Communications in Combinatorics and Optimization
Vol. 11, No. 4 (2026), pp. 1215-1219
https://doi.org/10.22049/cco.2026.31447.2855

CCO
Commun. Comb. Optim.

Research Article

A note on the domination number in bipartite graphs

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Received: 1 January 2026; Accepted: 19 April 2026
Published Online: 25 April 2026

In honour of Odile Favaron's 88th birthday

Abstract: Archdeacon et al. [J. Graph Theory 46 (2004), 207–210] proved that if G is a bipartite graph with partite sets X and Y whose vertices in Y are of minimum degree at least 3 then there exists a set $A \subseteq X$ of size at most $\frac{|X \cup Y|}{4}$ such that every vertex in Y is adjacent to a vertex in A. We generalize this result for all bipartite graphs with minimum degree $\delta \geq 3$ using the Brooks Theorem on the vertex coloring.

Keywords: domination number, bipartite graph.

AMS Subject Classification: 05C69

1. Introduction

For graph theory notation and terminology not given here we refer to [7]. We consider finite, undirected and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. The open neighborhood of a set S is $N(S) = \cup_{v \in S} N(S)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of G, respectively. A k-vertex coloring of G is an assignment of k colors to the vertices of G such that no pair of adjacent vertices receives the same color. The *chromatic number* of G is the minimum integer k such that a k-vertex coloring exists. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For a comprehensive survey on the subject of domination parameters in graphs the reader can refer to the two books [6, 7].

Fundamental results about the domination number are upper bounds in terms of the order and the minimum degree of the graph. One of the most important problems is the study of domination in bipartite graphs and numerous works have been proposed, see for example, [1, 3–5]. Archdeacon et al. [1] proved the following key theorem on the bipartite graphs which leaded to several upper bounds for various domination parameters, such as, total domination number, k-tuple total domination number, etc.

Theorem 1 (Archdeacon et al. [1]). *If G is a bipartite graph with partite sets X and Y whose vertices in Y are of minimum degree at least 3 then there exists a set $A \subseteq X$ of size at most $\frac{|X \cup Y|}{4}$ such that every vertex in Y is adjacent to a vertex in A.*

In this paper we will generalize Theorem 1 by proving that there exists a set $A \subseteq X$ of size at most $\frac{(\delta-2)|Y|+(\delta-2)^2|X|}{(\delta-1)^2}$ such that every vertex in Y is adjacent to a vertex in A. The set arises from our result is better than that of Theorem 1 is some particular cases. We use the Brooks Theorem which is stated as follows.

Theorem 2 (Brooks [2]). *If G is a connected graph with maximum degree Δ, then the chromatic number of G is at most Δ, unless G is a complete graph or an odd cycle, in which case the chromatic number is $\Delta + 1$.*

2. Main result

We prove the following generalization of Theorem 1.

Theorem 3. *Let $\delta \geq 3$ be an integer, and G be a bipartite graph with partite sets X and Y whose vertices in Y are of minimum degree at least δ. Then there exists a set $A \subseteq X$ of size at most*

$$\frac{(\delta-2)|Y| + (\delta-2)^2|X|}{(\delta-1)^2}$$

such that every vertex in Y is adjacent to a vertex in A.

Proof. Let G be a bipartite graph with partite sets X and Y whose vertices in Y are of minimum degree at least $\delta \geq 3$. Let $|V(G)| = n$ and $|E(G)| = m$. We prove by an induction on $n + m$. For the base step, if $G = K_{1,\delta}$, then a set A containing only one leaf of G is the desired set, and note that $|A| = 1$ while $\frac{(\delta-2)+(\delta-2)^2|X|}{(\delta-1)^2} = \delta - 2$. Suppose that the inductive hypothesis holds, and now consider the graph G. The result holds by Theorem 1 if $\delta = 3$. Thus assume that $\delta \geq 4$. Assume that there exists a vertex v in Y of degree at least $\delta + 1$, and e is an edge incident to v. Let $G' = G - e$. Then every vertex of Y has degree at least δ in G'. By the inductive hypothesis there is a set $A \subseteq X$ of size at most $\frac{(\delta-2)|Y|+(\delta-2)^2|X|}{(\delta-1)^2}$ such that in G' every vertex in Y is adjacent to a vertex in A. Clearly, in G every vertex in Y is adjacent to a vertex in A as well. Thus we may assume that every vertex in Y is of degree δ. If there is an isolated vertex $v \in X$, then we let $G' = G - v$, and using the inductive hypothesis the desired set A will be obtained. Thus we may assume that X contains no isolated vertex of G. Assume that there is a vertex $v \in X$ with $\deg(v) \geq \delta$. Let $G' = G - N[v]$. By the inductive hypothesis there is a set $A \subseteq X - \{v\}$ of size at most

$$\frac{(\delta-2)(|Y| - \deg(v)) + (\delta-2)^2(|X|-1)}{(\delta-1)^2}$$

such that in G' every vertex in Y is adjacent to a vertex in A. Let $A' = A \cup \{v\}$. Clearly in G every vertex of Y is adjacent to a vertex in A'. Furthermore,

$$\begin{aligned} |A'| = |A| + 1 &\leq \frac{(\delta-2)(|Y| - \deg(v)) + (\delta-2)^2(|X|-1)}{(\delta-1)^2} + 1 \\ &\leq \frac{(\delta-2)(|Y| - \delta) + (\delta-2)^2(|X|-1)}{(\delta-1)^2} + 1 \\ &\leq \frac{(\delta-2)|Y| + (\delta-2)^2|X| - \delta^2 + 4\delta - 3}{(\delta-1)^2} \\ &\leq \frac{(\delta-2)|Y| + (\delta-2)^2|X|}{(\delta-1)^2}. \end{aligned}$$

