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Research Article

New characterization of efficient closed and open dominated graphs

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Dedicated to Odile Favaron

Abstract: A graph G is an efficient closed dominated graph (ECD-graph) if there exists a subset of vertices whose closed neighborhoods partition $V(G)$ and is an efficient open dominated graph (EOD-graph) if there exists a subset of vertices whose open neighborhoods partition $V(G)$. We present a new characterization of ECD- and EOD-graphs that involves independent number and a vertex clique cover of some family of cliques of closed neighborhood graph and open neighborhood graph, respectively, that are intersection graphs of closed and open neighborhoods, respectively. Several consequences are presented as well, one of them with respect to the Vizing's conjecture and the other solves a conjecture on EOD-graphs among toruses $C_t \Box C_r$ posed by Kuziak et al. (Discrete Math. Theoret. Comput. Sci. 16 (2014) 105-120).

Keywords: Efficient closed dominated graph, efficient open dominated graph, domination number, independence number, clique cover, Vizing's conjecture.

AMS Subject classification: 05C69, 05C76

1. Introduction

Partitions are one of basic mathematical tools and one can benefit from (nice) partitions in connection with the equivalence relations that comes with the partition. But even if the equivalence relation is not known explicitly, one can benefit from a partition and its structure in particular when the partitions under consideration contains

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objects of the same type. We consider in this paper two such partition of vertex sets of graphs (when they exists). First are efficient closed dominated graphs (ECD-graphs for short) whose vertex set can be partitioned into closed neighborhoods and second are efficient open dominated graphs (EOD-graphs for short) with partition of vertex set into open neighborhoods. We characterize both mentioned classes.

The study of ECD-graphs was initiated by Biggs [6] over perfect codes in graphs and it present a generalization of the problem of the existence of (classical) error-correcting codes. The connection is that a perfect code of a graph G (if it exists) form a set of centers of closed neighborhoods which partition $V(G)$. The study of this topic later follow different directions, but the main brunches are for which graph classes is the decision problem—if a graph is an ECD-graph—an NP-complete problem and for which classes there exists a polynomial algorithm, as well as for which graph classes (families) we can construct perfect codes explicitly. We will mention just some references out of many on the topic. It is an NP-complete problem in general [4] and remains NP-complete on k-regular graphs ($k \geq 4$) [19], on planar graphs of maximum degree 3 [12, 19], as well as on bipartite and chordal graphs [27]. The existence of a perfect code can be decided in polynomial time on trees [12], interval graphs [20], circular-arc graphs [17] and hereditary efficiently dominated graphs [24]. Among construction of perfect codes we mention just the work on graph products [2, 16, 22, 25, 30, 32], that will be implicitly important also for the present paper. For more information and literature we recommend a survey [7].

The study of EOD-graphs was initiated later and it did not reach the same popularity as efficient close domination. The decision problem is NP-complete [13, 23]. EOD-trees were described recursively in [13] and EOD-Cayley graphs were considered in [9]. Again there is a lot of work done on graph products: see [1] for direct products, [10, 11, 18] for grid graphs, see [21] for the other standard graph products and disjunctive product and [29] for an approach on Cartesian product. In particular the following conjecture is from [21].

Conjecture 1. A Cartesian product of cycles $C_t \Box C_r$ is an EOD graph if and only if t and r are multiple of four.

We end the historical background by mentioning that in [15] the intersection of efficient closed and efficient open dominated graphs was studied.

In the following section we recall the important definitions. Two similar section follows, one on ECD-graphs and the other on EOD-graphs. In both we present a characterizations over the intersection graphs of closed, resp. open, neighborhoods and their independence number and a certain clique cover. Later we end the first with a remark toward (well known) Vizing's conjecture and in the second we confirm Conjecture 1 in affirmative.

