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Research Article

On the A_α-spectrum of superpower graphs associated with dihedral groups

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Abstract: The superpower graph $\mathcal{S}_\Gamma$ of a finite group Γ is an undirected simple graph whose vertices are the elements of the group Γ, and two distinct vertices $a, b \in \Gamma$ are adjacent if and only if the order of one vertex divides the order of the other vertex, which means that either $o(a)|o(b)$ or $o(b)|o(a)$. In this paper, we investigate the $A_\alpha-$spectral properties of the superpower graph of $D_p \times D_p, D_{p^k}, D_{pqr}$, and D_{p^2q}, where p, q, r are distinct primes. In particular, we determine the adjacency, the Laplacian, and the signless Laplacian spectra of these graphs and consequently prove that the superpower graphs of $D_p \times D_p$, and D_{p^k}, are Laplacian integral.

Keywords: A_α-adjacency matrix, Laplacian matrix, eigenvalues, power graph, dihedral group.

AMS Subject classification: 05C50, 05C25, 15A18

1. Introduction

Throughout the paper, all the groups and graphs taken here are assumed to be finite, and a graph means a simple undirected graph. This paper assumes only a foundational understanding of graph theory. Any book on graph theory, for instance, will have them [2]. Our group theory notations are taken from [4], and we refer to [3] for the algebraic graph theory concepts and notations. Graph theory has become a practical and powerful tool for understanding and describing relations, whether in social networks, biological systems, or theoretical physics. Within this broad area,

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graphs arising from algebraic structures, particularly groups, have attracted considerable attention from researchers. As a result, several graph constructions collectively termed *graphs of groups* have been introduced and studied, including Cayley graphs, commuting graphs, generating graphs, power graphs, and superpower graphs. These constructions provide a geometric viewpoint on properties of underlying groups, and spectral graph theory techniques can be employed to yield deeper algebraic insights. The natural connection between algebraic structures and their associated graphs tends to produce rich and intricate relationships, with the spectral characteristics of the graph revealing crucial facts about the structure of the group itself.

Among the various graph constructions on groups, *superpower graphs* represent a relatively new and intriguing class. The superpower graphs of finite groups are a quite recent development in the domain of graphs from groups, and they were first introduced by Hamzeh and Ashrafi [6], who called it the order superpower graph $\mathcal{S}_\Gamma$ of the power graph $\mathcal{G}_\Gamma$ of a finite group. The superpower graph, represented as $\mathcal{S}_\Gamma$, is defined as the graph in which vertices are the elements of the group Γ, and two distinct vertices $a, b \in \Gamma$ are adjacent if and only if the order of one vertex a divides the order of the other vertex b or the order of the vertex b divides the order of vertex a. Hamzeh et al. [5] call this graph the main supergraph, and they investigated its full automorphism group. Recently, Hamzeh and Ashrafi explored some characteristics of the order supergraph of a group, and precisely, they showed that $\mathcal{S}_\Gamma = \mathcal{G}_\Gamma$ if and only if Γ is cyclic [6]. They also investigated the 2-connectedness, Eulerianness, and Hamiltonianity of an order supergraph [7].

With these motivations, we consider the superpower graphs of any non-abelian finite group Γ. The dihedral groups, denoted as D_n, are simple non-abelian groups; they can be viewed as symmetries of regular $n-$gons. Their complicated algebraic nature and their extensive use in diverse areas of mathematics and physics make them the natural objects of detailed investigation in the context of graph theory. The direct product of groups gives a way of building more elaborate algebraic structures from more understandable ones, in which individual properties of a group can be combined and interact with one another in a bigger whole. Studying the spectral properties of superpower graphs of direct products, such as $D_p \times D_p$ can put light on how graph operations behave over more complex group structures. Sharp bounds for the vertex connectivity of superpower graphs $\mathcal{S}_{D_n}$ and $\mathcal{S}_{Q_{4n}}$ were established in [9].

The *adjacency matrix* $A(G)$ of a graph G with n vertices is the $n \times n$ symmetric $(0, 1)-$matrix with entries $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The adjacency matrix encapsulates the connectivity structure of the graph, and its spectrum is a fundamental tool in algebraic graph theory, revealing various combinatorial properties of G [3]. The *Laplacian matrix* $L(G)$ of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix of vertex degrees. It is a positive semidefinite matrix whose eigenvalues provide profound insights into the graph connectivity and the number of connected components [3]. The *signless Laplacian matrix* is defined as $Q(G) = D(G) + A(G)$. Its spectral properties are closely related to the bipartiteness of a graph structure, in particular, G is bipartite if and only if its signless Laplacian spectrum and Laplacian spectrum coincide. The set of all eigenvalues of a graph matrix M is known as the M-spectrum of G.

This paper significantly contributes to the understanding of the adjacency and the Laplacian spectral properties of superpower graphs of some finite groups. We establish a full spectrum analysis by looking at the adjacency matrix, the Laplacian matrix, the signless Laplacian, and the $A_\alpha-$matrix. Nikiforov [11] proposed the convex combinations of $A(G)$ and $D(G)$ defined by $A_\alpha(G) = \alpha D(G)+(1-\alpha)A(G)$, where $\alpha \in [0, 1]$, $A(G)$ is the adjacency matrix, and $D(G)$ is the diagonal degree matrix. For $\alpha = 0$, we get the adjacency matrix, for $\alpha = 1/2$, we get a scalar multiple of the signless Laplacian matrix, and from $A_\alpha(G) - A_\beta(G)$ where $\beta \in [0, 1]$, we obtain a scalar multiple of the Laplacian matrix and infinitely many other adjacency-type matrices [11]. By examining $A_\alpha(G)$, one can gain a deeper understanding of the spectral behavior of a graph. For more recent papers on the spectral properties of $A_\alpha(G)$, we refer the reader to [10, 12, 14, 17] and the references therein.

Since $A_\alpha(G)$ is a real symmetric matrix, its eigenvalues can be arranged in decreasing order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. In some theorems of this paper, the A_α-eigenvalues of the graph G are represented as

$$\mathrm{Spec}(A_\alpha(G)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}, \quad \text{where } k \leq n, \tag{1.1}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of $A_\alpha(G)$ with corresponding multiplicities at least $m_1, m_2, \ldots, m_k$, respectively. The multiset of all eigenvalues of the $A_\alpha(G)$ matrix is called the A_α-spectrum of G.

The paper is structured as follows: The primary findings related to the A_α-spectra of superpower graphs of $D_p \times D_p$ and D_{p^k} are given in Section 2. Section 3 gives the A_α-spectra of superpower graph of D_{pqr} and D_{p^2q}.

2. Adjacency spectra of superpower graphs of $D_p \times D_p$ and D_p^k

The objective of this section is to recall certain concepts and results from group theory and graph theory with the aim of achieving the objective of this work. In order to develop notations, we rewrite the standard definitions and conclusions from [2] for graph theory and [4] for group theory. All of the groups in the study are finite. For a group Γ, the order of an element x is represented by $o(x)$. The dihedral group of order $2n$ is also denoted as D_n. It is a non-commutative group formed by two elements $\langle a, b\rangle$ such that a and b meet the following properties: (i) $o(a) = n$, $o(b) = 2$ (ii) $ba = a^{-1}b = a^{n-1}b$. For more about the superpower graphs, we refer to [5–9], and the references cited therein.

Remark 1. Throughout this paper, O denotes the all-zero matrix of appropriate order.

We recall the following standard results, which will be used in the sequel.

Theorem 1. [8] *Let M be a block upper triangular matrix of the form*

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ O & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & M_{kk} \end{bmatrix}$$

where each M_{ii} is a square matrix. Then, the determinant of M is given by

$$\det(M) = \det(M_{11})\det(M_{22})\cdots\det(M_{kk}).$$

We start with the spectral properties of the superpower graph of the direct product of two dihedral groups $D_p \times D_p$ and the superpower graph D_{p^k}, where p is prime and k is a positive integer. In the following result, we calculate the A_α-spectrum of $\mathcal{S}_{D_p \times D_p}$ explicitly.

