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Research Article

Bounding the eviction number of a graph in terms of its independence number

Gary MacGillivray†, Christina M. Mynhardt*, Virgelot Virgile‡

Department of Mathematics and Statistics, University of Victoria, Victoria, Canada
†gmacgill@uvic.ca
*kieka@uvic.ca
‡virgilev@uvic.ca

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To our friend and colleague, Odile Favaron

Abstract: An eternal dominating family of a graph G in the eviction game is a collection $\mathcal{D}_k = \{D_1, D_2, \ldots, D_l\}$ of dominating sets of G such that (a) $|D_i| = |D_j|$ for all $i, j \in \{1, 2, \ldots, l\}$, and (b) for any $i \in \{1, 2, \ldots, l\}$ and any $v \in D_i$, either all neighbours of v belong to D_i, or there are a neighbour w of v not in D_i and an integer $j \in \{1, 2, \ldots, l\} \setminus \{i\}$ such that $D_i \cup \{w\} \setminus \{v\} = D_j$. The eviction number of G, denoted by $e^\infty(G)$, is the smallest cardinality of the sets in such an eternal dominating family.

We compare e^∞ to the independence number α. We show that the ratio α/e^∞ is unbounded and construct an infinite class of connected graphs for which $e^\infty/\alpha \approx 4/3$. As our main result, we use Ramsey numbers to show that for any integer $k \geq 1$, there exists a function $f(k)$ such that any graph with independence number k has eviction number at most $f(k)$.

Keywords: graph protection, eternal domination, eternal eviction, independence

AMS Subject classification: 05C69

1. Introduction

Graph protection involves the placement of mobile sensors on the vertices of a graph G to protect the vertices and edges of G against either single or longer sequences of "events" occurring at the vertices or edges. We refer to these sensors as *guards*, and to the events as *attacks*. A guard located on a vertex v of a graph G dominates v and all the neighbours of v; we say that v is *occupied* (by a guard). A vertex without

* *Corresponding Author*

a guard is said to be *unoccupied*. The initial challenge is to find a (usually smallest) subset S of occupied vertices of G such that every vertex is either in S or adjacent to a vertex in S, i.e., a *dominating set* of G. However, real-world systems seldom remain static. As situations evolve, the once minimum dominating set of guards may face unforeseen challenges. Imagine a situation where a sensor must be moved from its original position due to maintenance needs, adverse weather conditions, or other complications. The challenge then becomes:

"How do we reposition this sensor without compromising the overall coverage?"

This is the point where the movement of guards becomes relevant. We restrict our investigation to attack sequences of arbitrary length. Each such problem, called an *eternal domination problem*, can be modelled as a two-player game, alternating between a defender and an attacker: the defender chooses the initial configuration of guards as well as each configuration following an attack, and the attacker chooses the locations of the attack. We further restrict our attention to the *eviction game*, which was introduced by Klostermeyer et al. in [6].

In the eviction game, only vertices containing a guard may be attacked. In the standard version of *the eviction game*, which we will simply refer to as *eviction*, the guards start by choosing their opening configuration, which must induce a dominating set. We consider the case where at most one guard is located on each vertex. At each turn the attacker selects a vertex v on which there is a guard and the guard on v responds by moving to an unoccupied neighbour; that is, a neighbour of v on which there is no guard, if possible. If all the neighbours of v are occupied, we say that v is *surrounded*, and the guard on v must stay put. This implies that the attacker can only attack guards on vertices that are not surrounded. We sometimes say that an attacked guard is *evicted*. Only the guard that is attacked is allowed to move to a neighbour. The guards win the game if they are able to maintain a dominating set in the graph after responding to each attack; otherwise, the attacker wins. Assuming the guards move optimally, an *eternal dominating set* of a graph G (in the eviction game) is any opening configuration (a dominating set of G) from which they can defend any sequence of attacks on G. The *eviction number* of a graph G, denoted by $e^\infty(G)$, is the minimum cardinality of an eternal dominating set of G in the eviction game.