2. Preliminaries

Let G be a graph. The *distance* $d_G(u,v)$ between $u,v \in V(G)$ is the minimum number of edges on a path between u and v. The maximum distance between any pair of vertices of a graph G is the *diameter* of G. Two graphs G and H are isomorphic, $G \cong H$, if there exists a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. By union of graphs G and H we mean a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *open neighborhood* $N_G(v)$ of a vertex v in G is defined as the set of vertices adjacent to $v \in V(G)$, i.e. $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* $N_G[v]$ of $v \in V(G)$ is the set $N_G(v) \cup \{v\}$. A *clique* is a complete subgraph of G. Let $C_1, \ldots, C_k$ be a collection $\mathcal{C}$ of cliques of a graph. A vertex set A is a *vertex cover* of $\mathcal{C}$ if $A \cap C_i \neq \emptyset$ for every $i \in \{1, \ldots, k\}$. The *vertex cover number* $\tau(\mathcal{C})$ of $\mathcal{C}$ is the minimum cardinality of a vertex cover of $\mathcal{C}$. Notice that our vertex cover of $\mathcal{C}$ is a generalization of vertex cover of edges, where every clique is an edge. For a given graph G, a vertex set $S \subset V(G)$ is an *independent set* if there is no edge between any two vertices in S. A maximum cardinality of an independent set is the *independence number* of G denoted $\alpha(G)$. An independent set of cardinality $\alpha(G)$ in G is called an $\alpha(G)$-set. A subset D of vertices of G is called a *dominating set* of G if every vertex from $V(G) - D$ is adjacent to a vertex in D. A *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set.

A graph G is an *efficient closed dominated graph* if there exist a subset D of vertices whose closed neighborhoods partition $V(G)$. In other words, the union of closed neighborhoods centered in vertices of D equals to $V(G)$ and $N_G[u] \cap N_G[v] = \emptyset$ for every pair of different vertices $u,v \in D$. If such a set D exists, then it is called a 1-*perfect code* or an *efficient closed dominating set*. We use the later form and use abbreviation ECD-set for it and ECD-graph for a graph with an ECD-set. Clearly, any two vertices of an ECD-set D are at distance more than two. Set $P \subseteq V(G)$ is called a *packing* if any two different vertices from P are at distance more than two. The maximum cardinality of a packing in G is called the *packing number* of G and is denoted $\rho(G)$. So, $\rho(G)$ represents the biggest number of closed neighborhoods of a graph that have pairwise empty intersection. If G is an ECD-graph, then $\rho(G) = \gamma(G)$ is the cardinality of an ECD-set.

Similar is a graph G an *efficient open dominated graph* if there exist a subset D of vertices whose open neighborhoods partition $V(G)$. So, the union of open neighborhoods centered in vertices of D equals to $V(G)$ and $N_G(u) \cap N_G(v) = \emptyset$ for every pair of different vertices $u,v \in D$. If such a set D exists, then it is called an *efficient open dominating set*. We use the abbreviation EOD-set for it and EOD-graph for a graph with an EOD-set.

The *closed neighborhood graph* of a given graph H, denoted $CN(H)$, is defined as the intersection graph of closed neighborhoods of H. In other words, $V(CN(H)) = V(H)$ and different vertices $u,v \in V(CN(H))$ are adjacent whenever $N_H[u] \cap N_H[v] \neq \emptyset$. Notice that one can find also notation H^2 for $CN(H)$ in the literature. The *open* *neighborhood graph* of a given graph H, denoted $ON(H)$, is defined as the intersection graph of open neighborhoods of H. This means that $V(ON(H)) = V(H)$ and different vertices $u, v \in V(ON(H))$ are adjacent whenever $N_H(u) \cap N_H(v) \neq \emptyset$. See Figure 1 for an example of a graph H together with $CN(H)$ and $ON(H)$.