Theorem 2. *Let $G \cong \mathcal{S}_{D_p \times D_p}$, then the spectrum of $A_\alpha(G)$ is given as*

$$Spec(A_\alpha(G)) = \begin{pmatrix} (3p^2-2p)\alpha - 1 & 4p^2\alpha - 1 & (3p^2+1)\alpha - 1 \\ p^2 - 2 & 2p^2 - 2p - 1 & p^2 + 2p - 1 \end{pmatrix}$$

and the other 4 eigenvalues of G are the eigenvalues of the following matrix

$$\begin{bmatrix} (4p^2-1)\alpha & (p^2+2p)(1-\alpha) & (2p^2-2p)(1-\alpha) & (p^2-1)(1-\alpha) \\ (1-\alpha) & \eta_2 & (2p^2-2p)(1-\alpha) & 0 \\ (1-\alpha) & (p^2+2p)(1-\alpha) & \eta_3 & (p^2-1)(1-\alpha) \\ (1-\alpha) & 0 & (2p^2-2p)(1-\alpha) & \eta_4 \end{bmatrix},$$

where $\eta_2 = 3p^2\alpha + (p^2+2p-1)(1-\alpha), \eta_3 = (4p^2-1)\alpha + (2p^2-2p-1)(1-\alpha)$ and $\eta_4 = (p^2-2)(1-\alpha) + \alpha(3p^2-2p-1)$.

Proof. Let $G \cong \mathcal{S}_{D_p \times D_p}$ be the superpower graph of $D_p \times D_p$. Then, the vertices of G are represented as

$$D_p \times D_p = \begin{cases} (a,a),\ (a,a^2),\ \ldots,\ (a,a^{p-1}),\ (a,e), \\ (a,b),\ \ldots,\ (a,a^{p-1}b), \\ (a^2,a),\ (a^2,a^2),\ \ldots,\ (a^2,a^{p-1}),\ (a^2,e), \\ (a^2,b),\ \ldots,\ (a^2,a^{p-1}b), \\ \vdots \\ (a^{p-1},a),\ (a^{p-1},a^2),\ \ldots,\ (a^{p-1},a^{p-1}),\ (a^{p-1},e), \\ (a^{p-1},b),\ \ldots,\ (a^{p-1},a^{p-1}b), \\ (e,a),\ (e,a^2),\ \ldots,\ (e,a^{p-1}),\ (e,e), \\ (e,b),\ \ldots,\ (e,a^{p-1}b), \\ (b,a),\ (b,a^2),\ \ldots,\ (b,a^{p-1}),\ (b,e), \\ (b,b),\ \ldots,\ (b,a^{p-1}b), \\ \vdots \\ (a^{p-1}b,a),\ (a^{p-1}b,a^2),\ \ldots,\ (a^{p-1}b,a^{p-1}),\ (a^{p-1}b,e), \\ (a^{p-1}b,b),\ \ldots,\ (a^{p-1}b,a^{p-1}b). \end{cases}$$

The order of $D_p \times D_p$ is $4p^2$. So, the possible orders of elements of $D_p \times D_p$ are: $1, 2, p, 2p$. Now, we know that the order of element $(x, y) \in D_p \times D_p$ is $\mathrm{lcm}(o(x), o(y))$. We consider the following cases:

Case 1. Ordered pair of rotations: Every non-identity rotation a^i, $1 \leq i \leq p-1$, has order p in D_p. Thus, for $1 \leq i, j \leq p-1$, we have $\mathrm{lcm}(o(a^i), o(a^j)) = \mathrm{lcm}(p, p) = p$. For pairs of the form (e, a^j) or (a^i, e) with $1 \leq i, j \leq p-1$, the order is $\mathrm{lcm}(1, p) = p$.

Case 2. Ordered pair of a rotation and a reflection: For a pair of the form $(a^i, a^j b)$ or $(a^j b, a^i)$ where $1 \leq i \leq p-1$ and $0 \leq j \leq p-1$, since every non-identity rotation has order p and every reflection has order 2, we have $\mathrm{lcm}(p, 2) = 2p$ for $p \neq 2$.

Case 3. Ordered pair of reflections: For a pair $(a^i b, a^j b)$ where $0 \leq i, j \leq p-1$, since every reflection has order 2, we have $\mathrm{lcm}(2, 2) = 2$.

Case 4. The identity pair: The element (e, e) has order 1.

Therefore, $D_p \times D_p$ contains elements of orders $1, 2, p$, and $2p$ only. Counting precisely: there is exactly 1 element of order 1, there are $p^2 - 1$ elements of order p, there are $p^2 + 2p$ elements of order 2, and there are $2p(p-1)$ elements of order $2p$.

From the definition of the superpower graph $\mathcal{S}_{D_p \times D_p}$, the elements of $D_p \times D_p$ have orders $1, 2, p$, and $2p2p$. Consequently, the set of all elements of each fixed order forms a clique in $\mathcal{S}_{D_p \times D_p}$; specifically, the elements of orders $p, 2$, and $2p$ each induce a clique, while the identity element (the unique element of order 1) is adjacent to every other vertex. The block structure of $\mathcal{S}_{D_p \times D_p}$ is illustrated in (see Figure 1).

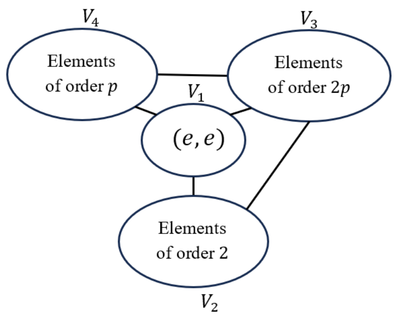

Figure 1. **Superpower graph of $D_p \times D_p$**

To determine the adjacency matrix, we partition the vertex set of $\mathcal{S}_{D_p \times D_p}$ as follows: $V_1 = \{(e, e)\}, V_2 =$ set of elements of order 2, $V_3 =$set of elements of order $2p$, $V_4 =$ set of elements of order p. With respect to this partition, the adjacency matrix of $\mathcal{S}_{D_p \times D_p}$ is given by

$$A(G) = \begin{bmatrix} O_1 & J_{1\times l} & J_{1\times m} & J_{1\times n} \\ J_{l\times 1} & (J-I)_l & J_{l\times m} & O_{l\times n} \\ J_{m\times 1} & J_{m\times l} & (J-I)_m & J_{m\times n} \\ J_{n\times 1} & O_{n\times l} & J_{n\times m} & (J-I)_n \end{bmatrix},$$

where $l = p^2 + 2p, m = 2p^2 - 2p$ and $n = p^2 - 1$. Also

$$D(G) = \begin{bmatrix} 4p^2-1 & O_{1\times l} & O_{1\times m} & O_{1\times n} \\ O_{l\times 1} & 3p^2 I_l & O_{l\times m} & O_{l\times n} \\ O_{m\times 1} & O_{m\times l} & (4p^2-1)I_m & O_{m\times n} \\ O_{n\times 1} & O_{n\times l} & O_{n\times m} & (3p^2-2p-1)I_n \end{bmatrix}.$$

Therefore, the A_α-matrix is given as

$$A_\alpha(G) = \begin{bmatrix} b_1 & (1-\alpha)J_{1\times l} & (1-\alpha)J_{1\times m} & (1-\alpha)J_{1\times n} \\ (1-\alpha)J_{l\times 1} & b_2 & (1-\alpha)J_{l\times m} & O_{l\times n} \\ (1-\alpha)J_{m\times 1} & (1-\alpha)J_{m\times l} & b_3 & (1-\alpha)J_{m\times n} \\ (1-\alpha)J_{n\times 1} & O_{n\times l} & (1-\alpha)J_{n\times m} & b_4 \end{bmatrix},$$

where

$$\begin{aligned} b_1 &= \alpha(4p^2-1) \\ b_2 &= \alpha(3p^2)I_l + (1-\alpha)(J-I)_l = (1-\alpha)J_l + ((3p^2+1)\alpha - 1)I_l \\ b_3 &= \alpha(4p^2-1)I_m + (1-\alpha)(J-I)_m = (1-\alpha)J_m + (4p^2\alpha - 1)I_m \\ b_4 &= \alpha(3p^2-2p-1)I_n + (1-\alpha)(J-I)_n = (1-\alpha)J_n + (\alpha(3p^2-2p)-1)I_n. \end{aligned}$$

Thus, the characteristic equation will be

$$\begin{vmatrix} b_1-\lambda & (1-\alpha)J_{1\times l} & (1-\alpha)J_{1\times m} & (1-\alpha)J_{1\times n} \\ (1-\alpha)J_{l\times 1} & b_2-\lambda I_l & (1-\alpha)J_{l\times m} & O_{l\times n} \\ (1-\alpha)J_{m\times 1} & (1-\alpha)J_{m\times l} & b_3-\lambda I_m & (1-\alpha)J_{m\times n} \\ (1-\alpha)J_{n\times 1} & O_{n\times l} & (1-\alpha)J_{n\times m} & b_4-\lambda I_n \end{vmatrix} = 0.$$