We focus on comparing the eviction number of G to its independence number $\alpha(G)$. As we show in Section 2, it is easy to see that the ratio α/e^∞ is unbounded. On the other hand, it is not so easy to determine whether the ratio e^∞/α is bounded or not. The cycle C_7 is an example of a graph whose eviction number exceeds its independence number: $\alpha(C_7) = 3$ and $e^\infty(C_7) = 4$. We construct an infinite class of connected graphs for which $e^\infty/\alpha \approx 4/3$. One of the difficulties one encounters when studying eviction is the anomaly that e^∞ could *increase* upon the addition of an edge. We illustrate this in Section 3. As our main result, we show in Section 4 that, for any integer $k \geq 1$, there exists a function $f(k)$ such that any graph with independence number k has eviction number at most $f(k)$. We state some open problems in Section 5.

2. Definitions and background

Concepts not defined here can be found in any standard text on graph theory, e.g. [3, 12]. For further background on graph protection, see [8, 9].

For any positive integers n and k, let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$.

We consider finite, simple graphs and, as usual, denote the vertex and edge sets of a graph G by $V(G)$ and $E(G)$, respectively. The *open neighbourhood* of $v \in V(G)$, denoted by $N(v)$, is the set of vertices that are adjacent to v. These vertices are known as the *neighbours* of v. The *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. Consider disjoint graphs G and H. The *disjoint union* of G and H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of k disjoint copies of a graph G by kG. The *join* of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

As stated above, we denote the independence number of G by $\alpha(G)$. Furthermore, we denote the clique number of G by $\omega(G)$, the domination number by $\gamma(G)$, the chromatic number by $\chi(G)$, and the clique covering number (the minimum cardinality of a partition of $V(G)$ such that each set in the partition induces a clique) by $\theta(G)$. Observe that $\alpha(G) = \omega(\overline{G})$ and $\chi(G) = \theta(\overline{G})$ for any graph G.
Let $\mathcal{D}_k$ be the collection of all dominating sets of G of fixed cardinality k. For $D \in \mathcal{D}_k$, we imagine that there is a single guard located on each vertex of D and therefore we think of D as a configuration of guards. We say that a (not necessarily dominating) set X *protects* a vertex v, or v is *protected* (by X), if v or one of its neighbours is occupied by a member of X.

As explained in the introduction, each eternal domination problem can be modelled as a two-player game, alternating between a *defender* and an *attacker*: the defender chooses $D_1 \in \mathcal{D}_k$ as well as each D_i, $i > 1$, while the attacker chooses the locations $r_1, r_2, \ldots$ of the attacks; we say the attacker *attacks* the vertices r_i. Thus, the game starts with the defender choosing D_1. For $i \geq 1$, the attacker attacks r_i and the defender *defends against* the attack by choosing $D_{i+1} \in \mathcal{D}_k$ subject to constraints that depend on the particular game. The defender wins the game if they can successfully defend the graph against any sequence of attacks, including sequences that are infinitely long, subject to the constraints of the game; the attacker wins otherwise. In other words, the attacker's goal is to force the defender into a configuration of guards that is not dominating. These dynamic models of domination were first defined and studied by Burger, Cockayne, Gründlingh, Mynhardt, Van Vuuren and Winterbach in [1, 2]. In particular, they studied the eternal domination number $\gamma^{\infty}(G)$ of a graph G, which is the smallest number of guards that can defend G against arbitrary sequences of attacks on unguarded vertices, where a single guard must move to the attacked vertex.

The definitions below of eternal dominating families of a graph in the eternal domination and the eviction games illustrate the difference between the two protection models.

An *eternal dominating family* of a graph G (in the eternal domination game) is a collection of sets $D_1, D_2, D_3, \ldots, D_l$ of G that satisfy the following properties.

1. For any $i, j \in [l]$, $|D_i| = |D_j|$.
2. For any $i \in [l]$ and any $w \in V(G) - D_i$, there exist $v \in D_i \cap N(w)$ and $j \in [l] - \{i\}$ such that $(D_i \cup \{w\}) - \{v\} = D_j$.

An *eternal dominating family* of a graph G (in the eviction game) is a collection of dominating sets $D_1, D_2, D_3, \ldots, D_l$ of G that satisfy the following properties.

3. For any $i, j \in [l]$, $|D_i| = |D_j|$.
4. For any $i \in [l]$ and any $v \in D_i$, either $N[v] \subseteq D_i$, or there exist $w \in N(v) - D_i$ and $j \in [l] - \{i\}$ such that $(D_i \cup \{w\}) - \{v\} = D_j$.