The *Cartesian product* $G \Box H$ of graphs G and H is a graph with $V(G \Box H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \Box H$ if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. All the vertices of $G \Box H$ that project to the vertex $g \in V(G)$ form an H-*layer* through g denoted by H^g. So, $H^g = \{(g, h) : h \in V(H)\}$. A subgraph of $G \Box H$ induced by H^g is isomorphic to H. Similar we define G-layer G^h through h. Again is a subgraph of $G \Box H$ induced by G^h isomorphic to G. Probably the most intriguing conjecture connected with the Cartesian product is the famous Vising's conjecture which states

$$\gamma(G \Box H) \geq \gamma(G)\gamma(H), \tag{2.1}$$

see the latest survey [8] for more information on the topic. The *direct product* $G \times H$ of graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \times H$ if $gg' \in E(G)$ and $hh' \in E(H)$. Again we can define G- and H-layers, but every layer is an independent set of $G \times H$ contrary to the Cartesian product. More about Cartesian and direct products, as well as other graph products, can be found in the book [14].

3. ECD-graphs

First we recall the characterization of closed neighborhood graphs from [26]. Notice that in [26] different notation was used, more adapted to notation $H^2 = CN(H)$.

Theorem 2. *A connected graph G with vertices labeled $v_1, \ldots, v_n$ is a closed neighborhood graph if and only if G contains a collection of n cliques labeled $C_1, \ldots, C_n$ whose union is G and such that the following conditions hold for every $i, j \in \{1, \ldots, n\}$*

(i) $v_i \in C_i$,

(ii) $v_i \in C_j \Leftrightarrow v_j \in C_i$.

From now on, we denote by $\mathcal{C}_G$ the collection of cliques described in Theorem 2. To underline the Theorem 2 observe the graph house H of Figure 1. Clearly $G = CN(H) \cong K_5$ as the diameter of H is two. But the collection of cliques $\mathcal{C}_G$ consists of five cliques: $C_a = \{a, b, c\}$, $C_b = \{a, b, c, d\}$, $C_c = \{a, b, c, e\}$, $C_d = \{b, d, e\}$ and $C_e = \{c, d, e\}$. Notice that a clique $C_x \in \mathcal{C}_G$ equals to $N_H[x]$ for every $x \in V(H) = V(G)$. Hence cliques of $\mathcal{C}_G$ are not always maximal cliques of G. Observe also that $\tau(\mathcal{C}_G) = 2$ as for instance $\{b, d\}$ is a clique cover of $\mathcal{C}_G$.

Theorem 3. *Let $G = CN(H)$, where H is a graph. Graph H is an ECD-graph with an ECD-set D if and only if D is an $\alpha(G)$-set and D covers every clique from $\mathcal{C}_G$.*