Now, applying the row transformation $R_{3p^2+i} \to R_{3p^2+i} - R_{3p^2+1}$ where $2 \le i \le 4p^2$. After this step, we apply the column transformation $C_{3p^2+1} \to C_{3p^2+1} + C_{3p^2+2} + \cdots + C_{4p^2}$. So, we have $[-\lambda - (2p-3p^2)\lambda - 1]^{p^2-2} = 0$ and we are left with

$$\begin{vmatrix} (4p^2-1)\alpha-\lambda & (1-\alpha)J_{1\times l} & (1-\alpha)J_{1\times m} & (p^2-1)(1-\alpha) \\ (1-\alpha)J_{l\times 1} & [(1-\alpha)J + ((3p^2+1)\alpha - 1 - \lambda)I]_l & (1-\alpha)J_{l\times m} & O_{l\times 1} \\ (1-\alpha)J_{m\times 1} & (1-\alpha)J_{m\times l} & [(1-\alpha)J + (4p^2\alpha - 1 - \lambda)I]_m & (p^2-1)(1-\alpha)J_{m\times 1} \\ (1-\alpha) & 0 & (1-\alpha)(2p^2-2p) & \eta_4 - \lambda \end{vmatrix} = 0,$$

where $\eta_4 = (p^2-2) + (2p^2-2p+1)\alpha$. Again, performing row operation and in another step column operation as $R_j \to R_j - R_2$, where $3 \le j \le p^2 + 2p + 1$. Next column operation as $C_2 \to C_2 + C_3 + \cdots + C_{p^2+2p+1}$ and $C_{p^2+2p+2} \to C_{p^2+2p+2} + C_{p^2+2p+3} + \cdots + C_{3p^2+1}$. So, we obtain $(-\lambda + (3p^2+1)\alpha - 1)^{(p^2+2p-2)}(-\lambda + 4p^2\alpha - 1)^{(2p^2-2p-1)} = 0$, and the remaining 4 zeros will count from the following 4×4 determinant.

$$\begin{vmatrix} (4p^2-1)\alpha-\lambda & (p^2+2p)(1-\alpha) & (2p^2-2p)(1-\alpha) & (p^2-1)(1-\alpha) \\ (1-\alpha) & (p^2+2p-1)(1-\alpha)+3p^2\alpha-\lambda & (2p^2-2p)(1-\alpha) & 0 \\ (1-\alpha) & (p^2+2p)(1-\alpha) & \eta_3-\lambda & (p^2-1)(1-\alpha) \\ (1-\alpha) & 0 & (1-\alpha)(2p^2-2p) & \eta_4-\lambda \end{vmatrix}.$$

where $\eta_3 = (4p^2-1)\alpha + (2p^2-2p-1)(1-\alpha)$. □

Using the complete description of the A_α-spectrum, which was just completed in the previous theorem, we will now deduce adjacency eigenvalues of $\mathcal{S}_{D_p\times D_p}$ using A_α-eigenvalues. This will provide us with information on the walk counts and cycles of $\mathcal{S}_{D_p\times D_p}$.

Corollary 1. *Let $G \cong \mathcal{S}_{D_p\times D_p}$, then the spectrum of $\mathcal{S}_{D_p\times D_p}$ is given as, -1 with algebraic multiplicity $4p^2-4$ and the remaining eigenvalues will be given by the equation*

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0$$

where $A = (4-4p^2), B = (p^4+2p^3-13p^2-2p+6), C = (2p^6+2p^5-3p^4+4p^3-11p^2-6p+4)$ and $D = (2p^6+2p^5-4p^4+2p^3-2p^2-4p+1)$.

Proof. By the definition of $A_\alpha(G)$, we have

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

For, $\alpha = 0$, we have

$$A_0(G) = A(G)$$

which is the adjacency matrix of graph G. In Theorem 2 with $\alpha = 0$, we have adjacency spectrum as -1 with algebraic multiplicity $4p^2-4$. The remaining 4 eigenvalues are the zeros of the following polynomial

$$\begin{vmatrix} -\lambda & (p^2+2p) & (2p^2-2p) & (p^2-1) \\ 1 & (p^2+2p-1)-\lambda & (2p^2-2p) & 0 \\ 1 & (p^2+2p) & (2p^2-2p-1)-\lambda & (p^2-1) \\ 1 & 0 & 2p^2-2p & (p^2-2)-\lambda \end{vmatrix} = 0.$$

Which, on solving, we will get the rest of the eigenvalues by the bi-quadratic equation below

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0$$

where $A = (4-4p^2), B = (p^4+2p^3-13p^2-2p+6), C = (2p^6+2p^5-3p^4+4p^3-11p^2-6p+4)$ and $D = (2p^6+2p^5-4p^4+2p^3-2p^2-4p+1)$. □

The adjacency spectrum obtained above by specializing $\alpha = 0$ captures information about the walk structure of the superpower graph of $D_p \times D_p$. We now derive the Laplacian spectrum, which reflects a different but related aspect of the graph's connectivity. Since $L = D - A$ and the degree structure of the superpower graph is already determined, the Laplacian eigenvalues follow naturally from the eigenvalues of A together with the degree sequence.

Corollary 2. *Let $G \cong D_p \times D_p$. Then, the Laplacian spectrum of the superpower graph of $D_p \times D_p$ is given as*

$$Spec(L(G)) = \begin{pmatrix} 3p^2-2p & 4p^2 & 3p^2+1 & 2p^2-2p+1 & 0 \\ p^2-2 & 2p^2-2p+1 & p^2+2p-1 & 1 & 1 \end{pmatrix}$$

Proof. From Theorem 2, we will obtain the Laplacian spectrum of G. Next, we have $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, and $A_\beta(G) = \beta D(G) + (1-\beta)A(G)$, for $\alpha, \beta \in [0,1]$. Therefore, the Laplacian matrix is given as

$$L(G) = D(G) - A(G) = \frac{A_\alpha(G) - A_\beta(G)}{\alpha - \beta}.$$

Ignoring the quantity, $\alpha - \beta$ in above equation, we have the Laplacian spectrum of $\mathcal{S}_{D_p \times D_p}$. That is $(3p^2-2p)\alpha - 1 - ((3p^2-2p)\beta - 1) = (3p^2-2p)(\alpha-\beta)$. Thus, $3p^2-2p$ is the Laplacian eigenvalue of G with multiplicity p^2-2. Similarly, we can find the other eigenvalues. The remaining four Laplacian eigenvalues of G are the eigenvalues of the following matrix

$$M = \begin{bmatrix} 4p^2-1 & -\left(p^2+2p\right) & -\left(2p^2-2p\right) & -\left(p^2-1\right) \\ -1 & 2p^2-2p+1 & -\left(2p^2-2p\right) & 0 \\ -1 & -\left(p^2+2p\right) & 2p^2+2p & -\left(p^2-1\right) \\ -1 & 0 & -\left(2p^2-2p\right) & 2p^2-2p+1 \end{bmatrix}.$$

The eigenvalues of the above matrix are $\left\{0, 4p^2, 4p^2, 2p^2-2p+1\right\}$. Thus, the complete spectrum of G is

$$\mathrm{Spec}(L(G)) = \begin{pmatrix} 3p^2-2p & 4p^2 & 3p^2+1 & 2p^2-2p+1 & 0 \\ p^2-2 & 2p^2-2p+1 & p^2+2p-1 & 1 & 1 \end{pmatrix}$$

□

A graph matrix is said to be integral, if its associated eigenvalues are integers. A graph is said to be Laplacian integral if all the eigenvalues of the matrix $L(G)$ are integers. Integral graphs are well studied and have many applications, see [16].

Corollary 3. *The superpower graph of $D_p \times D_p$ is a Laplacian integral.*

The following consequence of Theorem 2 gives the signless Laplacian spectra of $\mathcal{S}_{D_p \times D_p}$.