Item (2) implies that the D_i are dominating sets whereas (4) does not; thus we have to specify this explicitly. We proceed by stating some results on eviction obtained by Klostermeyer, Lawrence and MacGillivray in [6].

Proposition 1 ([6]). *If $k < |V(G)|$ guards can defend an arbitrarily long sequence of attacks in the eviction game on a graph G, then so can $k+1$ guards.*

We now examine how the eviction number of a graph G relates to some other parameters such as the domination number, the independence number and the clique covering number of G.

Proposition 2 ([6]). *For any graph G, $\gamma(G) \leq e^\infty(G) \leq \theta(G)$.*

Klostermeyer et al. [6] also determined the values of e^∞ for paths, cycles and complete bipartite graphs.

Proposition 3 ([6]). *For any integers $n, m \geq 1$,*

(i) $e^\infty(P_n) = \lceil \frac{n}{2} \rceil$,

(ii) $e^\infty(C_3) = 1$, $e^\infty(C_5) = 2$ *and* $e^\infty(C_n) = \lceil \frac{n}{2} \rceil$ *for any* $n \neq 3, 5$,

(iii) $e^\infty(K_{m,n}) = \max\{m, n\}$.

Since $\alpha(G)$ is a lower bound on $\gamma^\infty(G)$ (see [2]) and $\gamma(G) \leq e^\infty(G) \leq \theta(G)$, it is reasonable to ask whether $\alpha(G)$ is also a lower bound on $e^\infty(G)$. Observation 1 shows that this is not the case. Indeed, while fixing $e^\infty(G) = 1$, $\alpha(G)$ can be arbitrarily large. Hence the ratio α/e^∞ is unbounded.

Proposition 4. *Let G be a graph with at least two universal vertices. Then $e^{\infty}(G) = 1$.*

Proof. Let u, v be two universal vertices of G. By moving back and forth on the vertices u and v, one guard can dominate all of the vertices of G at each time $t = 1, 2, 3, \ldots$. □

Observation 1. Let G be the join of the graph K_2 with the graph $\overline{K_m}$ (see Figure 1). Then $\alpha(G) = m$ and $e^{\infty}(G) = 1$.

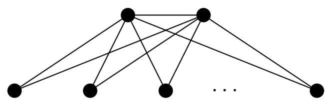


Figure 1. $K_2 \vee \overline{K_m}$

It is not so easy to determine whether the ratio e^{∞}/α is bounded or not. The cycle C_7 is an example of a graph with $\alpha = 3$ and $e^{\infty} = 4$. Therefore, disjoint unions of C_7 provide infinitely many (disconnected) graphs G for which $e^{\infty}(G)/\alpha(G) = \frac{4}{3}$. To see that a similar result holds for connected graphs, consider the graph G_k obtained by joining a new vertex v to each vertex of kC_7, and a new vertex w to v (see Figure 2 for the case where $k = 2$). The graph G_k satisfies the properties described in the following proposition.

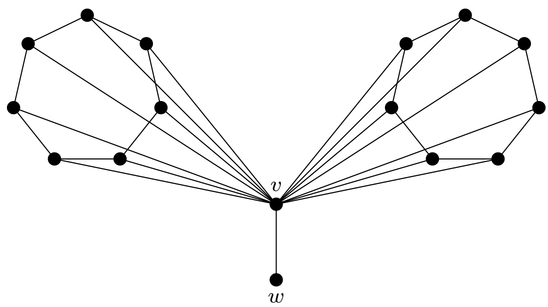


Figure 2. **Graph G_k for the case where $k = 2$.**

Proposition 5. *For any $k \geq 1$, $\alpha(G_k) = 3k + 1$ and $e^{\infty}(G_k) = \theta(G_k) = 4k + 1$.*

Proof. Since $\alpha(C_7) = 3$ and v is adjacent to all other vertices of G_k while w is adjacent to only v, $\alpha(G_k) = 3k + 1$. Since $\theta(C_7) = 4$ and v forms a clique with either w or a maximal clique of a copy of C_7, but not with both, $\theta(G_k) = 4k + 1$. Thus, by Proposition 2, we only need to show that $e^{\infty}(G_k) \geq 4k + 1$.

