3p+1}\}$ is an ILD-set. Therefore, $i_\ell(G) \leq k-p+3p+2 = k+2p+2$. This implies $i_\ell(G) = k+2p+2 = \ell$. □

Finally, we let $\mathcal{T}(n,\ell)$ be the class of trees T of n vertices having $i_\ell(T) = \ell$. Clearly, $\cup_{r=1}^{3}\mathcal{G}_r(\ell) \subseteq \mathcal{T}(n,\ell)$. We note that a star of n vertices is in the class $\mathcal{T}(n, n-1)$. By Lemmas 1 - 3 and Corollary 4, we have that

Corollary 5. *For a natural number $n \geq 12$, $\mathcal{T}(n,\ell) \neq \emptyset$ if and only if $\ell \in \{\lceil \frac{n+1}{3} \rceil, \ldots, n-1\}$.*

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