Corollary 4. *For the graph $G \cong \mathcal{S}_{D_p \times D_p}$, the signless Laplacian spectrum is given as*

$$Spec(Q(G)) = \begin{pmatrix} 3p^2-2p-2 & 4p^2-2 & 3p^2-1 \\ p^2-2 & 2p^2-2p-1 & p^2+2p-1 \end{pmatrix}$$

and the remaining 4 *eigenvalues are the zeros of the following polynomial*

$$\lambda^4 + (3+2p-13p^2)\lambda^3 + (4+4p-30p^2-24p^3+58p^4)\lambda^2 + (2-13p^2-42p^3+61p^4 + 90p^5-98p^6)\lambda + 16p^2-20p^3-60p^4+92p^5+36p^6-104p^7+40p^8.$$

Proof. For the case of the signless Laplacian, since $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. With $\alpha = \frac{1}{2}$, we have

$$2A_{\frac{1}{2}}(G) = D(G) + A(G)$$

Therefore, with this information, the signless Laplacian eigenvalues of G are

$$\text{Spec}(2A_\alpha(G)) = \text{Spec}(Q(G)) = \begin{pmatrix} \frac{(3p^2-2p)}{2}-1 & 2p^2-1 & \frac{(3p^2+1)}{2}-1 \\ p^2-2 & 2p^2-2p-1 & p^2+2p-1 \end{pmatrix}.$$

The other 4 eigenvalues are the eigenvalues of the following matrix

$$M = \begin{bmatrix} 4p^2-1 & p^2+2p & 2p^2-2p & p^2-1 \\ 1 & 4p^2+2p-1 & 2p^2-2p & 0 \\ 1 & p^2+2p & 2p^2-2p & p^2-1 \\ 1 & 0 & 2p^2-2p & 3p^2-2p-1 \end{bmatrix}.$$

□

We investigate the A_α-spectrum of the superpower graph associated with the dihedral group D_{p^k}, where p is a prime and k is a positive integer. The superpower graph is partitioned into cliques, and the structural properties of these groups are employed to derive the results. The findings contribute to a deeper understanding of the spectral properties of superpower graphs arising from dihedral groups.

Theorem 3. *Let $G \cong \mathcal{S}_{D_{p^k}}$ be the superpower graph of group D_{p^k} with odd prime p. Then the A_α-spectrum of G is given as*

$$((p^k+1)\alpha-1)^{(p^k-1)}, (p^k\alpha-1)^{(p^k-2)},$$

and the rest of the three eigenvalues are given by the 3×3 determinant below:

$$\begin{vmatrix} (2p^k-1)\alpha-\lambda & (p^k-1)(1-\alpha) & p^k(1-\alpha) \\ (1-\alpha) & p^k+\alpha-2-\lambda & 0 \\ (1-\alpha) & 0 & p^k+\alpha-1-\lambda \end{vmatrix}.$$

Proof. Let D_{p^k}, for $p \neq 2$ be a dihedral group of order $2p^k$ and $G \cong \mathcal{S}_{D_{p^k}}$ be its superpower graph. Now, to study the superpower graph of D_{p^k}, we start its construction explicitly. The group D_{p^k} is generated by two elements a and b with the relation $a^{p^k} = e, b^2 = e$ and $bab = a^{-1}$. Therefore, the group consists of p^k rotations and p^k reflections, which can be expressed as

$$\{a, a^2, \cdots, a^{p^k-1}, e\} \cup \{b, ab, a^2b, \cdots, a^{p^k-1}b\}.$$

By definition, the order of every reflection is 2, so by the definition of superpower graph reflection set will form a clique of order $p^k + 1$. Next, the set of rotations is isomorphic to $\mathbb{Z}_{p^k}$, and its order is p^k, and we know that the order of an element divides the order of the group. So, the order of every rotation will be of the form p^k, and by the definition of a superpower graph, the rotation set will form another clique of order p^k. Both cliques will share one common vertex, which will be the identity (See, Figure 2).

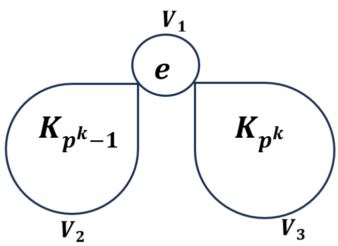

Figure 2. **Structure of the superpower graph of** D_{p^k}

Now, partitioning the vertices as

$$V_1 = \{e\},\ V_2 = \{a, a^2, a^3, \cdots, a^{p^k-1}\} \text{ and } V_3 = \{b, ab, a^2b, \cdots, a^{p^k-1}b\}.$$

The adjacency matrix of G with above partition can be put as

$$A(G) = \begin{bmatrix} 0 & J_{1\times(p^k-1)} & J_{1\times p^k} \\ J_{(p^k-1)\times 1} & (J-I)_{p^k-1} & O_{(p^k-1)\times(p^k)} \\ J_{p^k\times 1} & O_{p^k\times(p^k-1)} & (J-I)_{p^k} \end{bmatrix}.$$

Also, the diagonal matrix is

$$D(G) = \begin{bmatrix} (2p^k-1) & O_{1\times(p^k-1)} & O_{1\times p^k} \\ O_{(p^k-1)\times 1} & (p^k-1)I_{p^k-1} & O_{(p^k-1)\times(p^k)} \\ O_{p^k\times 1} & O_{p^k\times(p^k-1)} & (p^k)I_{p^k} \end{bmatrix}.$$

Therefore, characteristic equation of $A_\alpha(G)$ will be given as

$$\begin{vmatrix} (2p^k-1)\alpha-\lambda & (1-\alpha)J_{1\times(p^k-1)} & (1-\alpha)J_{1\times p^k} \\ (1-\alpha)J_{(p^k-1)\times 1} & [(1-\alpha)J+(p^k\alpha-1-\lambda)I]_{p^k-1} & O_{(p^k-1)\times(p^k)} \\ (1-\alpha)J_{p^k\times 1} & O_{p^k\times(p^k-1)} & [(1-\alpha)J+((p^k+1)\alpha-1-\lambda)I]_{p^k} \end{vmatrix}=0.$$

Now, applying $R_{p^k+2}-R_{p^k+1}, R_{p^k+3}-R_{p^k+1},\ldots,R_{2p^k+2}-R_{p^k+1}$within the last block, then applying the column operation $C_{p^k+1}\to C_{p^k+1}+C_{p^k+2}+\cdots+C_{2p^k}$, the determinant reduces to the block upper-triangular form

$$\begin{vmatrix} A & B \\ O & D \end{vmatrix}=0 \tag{1}$$

Where $A=\begin{vmatrix} (2p^k-1)\alpha-\lambda & (1-\alpha)J_{1\times(p^k-1)} & p^k(1-\alpha) \\ (1-\alpha)J_{(p^k-1)\times 1} & [(1-\alpha)J+(p^k\alpha-1-\lambda)I]_{p^k-1} & O_{(p^k-1)\times 1} \\ (1-\alpha) & O_{1\times(p^k-1)} & p^k-1+\alpha-\lambda \end{vmatrix}=$ 0, $B=\begin{vmatrix} 1-\alpha & 1-\alpha & \cdots & 1-\alpha \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1-\alpha & 1-\alpha & \cdots & 1-\alpha \end{vmatrix}$, O is the zero matrix defined in section 2, and D is the diagonal matrix with diagonal entries $[-\lambda+(p^k+1)\alpha-1]$ and is of order p^k-1. By the block-triangular determinant formula, (1) becomes (see 1)

$$|A|.|D|=0$$

The determinant of D gives immediately

$$[-\lambda+(p^k+1)\alpha-1]^{p^k-1}=0$$

Next, we compute the determinant of A. Expanding explicitly, $|A|$ is given by.

$$|A|=\begin{vmatrix} (2p^k-1)\alpha-\lambda & 1-\alpha & 1-\alpha & \cdots & 1-\alpha & p^k(1-\alpha) \\ (1-\alpha) & (p^k-1)\alpha-\lambda & 1-\alpha & \cdots & 1-\alpha & 0 \\ 0 & 1-p^k\alpha+\lambda & p^k\alpha-1-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1-p^k\alpha+\lambda & 0 & \cdots & p^k\alpha-1-\lambda & 0 \\ (1-\alpha) & 0 & 0 & \cdots & 0 & p^k\alpha-\lambda \end{vmatrix}.$$

Now, applying row operations $R_3-R_2, R_4-R_2,\cdots,R_{p^k}-R_2$, we get

$$|A|=\begin{vmatrix} (2p^k-1)\alpha-\lambda & 1-\alpha & 1-\alpha & \cdots & 1-\alpha & p^k(1-\alpha) \\ (1-\alpha) & (p^k-1)\alpha-\lambda & 1-\alpha & \cdots & 1-\alpha & 0 \\ (1-\alpha) & (1-\alpha) & (p^k-1)\alpha-\lambda & \cdots & 1-\alpha & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1-\alpha & 1-\alpha & 1-\alpha & \cdots & (p^k-1)\alpha-\lambda & 0 \\ (1-\alpha) & 0 & 0 & \cdots & 0 & p^k\alpha-\lambda \end{vmatrix}.$$