Suppose the assumption is false. Then G_k can be defended by $4k$ guards (by Proposition 1). Consider a configuration of these $4k$ guards on G_k. If w is occupied and v is unoccupied, then evict the guard on w to v. If v and w are both occupied, evict the guard on v to an unoccupied vertex of one of the copies of C_7 and then evict the guard on w to v. So, we may assume without loss of generality that v is occupied and w is unoccupied. Since there are $4k$ guards located on G_k, there is a copy of C_7, which we will refer to as C_7^*, on which are located fewer than four guards. Evict the guards located on C_7^* in a way such that a vertex of C_7^* is not dominated by any guard located on a vertex of this subgraph. (This is possible because $e^\infty(C_7) = 4$, as stated in Proposition 3.) Now, evict the guard located on v. If the guard moves to w, then a vertex of C_7^* is not dominated. If the guard moves somewhere else, then w is not dominated. This contradicts our assumption that $4k$ guards can defend the graph. □

As it turns out, $\alpha(G)$ is a lower bound on $e^\infty(G)$ when G belongs to a specific graph class, as stated below.

Proposition 6 ([6]). *If G is a triangle-free graph, then $e^\infty(G) \geq \alpha(G)$.*

Although $\alpha(G)$ is neither an upper bound nor a lower bound on $e^\infty(G)$, Klostermeyer et al. show that $e^\infty(G)$ is bounded for the first three values of α.

Theorem 2 ([6]). *1. If $\alpha(G) = 1$, then $e^\infty(G) = 1$.*

2. If $\alpha(G) = 2$, then $e^\infty(G) \leq 2$.

3. If $\alpha(G) = 3$, then $e^\infty(G) \leq 5$.

It is unknown whether there exists a graph G such that $\alpha(G) = 3$ and $e^\infty(G) = 5$. There are also no results in the literature bounding the eviction number in terms of the independence number when the latter is at least 4.

Klostermeyer et al. [7] proved that $\gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}$ for any graph G. Thus, the eternal domination number of a graph is bounded by a function of its independence number. Since no such bound is known in general for the eviction number of the graph, Klostermeyer et al. asked the following questions.

Question 1 ([6]). Does there exist a constant c such that $e^\infty(G) \leq c\alpha(G)$ for all graphs G?

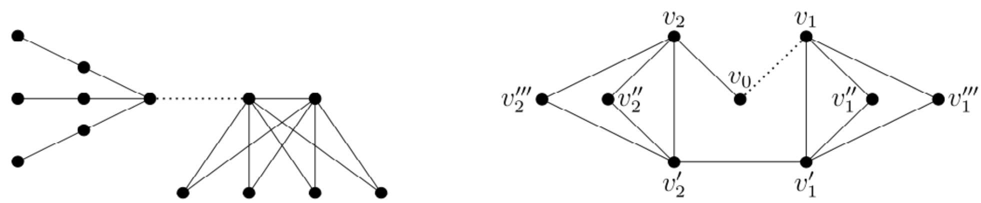
Question 2 ([6]). Does there exist a graph G such that $\gamma^\infty(G) < e^\infty(G)$?

The results in the next two sections are motivated by Question 1, which still remains unanswered. We aim to show the existence of a function f such that any graph with independence number k has eviction number at most $f(k)$. We begin by illustrating one of the difficulties we encountered when trying to determine the eviction number of a graph by considering its subgraphs.

3. Eviction is different

For almost any domination-type parameter π, adding an edge to a graph G can only result in a graph G' with $\pi(G') \leq \pi(G)$, and usually $\pi(G') \in \{\pi(G), \pi(G)-1, \pi(G)-2\}$. This is not the case with the eviction number.

Consider the graph $G \cong K_1 + (K_2 \vee \overline{K_t}), t \geq 2$. Observe that $e^\infty(G) = 2$ and the graph G' obtained from G by adding an edge from the isolated vertex of G to one of the vertices of degree $t+1$ has eviction number $t+1$. The problem occurs when the attacker can force a guard to be surrounded. This is trivially the case for K_1, but there are infinitely many graphs with this property. For example, the spider Sp$(2;k)$, which is obtained from the star $K_{1,k}$ by subdividing each edge exactly once, has eviction number $k+1$. The attacker can force guards to be on the central vertex c and all its neighbours. Now, when c is joined to a vertex of another graph, for example $K_2 \vee \overline{K_t}$, the eviction number of the resulting graph can be arbitrarily higher than the eviction number of Sp$(2;k) + (K_2 \vee \overline{K_t})$ (see Figure 3(a)).