Thus we may assume that every vertex of X has degree at most $\delta - 1$ in G. For $i = 1, 2, \ldots, \delta - 1$, let X_i be the set of vertices of X of degree i. Note that counting the number of edges having one end in Y and one end in $\cup_{i=1}^{\delta-1} X_i$, we obtain that

$$\delta|Y| \leq |X_1| + 2|X_2| + \cdots + (\delta-1)|X_{\delta-1}|.$$

Now we form a graph G' from G, where $V(G') = X_{\delta-1}$, and $uv \in E(G')$ if and only if u and v have a common neighbor in Y. Clearly $\Delta(G') \leq (\delta-1)^2$.

Suppose that G' is a complete graph. Then G' has order $(\delta-1)^2+1$ and every vertex of G' has degree $(\delta-1)^2$. Let $Y' = N_G(X_{\delta-1})$, and m' be the number of edges in G having one end in $X_{\delta-1}$ and the other end in Y'. For each vertex of $X_{\delta-1}$ all of its $\delta - 1$ neighbors belong to Y'; so $m' = ((\delta-1)^2+1)(\delta-1)$. On the other hand for each vertex of Y' all of its δ neighbors belong to $X_{\delta-1}$; so $m' = \delta|Y'|$. Thus $\delta|Y'| = ((\delta-1)^2+1)(\delta-1)$ which implies that $|Y'| = \frac{((\delta-1)^2+1)(\delta-1)}{\delta} = \delta^2 - 3\delta + 4 - \frac{2}{\delta}$, a contradiction, since $|Y'|$ is an integer. We deduce that G' is not a complete graph. Let I be a maximum independent set in G'. If G' is not an odd cycle then by Brooks theorem $|I| \geq \frac{|X_{\delta-1}|}{(\delta-1)^2}$, and if G' is an odd cycle then $|I| > \frac{|X_{\delta-1}|}{2} \geq \frac{|X_{\delta-1}|}{(\delta-1)^2}$. Let A be the vertices of G corresponded to I. Then the vertices of A have disjoint neighborhoods, and each of them is adjacent to $\frac{(\delta-1)|X_{\delta-1}|}{(\delta-1)^2}$ vertices of Y. For each vertex of Y which is not dominated by A, we pick one of its neighbors in X, and let B be the set of these new vertices. Then $A \cup B$ dominates every vertex of Y. We now compute the cardinality of $A \cup B$. Clearly $|B| \leq |Y| - (\delta - 1)|A|$. Thus,

$$\begin{aligned}
|A \cup B| &\leq |A| + |B| \\
&= |A| + |Y| - (\delta - 1)|A| \\
&= |Y| - (\delta - 2)|A| \\
&\leq |Y| - (\delta - 2)\frac{|X_{\delta-1}|}{(\delta-1)^2} \\
&= \frac{(\delta-1)^2|Y| - (\delta-2)|X_{\delta-1}|}{(\delta-1)^2} \\
&= \frac{(\delta-2)|Y| + ((\delta-1)^2 - (\delta-2))|Y| - (\delta-2)|X_{\delta-1}|}{(\delta-1)^2} \\
&\leq \frac{(\delta-2)|Y| + (\frac{\delta^2-3\delta-1}{\delta})(|X_1| + 2|X_2| + ... + (\delta-1)|X_{\delta-1}|) - (\delta-2)|X_{\delta-1}|}{(\delta-1)^2} \\
&\leq \frac{(\delta-2)|Y| + ((\frac{\delta^2-3\delta-1}{\delta})(\delta-1) - (\delta-2))|X|}{(\delta-1)^2} \\
&\leq \frac{(\delta-2)|Y| + (\frac{\delta^3-5\delta^2+4\delta+1}{\delta})(\delta-1)|X|}{(\delta-1)^2} \\
&\leq \frac{(\delta-2)|Y| + (\delta-2)^2|X|}{(\delta-1)^2},
\end{aligned}$$

as desired. □

Note that the set arising by Theorem 3 is better than that of Theorem 1 if $(3\delta^2 - 14\delta + 15)|X| \leq (\delta^2 - 6\delta + 9)|Y|$. For some examples: if $\delta = 4$ then in the case $|Y| \geq 7|X|$; if $\delta = 5$ then in the case $|Y| \geq 5|X|$, and if $\delta = 6$ then in the case $|Y| \geq \frac{39}{9}|X|$ the set in Theorem 3 is better than that of Theorem 1.

Conflict of Interest: The author declares that he has no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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