Again, applying the column operation $C_2 \to C_2 + C_3 + \cdots + C_{p^k}$, we obtain

$$|A| = [-\lambda + (p^k\alpha - 1)]^{p^k-2} \begin{vmatrix} (2p^k-1)\alpha - \lambda & 1-\alpha & p^k(1-\alpha) \\ (1-\alpha) & p^k+\alpha-2-\lambda & 0 \\ (1-\alpha) & 0 & p^k+\alpha-1-\lambda \end{vmatrix}.$$

The factor $[p^k\alpha-1-\lambda]^{p^k-2} = 0$ gives the eigenvalue $p^k\alpha-1$ with algebraic multiplicity p^k-2. Therefore, the eigenvalues of $A_\alpha(G)$ are $(p^k+1)\alpha-1$ with algebraic multiplicity p^k-1, $p^k\alpha-1$ with algebraic multiplicity p^k-2, and the remaining three eigenvalues are the roots of

$$\begin{vmatrix} (2p^k-1)\alpha - \lambda & (p^k-1)(1-\alpha) & p^k(1-\alpha) \\ (1-\alpha) & p^k+\alpha-2-\lambda & 0 \\ (1-\alpha) & 0 & p^k+\alpha-1-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, the cubic polynomial in λ is

$$\begin{aligned} &(-2p^k+1)\lambda^3 + (4p^{2k} - 8p^k + 3 + (4p^k-1)\alpha)\lambda^2 + \big(-2p^{3k} + 11p^{2k} - 11p^k + 3 \\ &+ (4p^{2k} - 6p^k)\alpha^2 + (-12p^{2k} + 14p^k - 2)\alpha\big)\lambda + \big(-4p^{3k} + 10p^{2k} - 6p^k + 4p^k\alpha^3 \\ &- 4p^{2k}\alpha^3 + (-4p^{3k} + 18p^{2k} - 12p^k)\alpha^2 + (8p^{3k} - 23p^{2k} + 13p^k - 1)\alpha + 1\big) = 0 \end{aligned}$$

□

The following is an immediate consequence of Theorem 2, and gives the adjacency, the Laplacian, and the signless Laplacian spectra of graphs.

Corollary 5. *Let $G \cong S_{D_{p^k}}$ be the superpower graph of group D_{p^k} with odd prime p. Then the following hold.*

1. *The adjacency spectrum of G consist of the eigenvalue -1 with multiplicity $2p^k-3$, and the eigenvalues of the following characteristic equation*

$$\lambda^3 + (3-2p^k)\lambda^2 + (p^{2k} - 5p^k + 3)\lambda + 2p^{2k} - 4p^k + 1 = 0.$$

2. *The Laplacian spectrum of G consist of the eigenvalue p^k+1 with multiplicity p^k-1, the eigenvalue p^k with multiplicity p^k-2 and the simple eigenvalues $\{1, 0, 2p^k\}$.*

3. *The signless Laplacian spectrum of G consists of the eigenvalue p^k-1 with multiplicity p^k-1, the eigenvalues p^k-2 with multiplicity p^k-2, and the eigenvalues*

$$\left\{2\left(p^k-1\right), \frac{1}{2}\left(4p^k - 3 \pm \sqrt{8p^k+1}\right)\right\}.$$

3. A_α-spectrum of superpower graph of D_{pqr} and D_{p^2q}

In this section, we will investigate the spectral results of D_{pqr} and D_{p^2q}, where p, q, r are distinct primes. Before proceeding further, we need the following result.

Theorem 4. [15] *Suppose G is a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $B = \{v_1, v_2, \ldots, v_k\}$ is a clique set of G satisfying $N(v_i) \setminus B = N(v_j) \setminus B$ for all $i, j \in \{1, 2, \ldots, k\}$. Then $\alpha(\omega + \beta) - 1$ is an A_α-eigenvalue of G with multiplicity at least $k - 1$, where $\omega = |B|$ is the order of the clique B and β is the total number of vertices in $V(G) \setminus B$ that are edge connected to every vertex of B.*

The above result is useful for the computation of certain A_α-eigenvalues of a graph G that satisfies the given hypothesis. We now apply this result to compute the A_α-eigenvalues of superpower graphs of D_n for $n \in \{pqr, p^2q\}$.

Theorem 5. *Let $G \cong D_{pqr}$ be the dihedral group of order $2pqr$, where p, q and r are distinct primes and $p, q, r \neq 2$. Then the A_α-spectrum of the superpower graph of G consists of the eigenvalues $\alpha(pqr) - 1, \alpha((p-1)(q-1)r + p + q - 1) - 1, \alpha((q-1)(r-1)p + q + r - 1) - 1, \alpha((p-1)(r-1)q + p + r - 1) - 1, \alpha(pqr - qr + 1) - 1, \alpha(pqr - pr + 1) - 1, \alpha(pqr - pq + 1) - 1$ and $\alpha(pqr + 1) - 1$ with multiplicities at least $\phi(pqr) - 1, \phi(pq) - 1, \phi(qr) - 1, \phi(pr) - 1, \phi(p) - 1, \phi(q) - 1, \phi(r) - 1$ and $pqr - 1$, respectively. The remaining eigenvalues of $A_\alpha(G)$ are the eigenvalues of the matrix given in (3.1).*

Proof. Let D_{pqr} be the dihedral group of order $2pqr$, with distinct primes p, q and r, and G be its superpower graph. This group contains exactly pqr rotations and pqr reflections, which can be expressed as

$$\{a, a^2, a^3, \cdots, a^{pqr-1}, e\} \text{ and } \{b, ab, a^2b, \cdots, a^{pqr-1}b\}, \text{ respectively.}$$

Clearly, the identity element e is adjacent to every other vertex of G. Moreover, the induced subgraph formed by the set of rotations of D_{pqr} is isomorphic to the cyclic group $\mathbb{Z}_{pqr}$. Clearly, in a cyclic group $\mathbb{Z}_{pqr}$ for every $d|pqr$, there is $a \in \mathbb{Z}_{pqr}$ such that $o(a) = d$. So, the possible divisors of pqr are p, q, r, pq, qr, pr, and pqr. By the definition of the superpower graph of a finite group G, the elements having same order will form a clique. Thus, the elements having orders p, q, r, pq, qr, pr, and pqr form distinct cliques. Also, by the definition of the superpower graph, the elements of order p are adjacent to every element of order $pqr, pq, pr,$ and e as $p|pqr$, $p|pq$, $p|pr$. The elements of order q in $\mathbb{Z}_{pqr}$ are adjacent to every element of order pqr, qr, pq and e as $q|pqr$, $q|qr$, $q|pq$. A similar idea works for r, pq, qr, pr and p, q, r. The structure of G is given in Figure 3 with ellipses representing cliques of specific orders. We denote the cliques with ω_i as represented in Figure 3.
Now, in the superpower graph of D_{pqr}, there are exactly $\phi(pqr)$ elements of order pqr, similarly $\phi(p)$ elements of order p, $\phi(q)$ elements of order q, $\phi(r)$ elements of order r, $\phi(pq)$ elements of order pq, $\phi(qr)$ elements of order qr, $\phi(pr)$ elements of order pr,