(a) Sp$(2;k) + (K_2 \vee \overline{K_{2,t}})$ (b) G_2

Figure 3. **Graphs whose eviction number increases by adding the dotted edge.**

The graph G_2 in Figure 3(b) is an example of a connected graph for which adding an edge increases the eviction number. We first show that three guards can defend the subgraph induced by the vertices $v_0, v_2, v_2', v_2'', v_2'''$. Let

$$\begin{aligned} X_1 &= \{v_0, v_2, v_2'\},\ X_2 = \{v_0, v_2', v_2''\},\ X_3 = \{v_2, v_2', v_2''\}, \\ X_4 &= \{v_0, v_2, v_2''\},\ X_5 = \{v_2, v_2'', v_2'''\} \text{ and } X_6 = \{v_0, v_2'', v_2'''\}. \end{aligned}$$

For $i \in \{2,3,4\}$, let $X_i' = (X_i \setminus \{v_2''\}) \cup \{v_2'''\}$.

- Begin by placing a guard on each vertex in X_1. Since the guard on v_0 is surrounded, the attacker can only attack v_2 or v_2'. Either guard then moves to v_2''; in the former case, the guards are now on X_2, and in the latter case they are on X_4.

- Suppose the guards are on X_2. When v_0 or v_2' is attacked, its guard moves to v_2 and the guards are on X_3 or X_4, respectively. When v_2'' is attacked, its guard moves to v_2 and the guards are on X_1.
- Suppose the guards are on X_3 (where the guard on v_2'' is surrounded). When v_2 is attacked, its guard moves to v_0, thus forming X_2, and when v_2' is attacked, the guard moves to v_2''', thus forming X_5.
- Suppose the guards are on X_4 (where the guard on v_0 is surrounded). If v_2 or v_2'' is attacked, its guard moves to v_2', thus forming X_2 or X_1, respectively.
- Suppose the guards are on X_5. If v_2 is attacked, its guard moves to v_0, forming X_6. If v_2'' or v_2''' is attacked, the guard there moves to v_2', forming X_3' or X_3.
- Suppose the guards are on X_6. An attack on v_0 restores X_5, and an attack on v_2'' or v_2''' results in one of X_2, X_2', X_4, or X_4'.

Attacks on $X_i', i \in \{2,3,4\}$, are defended similar to attacks on $X_i, i \in \{2,3,4\}$. Hence three guards can defend this subgraph.

By Observation 1, one guard can defend the subgraph induced by the vertices v_1, v_1', v_1'', v_1'''. However, the graph G_2' obtained from G_2 by the addition of the edge v_0v_1 cannot be defended by four guards. To see this, suppose four guards are initially located on the vertices of G_2'. We may assume without loss of generality that the vertex v_0 is occupied since its neighbourhood is an independent set. We may further assume without loss of generality that there is one guard located on each of v_1', v_2 and v_2'. After the sequence of attacks $v_0, v_1', v_1, v_2', v_2$, a vertex of G_2' is not dominated. The tedious details are left to the reader.

A graph may have several vertices on which guards are surrounded at the same time. By Proposition 3, the complete bipartite graph $K_{r,r+s}$, where $s > 0$, has eviction number $r+s$. The attacker can force r guards to be located on the vertices of degree $r+s$ and s guards to be located on the vertices of degree r; these s guards are all surrounded.

The above paragraphs illustrate that, when determining the eviction number of a graph G by considering how guards can defend various subgraphs of G, it is important to establish that the guards can always move within these specific subgraphs.

4. The function f

We begin our result on the function f by stating a special case of Ramsey's Theorem [10]. We use this theorem to prove Lemma 1, on which our proof of the main theorem depends.

Theorem 3 (Ramsey's Theorem for a Graph and its Complement). *For any positive integers k and l, there exists a positive integer n such that every graph of order n contains an independent set of size k or a clique of size l.*

The *Ramsey number* $r(k, l)$ is the minimum integer n such that any graph on at least n vertices contains an independent set of size k or a clique of size l. Theorem 3 ensures that $r(k, l)$ is well defined.