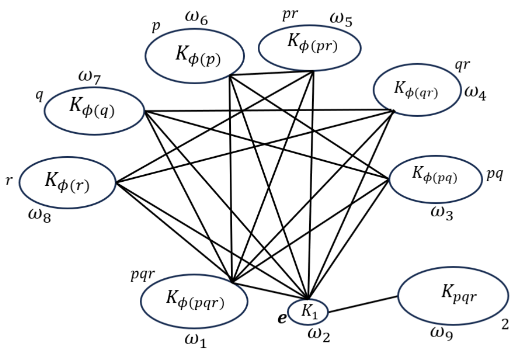

Figure 3. **Structure of superpower graph of** D_{pqr}

there are exactly pqr elements of order 2 and the identity element e is of order 1. As, $\Omega_i, 1 \le i \le 9$ denote the clique sets of different orders in G. Thus, indexing the vertices of G as $V_1 = \Omega_1, V_2 = \Omega_2, V_3 = \Omega_3, V_4 = \Omega_4, V_5 = \Omega_5, V_6 = \Omega_6, V_7 = \Omega_7, V_8 = \Omega_8$ and $V_9 = \Omega_9$. In order to make calculations easy, we let $|\Omega_i| = \omega_i$ and d_i be the common degree of Ω_i. So with this indexing, the A_α-adjacency matrix is written as

$$A_\alpha(G) = \begin{bmatrix} \eta_1 & \beta J_{\omega_1\times 1} & \beta J_{\omega_1\times\omega_3} & \beta J_{\omega_1\times\omega_4} & \beta J_{\omega_1\times\omega_5} & \beta J_{\omega_1\times\omega_6} & \beta J_{\omega_1\times\omega_7} & \beta J_{\omega_1\times\omega_8} & 0_{\omega_1\times\omega_9} \\ \beta J_{1\times\omega_1} & \alpha d_2 & \beta J_{1\times\omega_3} & \beta J_{1\times\omega_4} & \beta J_{1\times\omega_5} & \beta J_{1\times\omega_6} & \beta J_{1\times\omega_7} & \beta J_{1\times\omega_8} & \beta J_{1\times\omega_9} \\ \beta J_{\omega_3\times\omega_1} & \beta J_{\omega_3\times 1} & \eta_3 & 0_{\omega_3\times\omega_4} & 0_{\omega_3\times\omega_5} & 0_{\omega_3\times\omega_6} & \beta J_{\omega_3\times\omega_7} & \beta J_{\omega_3\times\omega_8} & 0_{\omega_3\times\omega_9} \\ \beta J_{\omega_4\times\omega_1} & \beta J_{\omega_4\times 1} & 0_{\omega_4\times\omega_3} & \eta_4 & 0_{\omega_4\times 5} & J_{\omega_4\times\omega_6} & \beta J_{\omega_4\times\omega_7} & 0_{\omega_4\times\omega_8} & 0_{\omega_4\times\omega_9} \\ \beta J_{\omega_5\times\omega_1} & \beta J_{\omega_5\times 1} & 0_{\omega_5\times\omega_3} & 0_{\omega_5\times 4} & \eta_5 & \beta J_{\omega_5\times\omega_6} & 0_{\omega_5\times\omega_6} & \beta J_{\omega_5\times\omega_8} & 0_{\omega_5\times\omega_9} \\ \beta J_{\omega_6\times\omega_1} & \beta J_{\omega_6\times 1} & 0_{\omega_6\times\omega_3} & \beta J_{\omega_6\times\omega_4} & \beta J_{\omega_6\times\omega_5} & \eta_6 & 0_{\omega_6\times\omega_7} & 0_{\omega_6\times\omega_8} & 0_{\omega_6\times\omega_9} \\ \beta J_{\omega_7\times\omega_1} & \beta J_{\omega_7\times 1} & \beta J_{\omega_7\times\omega_3} & \beta J_{\omega_7\times\omega_4} & 0_{\omega_7\times\omega_5} & 0_{\omega_7\times\omega_6} & \eta_7 & 0_{\omega_7\times\omega_8} & 0_{\omega_7\times\omega_9} \\ \beta J_{\omega_8\times\omega_1} & \beta J_{\omega_8\times 1} & \beta J_{\omega_8\times 3} & 0_{\omega_8\times\omega_4} & \beta J_{\omega_8\times 5} & 0_{\omega_8\times\omega_6} & 0_{\omega_8\times\omega_7} & \eta_8 & 0_{\omega_8\times\omega_9} \\ 0_{\omega_9\times\omega_1} & 0_{\omega_9\times 1} & 0_{\omega_9\times\omega_3} & 0_{\omega_9\times\omega_4} & 0_{\omega_9\times\omega_5} & 0_{\omega_9\times\omega_6} & 0_{\omega_9\times\omega_7} & 0_{\omega_9\times\omega_8} & \eta_9 \end{bmatrix},$$

where $\beta = 1 - \alpha$, and $\eta_i = \alpha d_i I_{\omega_i} + (1-\alpha)(J - I)_{\omega_i}$.
Furthermore, $\omega_1 = \phi(pqr), \omega_2 = 1, \omega_3 = \phi(pq), \omega_4 = \phi(qr), \omega_5 = \phi(pr), \omega_6 = \phi(r), \omega_7 = \phi(q), \omega_8 = \phi(p)$, and $\omega_p = pqr$. Now, we use Theorem 4 to calculate the A_α-spectrum of the superpower graph of D_{pqr}. Clearly, each of Ω_i satisfies the required conditions of Theorem 4. Thus, we consider the following cases:

1. First, we consider the identity element Ω_2. The common neighborhood of Ω_2 are the vertices in $\Omega_1, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7$,$\Omega_8$ and Ω_9. Hence, $\alpha(\omega + \beta) - 1$ is an A_α-eigenvalue with multiplicity atleast $k - 1$ see 4, thus A_α-spectrum of G consists of the eigenvalues

   $$\{\alpha(\phi(pqr) + \phi(p) + \phi(q) + \phi(r) + \phi(pq) + \phi(qr) + \phi(pr) + 1) - 1\} = \alpha(pqr) - 1$$

   with multiplicity at least $\phi(pqr) - 1$.

2. Consider the cliques Ω_3, Ω_4, and Ω_5 as in Figure 3. The common neighbors of each Ω_i with $i = 3,4,5$ are as follows:

   (a) For Ω_3, the neighbors are $\Omega_1, \Omega_2, \Omega_6, \Omega_7$.

   (b) For Ω_4, the neighbors are $\Omega_1, \Omega_2, \Omega_7, \Omega_8$.

   (c) For Ω_5, the neighbors are $\Omega_1, \Omega_2, \Omega_6, \Omega_8$.

   Applying the theorem 4, the A_α-eigenvalues of G corresponding to these Ω_i are:

   (a) $\alpha\big(\phi(pq)+\phi(pqr)+\phi(p)+\phi(q)+1\big)-1 = \alpha\big((p-1)(q-1)r+p+q-1\big)-1$, with multiplicity at least $\phi(pq)-1$.

   (b) $\alpha\big(\phi(qr)+\phi(pqr)+\phi(q)+\phi(r)+1\big)-1 = \alpha\big((q-1)(r-1)p+q+r-1\big)-1$, with multiplicity at least $\phi(qr)-1$.

   (c) $\alpha\big(\phi(pr)+\phi(pqr)+\phi(p)+\phi(r)+1\big)-1 = \alpha\big((p-1)(r-1)q+p+r-1\big)-1$, with multiplicity at least $\phi(pr)-1$.

3. Now, considering the other cliques Ω_i, with $i = 6,7,8$ (see, Figure 3). The common neighbors of each Ω_i are as follows:

   (a) For Ω_6, the neighbors are $\Omega_1, \Omega_3, \Omega_5, \Omega_2$.

   (b) For Ω_7, the neighbors are $\Omega_1, \Omega_3, \Omega_4, \Omega_2$.

   (c) For Ω_8, the neighbors are $\Omega_1, \Omega_4, \Omega_5, \Omega_2$.

   By Theorem 4, the eigenvalues associated with these Ω_i are:

   (a) $\alpha\big(\phi(p)+\phi(pqr)+\phi(pq)+\phi(pr)+1\big)-1 = \alpha\big(pqr-qr+1\big)-1$, with multiplicity at least $\phi(p)-1$.

   (b) $\alpha\big(\phi(q)+\phi(pqr)+\phi(pq)+\phi(qr)+1\big)-1 = \alpha\big(pqr-pr+1\big)-1$, with multiplicity at least $\phi(q)-1$.

   (c) $\alpha\big(\phi(r)+\phi(pqr)+\phi(qr)+\phi(pr)+1\big)-1 = \alpha\big(pqr-pq+1\big)-1$, with multiplicity at least $\phi(r)-1$.

4. For Ω_9, the neighboring set of vertices is only Ω_1. So, by Theorem 4, the A_α-eigenvalue of G is $\alpha\big(pqr+1\big)-1$, with multiplicity at least $pqr-1$.