Lemma 1. *For any integer $k \geq 1$, there exists a constant $c(k) = c_k$ such that if G is a graph with independence number k which has at least c_k disjoint maximum independent sets, then G can be defended by k^2 guards. Moreover, no guard is ever prevented from moving by being surrounded.*

Proof. Let $k \geq 1$ and let $l_0, l_1, l_2, l_3, \ldots, l_k$ be sufficiently large positive integers (which will be chosen later). Let G be a graph with independence number k. Suppose G has at least $c_k = l_0$ disjoint maximum independent sets $R_1, R_2, R_3, \ldots, R_{c_k}$, where $R_i = \{v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,k}\}$ for each $i = 1, 2, 3, \ldots, c_k$.

If $l_0 \geq r(k+1, l_1)$, Ramsey's Theorem guarantees that there exist l_1 vertices in the set $\bigcup_{i\in[l_0]}\{v_{i,1}\}$ which induce a complete subgraph of G. Without loss of generality, let $C_1 = \{v_{1,1}, v_{2,1}, v_{3,1}, \ldots, v_{l_1,1}\}$ be such a set. If $l_1 \geq r(k+1, l_2)$, Ramsey's Theorem guarantees that there exist l_2 vertices in the set $\bigcup_{i\in[l_1]}\{v_{i,2}\}$ which induce a complete subgraph of G. Without loss of generality, let $C_2 = \{v_{1,2}, v_{2,2}, v_{3,2}, \ldots, v_{l_2,2}\}$ be such a set. Likewise, for each $j \in \{3, 4, \ldots, k-1\}$, if $l_{j-1} \geq r(k+1, l_j)$, Ramsey's Theorem guarantees that there exist l_j vertices in the set $\bigcup_{i\in[l_{j-1}]}\{v_{i,j}\}$ which induce a complete subgraph of G. Finally, if $l_{k-1} \geq r(k+1, k+1)$, then there exist $l_k = k+1$ vertices in the set $\bigcup_{i\in[l_{k-1}]}\{v_{i,k}\}$ which induce a complete subgraph of G. To summarize, let

$$\begin{aligned}
l_k &= k+1 \\
l_{k-1} &= r(k+1, l_k) &&= r(k+1, k+1) \\
l_{k-2} &= r(k+1, l_{k-1}) &&= r(k+1, r(k+1, k+1)) \\
&\vdots \\
l_0 &= r(k+1, l_1) &&= \underbrace{r(k+1, r(k+1, r(k+1, \ldots)))}_{k \text{ times}}.
\end{aligned}$$

As explained above, if G has at least $c_k = l_0$ disjoint independent sets of size k, then G has a subset of vertices $S^* = \bigcup_{i\in[k+1], j\in[k]}\{v_{i,j}\}$, where

(i) $R_i = \bigcup_{j\in[k]}\{v_{i,j}\}$ is a maximum independent set for each $i \in [k+1]$,

(ii) $C'_j = \bigcup_{i\in[k+1]}\{v_{i,j}\}$ is a clique of size $k+1$ for each $j \in [k]$.

We represent S^* as an array with rows $R_i, i \in [k+1]$, and columns $C'_j, j \in [k]$, as

$$\begin{bmatrix}
v_{1,1} & v_{1,2} & v_{1,3} & \cdots & v_{1,k} \\
v_{2,1} & v_{2,2} & v_{2,3} & \cdots & v_{2,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_{k+1,1} & v_{k+1,2} & v_{k+1,3} & \cdots & v_{k+1,k}
\end{bmatrix}.$$

In this case, place k guards in C'_j for each $j \in [k]$ so that the subset S^* contains exactly k^2 vertices. For any $j \in [k]$, if a guard in C'_j is attacked, the guard can always relocate to the only unoccupied vertex in C'_j. Since there are exactly $k+1$ rows $R_1, R_2, R_3, \ldots, R_{k+1}$ and there are exactly k^2 guards located in $S^* = \bigcup_{i \in [k+1]} R_i$ at each time $t = 1, 2, 3, \ldots$, by the Generalized Pigeonhole Principle, there is an integer $m \in [k+1]$ such that row m contains at least $\lceil \frac{k^2}{k+1} \rceil = k$ guards; that is, all the vertices in R_m are occupied. Since R_m is a maximum independent (and hence dominating) set of G, all the vertices in G are dominated by R_m. This completes the proof. □

Observe that our proof of Lemma 1 shows that

$$c_1 \leq r(2,2) = 2 \text{ and } c_2 \leq r(3, r(3,3)) = 18.$$

We are now ready to prove our main theorem.