Let X be the eigenvector of A_α matrix of G with $x_i = X(v_i)$, for $i = 1,2,3,\ldots,2pqr$. Then by the common neighborhood properties of cliques Ω_i, each component in Ω_i is equal to x_i, that is

$$X = \Big(\underbrace{x_1,\ldots,x_1}_{\phi(pqr)}, x_2, \underbrace{x_3,\ldots,x_3}_{\phi(pq)}, \underbrace{x_4,\ldots,x_4}_{\phi(qr)}, \underbrace{x_5,\ldots,x_5}_{\phi(pr)}, \underbrace{x_6,\ldots,x_6}_{\phi(p)}, \underbrace{x_7,\ldots,x_7}_{\phi(q)}, \underbrace{x_8,\ldots,x_8}_{\phi(r)}, \underbrace{x_9,\ldots,x_9}_{pqr}\Big).$$

Therefore, from the eigenequation $A_\alpha X = \lambda X$, we have

$$\begin{aligned}
\lambda x_1 =&(\alpha d_1+(1-\alpha)(\omega_1-1))x_1+(1-\alpha)x_2+(1-\alpha)\omega_3x_3+(1-\alpha)\omega_4x_4+(1-\alpha)\omega_5x_5\\
&+(1-\alpha)\omega_6x_6+(1-\alpha)\omega_7x_7+(1-\alpha)\omega_8x_8+(1-\alpha)\omega_9x_9,\\
\lambda x_2 =&(1-\alpha)\omega_1x_1+0x_2+(1-\alpha)\omega_3x_3+(1-\alpha)\omega_4x_4+(1-\alpha)\omega_5x_5+(1-\alpha)\omega_6x_6\\
&+(1-\alpha)\omega_7x_7+(1-\alpha)\omega_8x_8+(1-\alpha)\omega_9x_9,\\
\lambda x_3 =&(1-\alpha)\omega_1x_1+(1-\alpha)x_2+(\alpha d_3+(1-\alpha)(\omega_3-1))x_3+(1-\alpha)\omega_7x_7+(1-\alpha)\omega_8x_8,\\
\lambda x_4 =&(1-\alpha)\omega_1x_1+(1-\alpha)x_2+(\alpha d_4+(1-\alpha)(\omega_4-1))x_4+(1-\alpha)\omega_6x_6+(1-\alpha)\omega_7x_7,\\
\lambda x_5 =&(1-\alpha)\omega_1x_1+(1-\alpha)x_2+(\alpha d_5+(1-\alpha)(\omega_5-1))x_5+(1-\alpha)\omega_6x_6+(1-\alpha)\omega_8x_8,\\
\lambda x_6 =&(1-\alpha)\omega_1x_1+(1-\alpha)x_2+(1-\alpha)\omega_4x_4+(1-\alpha)\omega_5x_5+(\alpha d_6+(1-\alpha)(\omega_6-1))x_6,\\
\lambda x_7 =&(1-\alpha)\omega_1x_1+(1-\alpha)x_2+(1-\alpha)\omega_3x_3+(1-\alpha)\omega_4x_4+(\alpha d_7+(1-\alpha)(\omega_7-1))x_7,\\
\lambda x_8 =&(1-\alpha)\omega_1x_1+(1-\alpha)x_2+(1-\alpha)\omega_3x_3+(1-\alpha)\omega_5x_5+(\alpha d_8+(1-\alpha)(\omega_8-1))x_8,\\
\lambda x_9 =&(1-\alpha)x_2+(\alpha d_9+(1-\alpha)(\omega_9-1))x_9.
\end{aligned}$$

The coefficient matrix of the right side of the above equations is:

$$\begin{bmatrix}
b_1 & 1-\alpha & (1-\alpha)\omega_3 & (1-\alpha)\omega_4 & (1-\alpha)\omega_5 & (1-\alpha)\omega_6 & (1-\alpha)\omega_7 & (1-\alpha)\omega_8 & 0\\
(1-\alpha)\omega_1 & \alpha d_2 & (1-\alpha)\omega_3 & (1-\alpha)\omega_4 & (1-\alpha)\omega_5 & (1-\alpha)\omega_6 & (1-\alpha)\omega_7 & (1-\alpha)\omega_8 & (1-\alpha)\omega_9\\
(1-\alpha)\omega_1 & (1-\alpha) & b_3 & 0 & 0 & 0 & (1-\alpha)\omega_7 & (1-\alpha)\omega_8 & 0\\
(1-\alpha)\omega_1 & (1-\alpha) & 0 & d_4 & 0 & (1-\alpha)\omega_6 & (1-\alpha)\omega_7 & 0 & 0\\
(1-\alpha)\omega_1 & (1-\alpha) & 0 & 0 & d_5 & (1-\alpha)\omega_6 & 0 & (1-\alpha)\omega_8 & 0\\
(1-\alpha)\omega_1 & (1-\alpha) & 0 & (1-\alpha)\omega_4 & (1-\alpha)\omega_5 & b_6 & 0 & 0 & 0\\
(1-\alpha)\omega_1 & (1-\alpha) & (1-\alpha)\omega_3 & (1-\alpha)\omega_4 & 0 & 0 & b_7 & 0 & 0\\
(1-\alpha)\omega_1 & (1-\alpha) & (1-\alpha)\omega_3 & 0 & (1-\alpha)\omega_5 & 0 & 0 & b_8 & 0\\
0 & (1-\alpha) & 0 & 0 & 0 & 0 & 0 & 0 & b_9
\end{bmatrix}, \tag{3.1}$$

where $b_i=\alpha d_i+(1-\alpha)(\omega_i-1)$, for $i=1,3,4,\dots,9$. The remaining A_α-eigenvalues of G are the eigenvalues of the matrix given in (3.1). □

We note that the matrix given in (3.1) is same as the equitable quotient matrix of G. Its eigenvalues are simple and hard to locate or find. For more about the quotient matrix, we refer the reader to [1, 13].

We now consider a mixed prime-power case in addition to the previous ones. In particular, we calculate the A_α-spectrum of the superpower graph of D_{p^2q}, including a prime square and a distinct prime in the group order.

Theorem 6. *Let $G\cong\mathcal{S}_{D_{p^2q}}$ be the superpower graph of the dihedral group of order $2p^2q$, where p and q are distinct odd primes. Then the A_α-spectrum of the superpower graph of G consists of the eigenvalues $\alpha p^2q-1, \alpha(p^2q-q+1)-1, \alpha(pq(p-1)+p)-1, \alpha(p^2q-p^2+p)-1, \alpha(p^2q-p^2+1)-1$ and $\alpha(p^2q+1)-1$ with multiplicities at least $\phi(p^2q)-1, \phi(p)-1, \phi(p^2)-1, \phi(pq)-1, \phi(q)-1$ and p^2q-1, respectively. The other* *eigenvalues of G are the eigenvalues of the following matrix*

$$\begin{bmatrix} b_1 & 1-\alpha & (1-\alpha)\omega_3 & (1-\alpha)\omega_4 & (1-\alpha)\omega_5 & (1-\alpha)\omega_6 & 0 \\ (1-\alpha)\omega_1 & \alpha d_2 & (1-\alpha)\omega_3 & (1-\alpha)\omega_4 & (1-\alpha)\omega_5 & (1-\alpha)\omega_6 & (1-\alpha)\omega_7 \\ (1-\alpha)\omega_1 & (1-\alpha) & b_3 & (1-\alpha)\omega_4 & (1-\alpha)\omega_5 & 0 & 0 \\ (1-\alpha)\omega_1 & (1-\alpha) & (1-\alpha)\omega_3 & d_4 & 0 & 0 & 0 \\ (1-\alpha)\omega_1 & (1-\alpha) & (1-\alpha)\omega_3 & 0 & d_5 & (1-\alpha)\omega_6 & 0 \\ (1-\alpha)\omega_1 & (1-\alpha) & 0 & 0 & (1-\alpha)\omega_5 & b_6 & 0 \\ 0 & (1-\alpha) & 0 & 0 & 0 & 0 & b_7 \end{bmatrix}, \quad (3.2)$$

where $b_i = \alpha d_i + (1-\alpha)(\omega_i - 1)$, *and* d_i *is the distinct degree sequence of* G.

Proof. Let $G \cong \mathcal{S}_{D_{p^2q}}$ be the superpower graph of the dihedral group D_{p^2q} of order $2p^2q$. First, we give the structural analysis of G. We note that the group D_{p^2q} has exactly p^2q rotations and p^2q reflections. The rotation part as a vertex subset is isomorphic to the cyclic group $\mathbb{Z}_{p^2q}$. The rotation part contains elements of order p, p^2, pq, q and p^2q. By the definition of the superpower graph of a finite group G, elements of orders p, p^2, pq, q, and p^2q correspond to distinct cliques. Let $\Omega_i, (i = 1, 3, 4, 5, 6, 7)$ denote the induced cliques in G generated by the elements of orders p^2q, p, p^2, pq, q and 2, respectively. Let ω_i $(i = 1, 3, 4, 5, 6, 7)$ denote the cardinalities of Ω_i, and ω_2 denote the identity element of the superpower graph G. Thus with the definition of a superpower graph, every element of Ω_3 will be adjacent to every element of $\Omega_1, \Omega_4, \Omega_5$ and identity, every element of Ω_4 will be adjacent to every element of Ω_1, Ω_3 and identity, every element of Ω_5 will be adjacent to every element of $\Omega_1, \Omega_3, \Omega_6$ and identity, and similarly for Ω_6, its every element will be adjacent to every element of Ω_1, Ω_5 and identity and lastly, every element of Ω_7 will be adjacent to identity. The block representation of G with circles denoting cliques of specific sizes can be seen in Figure 4.