Theorem 4. *There exists a function f such that if G is a graph with independence number $k \geq 1$, then $e^{\infty}(G) \leq f(k)$. In particular,*

$$f(1) = 1 \text{ and } f(k) \leq \frac{2kc_k(k^{k-1} - 1)}{k-1} \text{ when } k \geq 2,$$

where c_k is as in Lemma 1.

Proof. The cases $k = 1, 2, 3$ are clear (see Theorem 2). Let G be a graph such that $\alpha(G) = k \geq 4$. We may assume that $|V| > f(k)$; otherwise, the theorem clearly holds. If G has at least c_k disjoint independent sets of size k, then, by Lemma 1, G can be defended by k^2 guards. So, we may further assume that G has fewer than c_k disjoint maximum independent sets. We first prove the following claim.

Claim 1. *There exist a positive integer $l < k$ and a set $S \subseteq V(G)$ such that $\alpha(G - S) = l$ and $G - S$ has at least $c_l + |S|$ disjoint independent sets of size l.*

Proof. Let $G_0, G_1, G_2, \ldots, G_{k-1}$ be a sequence of subgraphs of G (where $G = G_0$) that satisfy the following conditions for each $i \in \{0, 1, 2, \ldots, k-2\}$:

(i) $\alpha(G_i) = k - i$.

(ii) $G_{i+1} = G_i - S_i$, where S_i is a smallest subset of vertices of G_i such that $\alpha(G_i - S_i) = k - i - 1$.

Since G_0 has fewer than c_k disjoint independent sets of cardinality k, we have $|S_0| < kc_k$.

If G_1 has at least $c_{k-1} + |S_0|$ disjoint independent sets of cardinality $k-1$, then we are done, hence suppose this is not the case. Then

$$|S_1| < (k-1)(c_{k-1} + |S_0|) < (k-1)(c_{k-1} + kc_k) < k^2 c_k.$$

If G_2 has at least $c_{k-2} + |S_0| + |S_1|$ disjoint independent sets of cardinality $k-2$, then we are done, hence suppose this is not the case. Then

$$|S_2| < (k-2)(c_{k-2} + |S_0| + |S_1|) < (k-2)(c_{k-2} + kc_k + k^2 c_k) < k^3 c_k.$$

Likewise, for each $i \in \{3, 4, 5, \ldots, k-2\}$, if G_i has at least $c_{k-i} + |S_0| + |S_1| + \cdots + |S_{i-1}|$ disjoint independent sets of cardinality $k-i$, then we are done, hence suppose this is not the case. Then

$$\begin{aligned} |S_i| &< (k-i)(c_{k-i} + |S_0| + |S_1| + \cdots + |S_{i-1}|) \\ &< (k-i)(c_{k-i} + kc_k + k^2 c_k + \cdots + k^i c_k) \\ &< k^{i+1} c_k. \end{aligned}$$

Since G is a graph of order at least $1 + \frac{2kc_k(k^{k-1}-1)}{(k-1)}$, G has a subset $S = \bigcup_{i=0}^{k-2} S_i$ of cardinality at most

$$\begin{aligned} |S_0| + |S_1| + \cdots + |S_{k-2}| &< c_k(k + k^2 + k^3 + \ldots + k^{k-2} + k^{k-1}) \\ &= \frac{kc_k(k^{k-1}-1)}{k-1} \end{aligned}$$

such that $G - S$ is a graph on at least $1 + |S|$ vertices with independence number 1. This completes the proof of the claim. $\diamondsuit$