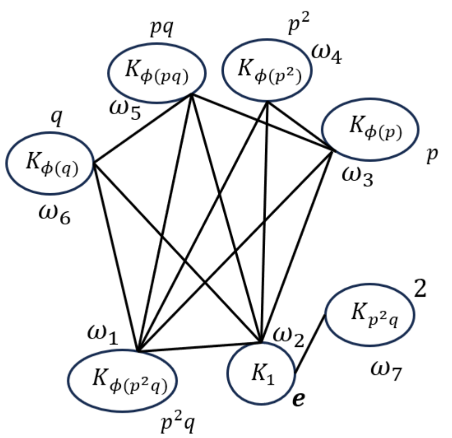

Figure 4. **Block representation of the superpower graph of** D_{p^2q}

Let d_i denote the common degree in clique Ω_i. With the vertex labelling $V_1 = \omega_2, V_2 = \omega_1, V_3 = \omega_3, V_4 = \omega_4, V_5 = \omega_5, V_6 = \omega_6,$ and $V_7 = \omega_7$, the A_α-adjacency matrix is given as

$$A_\alpha(G) = \begin{bmatrix} \eta_1 & \beta J_{\omega_1\times 1} & \beta J_{\omega_1\times\omega_3} & \beta J_{\omega_1\times\omega_4} & \beta J_{\omega_1\times\omega_5} & \beta J_{\omega_1\times\omega_6} & 0_{\omega_1\times\omega_7} \\ \beta J_{1\times\omega_1} & \alpha d_2 & \beta J_{1\times\omega_3} & \beta J_{1\times\omega_4} & \beta J_{1\times\omega_5} & \beta J_{1\times\omega_6} & 0_{1\times\omega_7} \\ \beta J_{\omega_3\times\omega_1} & \beta J_{\omega_3\times 1} & \eta_3 & \beta J_{\omega_3\times\omega_4} & \beta J_{\omega_3\times\omega_5} & 0_{\omega_3\times\omega_6} & 0_{\omega_3\times\omega_7} \\ \beta J_{\omega_4\times\omega_1} & \beta J_{\omega_4\times 1} & \beta J_{\omega_4\times\omega_3} & \eta_4 & 0_{\omega_4\times\omega_5} & 0_{\omega_4\times\omega_6} & 0_{\omega_4\times\omega_7} \\ \beta J_{\omega_5\times\omega_1} & \beta J_{\omega_5\times 1} & \beta J_{\omega_5\times\omega_3} & 0_{\omega_5\times\omega_4} & \eta_5 & \beta J_{\omega_5\times\omega_6} & 0_{\omega_5\times\omega_7} \\ \beta J_{\omega_6\times\omega_1} & \beta J_{\omega_6\times 1} & 0_{\omega_6\times\omega_3} & 0_{\omega_6\times\omega_4} & \beta J_{\omega_6\times\omega_5} & \eta_6 & 0_{\omega_6\times\omega_7} \\ 0_{\omega_7\times\omega_1} & 0_{\omega_7\times 1} & 0_{\omega_7\times\omega_3} & 0_{\omega_7\times\omega_4} & 0_{\omega_7\times\omega_5} & 0_{\omega_7\times\omega_6} & \eta_7 \end{bmatrix}$$

where $\eta_i = \alpha d_i + (1-\alpha)(J-I)_{\omega_i}$, and $\beta = 1-\alpha$.
As in the previous theorem, we have, $\phi(p^2q) = (p^2-p)(q-1)$ elements of order p^2q, $\phi(p) = p-1$ elements of order p, $\phi(p^2) = p^2-p$ elements of order p^2, $\phi(pq) = (p-1)(q-1)$ elements of order pq, $\phi(q) = q-1$ elements of order q and lastly we have exactly p^2q elements of order 2. Now, by 4, the A_α-spectrum is obtained as follows.

1. For Ω_1, the neighbors are $\Omega_3, \Omega_4, \Omega_5, \Omega_6$ and identity, therefore, the eigenvalues will be

$$\{\alpha(\phi(p^2q) + \phi(p) + \phi(p^2) + \phi(pq) + \phi(q) + 1) - 1\} = \alpha p^2q - 1,$$

with multiplicity at least $\phi(p^2q) - 1$.

2. Considering Ω_3, the neighbors are $\Omega_1, \Omega_4, \Omega_5$ and identity, therefore the eigenvalues will be

$$\{\alpha(\phi(p) + \phi(p^2q) + \phi(p^2) + \phi(pq) + 1) - 1\} = \alpha(p^2q - q + 1) - 1,$$

with multiplicity at least $\phi(p) - 1$.

3. Considering the clique Ω_4, the neighbors are Ω_1, Ω_3 and identity element, therefore the eigenvalues will be

$$\{\alpha(\phi(p^2) + \phi(p^2q) + \phi(p) + 1) - 1\} = \alpha(pq(p-1) + p) - 1,$$

with multiplicity at least $\phi(p^2) - 1$.

4. Similarly for Ω_5, neighbors are $\Omega_1, \Omega_3, \Omega_6$ and identity, so the A_α-eigenvalues will be

$$\{\alpha(\phi(pq) + \phi(p^2q) + \phi(p) + \phi(q) + 1) - 1\} = \alpha(p^2q - p^2 + p) - 1,$$

with multiplicity at least $\phi(pq) - 1$.

5. Considering the clique Ω_6, the neighbors are Ω_1, Ω_5 and identity element so the A_α-eigenvalues are

$$\{\alpha(\phi(q) + \phi(p^2q) + \phi(pq) + 1) - 1\} = \alpha(p^2q - p^2 + 1) - 1$$

having multiplicity at least $\phi(q) - 1$.

6. Finally, for the Ω_7 the only neighbor is the identity element, so the A_α-eigenvalues will be

$$\alpha(p^2q + 1) - 1,$$

with multiplicity at least $p^2q - 1$.

Now, proceeding as in Theorem 5, the remaining eigenvalues of G are the eigenvalues of the matrix given in (3.2). □

Remark 2. Similar to Corollary 5, we can obtain the consequences of Theorems 5 and 6, thereby we can easily find the adjacency, Laplacian and the signless Laplacian eigenvalues of superpower graphs of D_ζ for $\zeta \in \{pqr, p^2q\}$, where p, q, r are primes.

One of the interesting problem: is the characterization of matrices such that their equitable quotient matrices contains all the distinct eigenvalues with respect to the smallest equitable partition. A similar analysis can be investigated for the quotient matrices given in (3.1) and (3.2), (see [13]).

4. Conclusion

In this study, we have investigated the spectral properties of the superpower graphs of the direct product of two dihedral groups, particularly on the group $G = D_p \times D_p$, where D_p is the dihedral group of order $2p$ with $p \neq 2$. Also, we computed the characteristic polynomial and determined the adjacency and Laplacian spectra of the superpower graphs $\mathcal{S}_G$, thereby contributing to the broader understanding of the structural and spectral behavior of such graphs.
Furthermore, we extended our investigation to the A_α-spectrum of superpower graphs of finite groups, analyzing both $\mathcal{S}_{D_{p^k}}, \mathcal{S}_{D_p \times D_p}, \mathcal{S}_{D_{pqr}}$ and $\mathcal{S}_{D_{p^2q}}$ for p, q, r being distinct odd primes. This has been shown that investigating the A_α-matrix, which interpolates between the adjacency and signless Laplacian matrices, provides a more comprehensive understanding of spectrum analysis and can act as a connection between various spectral characteristics.
These observations are useful in the context of algebraic and spectral graph theory, as it is related to group-based graphs, and they also present various opportunities for further development. To understand further, one may specifically examine various families of non-abelian groups and corresponding graph-theoretic constructions, such as the symmetric groups, di-cyclic groups, or even bigger direct products. Moreover,

some new structural or spectral bound characterizations might be obtained by a detailed study of the relationships between spectral properties and structural features.

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