Now, consider a smallest such subset of vertices S such that $\alpha(G - S) = l < k$ and $G - S$ has at least $c_l + |S|$ disjoint independent sets of size l.
Let M be a smallest matching in G that covers the largest number of vertices in S. Let S' be the set of vertices in $G - S$ that belong to an edge of the matching M. Since $|S'| \leq |S|$, $\alpha(G - (S \cup S')) = l$ and $G - (S \cup S')$ has at least c_l disjoint independent sets of size l. As a consequence of Lemma 1, l^2 guards (and therefore at most k^2 guards) can defend $G - (S \cup S')$ and any guard that is evicted can always move to an unoccupied vertex in that subgraph.
Since $|S \cup S'| < \frac{2kc_k(k^{k-1}-1)}{k-1}$, in the rest of the proof, we will show that there exists a strategy with no more than $\frac{kc_k(k^{k-1}-1)}{k-1}$ guards to defend G', the subgraph of G induced by $S \cup S'$, where it is always possible for any guard located on G' to move at each step of the game to a vertex of G'. Let the initial configuration of the guards be such that there is exactly one guard on each edge of M and one guard on each vertex that is not covered by M, where such a vertex necessarily belongs to S. We will maintain the invariant that there is a guard on exactly one vertex of each edge of a matching M that covers the largest number of vertices of G' and one guard on each of the vertices that are not covered by the matching. The invariant is clearly initially satisfied. Suppose at some time $t = 1, 2, 3, \ldots$ a guard located on x_i, a vertex that is covered by M, is attacked. Observe that the guard located on x_i has at least one

unoccupied neighbour y_i, which is matched to x_i by M, since there is only one guard on each edge of the matching. Move the guard to its neighbour y_i. The invariant obviously still holds. Now, suppose a guard located on a vertex z_i in G' that is not covered by M is attacked. Note that our choice of M implies that $z_i \in S$. We consider two cases:

Case 1. If z_i has no unoccupied neighbour in G, then there is nothing to do.

Case 2. If z_i has an unoccupied neighbour x_i, then x_i is covered by M and therefore belongs to G'; otherwise we could find a matching that covers more vertices of S. Then, there exists $y_i \in S$ such that x_i is matched to y_i by M and there is a guard on y_i. In this case, move the guard on z_i to x_i and consider the new matching $M' = (M \cup \{x_i z_i\}) \setminus \{x_i y_i\}$. The invariant clearly still holds.

Since G is a graph on at least $1 + \frac{2kc_k(k^{k-1}-1)}{k-1}$ vertices, $V(G)$ can be partitioned into two sets V_1 and V_2 in a way such that at most $k^2 < \frac{kc_k(k^{k-1}-1)}{k-1}$ guards can effectively defend the subgraph of G induced by V_1 and at most $\frac{kc_k(k^{k-1}-1)}{k-1}$ guards can effectively defend the subgraph induced by V_2. This completes the proof. □

5. Open problems

As shown by Klostermeyer et al. in [6], if G is a graph with independence number 3, then G has eviction number at most 5. However, it is unknown whether there exists a graph G such that $\alpha(G) = 3$ and $e^{\infty}(G) = 5$. This naturally gives rise to the following question.

Question 3. Does there exist a graph G such that $\alpha(G) = 3$ and $e^{\infty}(G) = 5$?

In Proposition 5 we constructed an infinite class of connected graphs for which $e^{\infty}/\alpha \approx 4/3$. We do not know of any graph for which $e^{\infty}/\alpha > 4/3$. The next question is more general than Question 3.

Question 4. Does there exist a graph such that $e^{\infty}/\alpha > 4/3$?

The upper bound on the function $f(k)$ in Theorem 4 is much larger than 4/3, the largest known value of e^{∞}/α, and almost certainly excessively large.

Problem 1. Improve the bound on the function $f(k)$ given in Theorem 4.

A *cograph* (or *complement reducible graph*) is a graph that can be generated from the trivial graph K_1 by complementation and disjoint union. These graphs are also known under various characterizations, among which are the following.

Proposition 7 ([4, 5]). (1) *A cograph is a graph that does not contain P_4 as an induced subgraph.*

(2) *A cograph is a graph that can be generated from the following operations:*

(*i*) *K_1 is a cograph.*

(*ii*) *If G_1 and G_2 are cographs, then so is $G_1 + G_2$.*

(*iii*) *If G_1 and G_2 are cographs, then so is $G_1 \vee G_2$.*

Virgile [11] showed that EVICTION ETERNAL DOMINATING SET is EXP-TIME-complete and that the eviction number of cographs can be computed in polynomial time. Clearly, the same holds for the graphs listed in Proposition 3.

Problem 2. Find further classes of graphs for which the eviction number can be computed in polynomial time.

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