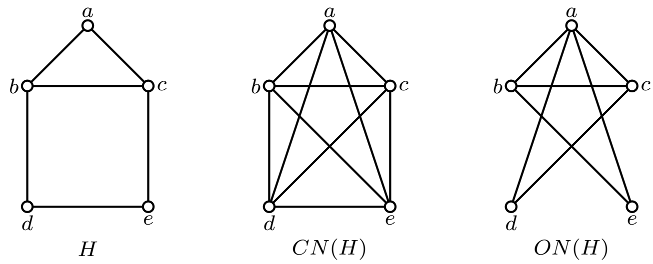

Figure 1. **A graph house H together with $CN(H)$ and $ON(H)$.**

Proof. Let H be an ECD-graph with an ECD-set D and let $G = CN(H)$. If $V(H) = \{v_1, \ldots, v_n\}$, then $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{v_iv_j : 1 \leq d_H(v_i, v_j) \leq 2\}$. If there exist $v_i, v_j \in D$ such that $v_iv_j \in E(G)$, then $d_H(v_i, v_j) \leq 2$, and this yields a contradiction with D being an ECD-set in H. Therefore $d_G(v_i, v_j) \geq 2$ and D is an independent set in G.
Let us prove now that D is in fact a maximum independent set in G. Assume there exits an independent set P in G of cardinality $|P| > |D|$. Suppose first that $D \cap P = \emptyset$. Since D is an ECD-set and $|P| > |D|$, there exist at least two vertices $p_i, p_j \in P$ which are neighbors of the same vertex $v_i \in D$ in H. Therefore $p_ip_j \in E(G)$ and this contradicts the fact that P is an independent set in G. Hence we may assume that $D \cap P \neq \emptyset$. Since D is an ECD-set in H, we have $N_H[v_j] \cap D = \{v_j\}$ for every vertex v_j in D. Note that for every $p \in P - D$ there exists a unique vertex $v_j \in D$, such that $p \in N_H[v_j]$. We have $N_H[v_j] \cap (P - D) = \{p\}$ because P is an independent set of G. This brings a bijection between $P - D$ and $D - P$. Thus, we conclude that $|P| \leq |D|$ in any case, which is a contradiction.
Finally, note that for every clique $C_i \in \mathcal{C}_G$, we have $V(C_i) = N_H[v_i]$, with $v_i \in V(H) - D$. Since D is an ECD-set in H, every vertex v_i in H is adjacent to exactly one vertex in D. Therefore, each clique in $\mathcal{C}_G$ contains exactly one vetex of D and thus, D covers $\mathcal{C}_G$.
Let us prove the second implication. Let D be a maximum independent set of G, such that D covers every clique from $\mathcal{C}_G$. Assume that D is not an ECD-set in H.
If there exits $v_k \in V(H)$ such that $v_k \in N_H[v_i] \cap N_H[v_j]$ for some $v_i, v_j \in D$, $v_i \neq v_j$, then $v_iv_j \in E(G)$, which yields a contradiction with D being an independent set in G. Therefore, $N_H[v_i] \cap N_H[v_j] = \emptyset$ for any $v_i, v_j \in D$, $v_i \neq v_j$. On the other hand, if there exits $x \in V(H)$ such that $x \notin \cup_{v_i \in D} N_H[v_i]$, then this contradicts the fact that D covers all cliques in $\mathcal{C}_G$. Thus, we conclude that D is an ECD-set in H. □

Recall the graph H from Figure 1 where $CN(H)$ is complete graph on five vertices. We have $\alpha(CN(H)) = 1$ and $\tau(\mathcal{C}_G) = 2$. Hence, by Theorem 3 there is no ECD-set of H and H is not an ECD-graph.
As every ECD-set of an ECD-graph H is also a packing of H, Theorem 3 immediately implies the following result.

Corollary 1. *If H is an ECD-graph, then $\gamma(H) = \rho(H) = \alpha(CN(H))$.*

The last equality in above corollary holds for any graph as shown next.

Proposition 1. *If H is a graph, then $\rho(H) = \alpha(CN(H))$.*

Proof. Let D be a packing set of H of maximum cardinality. We claim that D is an $\alpha(CN(H))$-set. As $d_H(u, v) \geq 3$ for all pairs of different vertices $u, v \in D$, we have that u and v are non adjacent in $CN(H)$. Therefore D is an independent set in $CN(H)$. Suppose that there exists an independent set D' of $CN(H)$ of greater cardinality than D. Since $u', v' \in D'$, $u' \neq v'$, are not adjacent in $CN(H)$, they must be at distance at least three in H. Hence D' is a packing set with $|D| < |D'|$, which is a contradiction with the choice of D. So D is an $\alpha(CN(H))$-set and the result follows. □

Corollary 2. *If H is a graph, then $\gamma(H) \geq \alpha(CN(H))$.*

Proof. It is well known that $\gamma(H) \geq \rho(H)$ and by Proposition 1 we have $\gamma(H) \geq \alpha(CN(H))$. □

The Corollary 2 can be used in two ways. Either to bound $\alpha(CN(H))$ from above when $\gamma(H)$ is known or to bound $\gamma(H)$ from below if we know $\alpha(CN(H))$. One can expect that the second option is not realistic at the time being as we do not know any results on $\alpha(CN(H))$.
If we replace graph H in Corollary 2 by a Cartesian product of two graphs G and H, then we obtain

$$\gamma(G\Box H) \geq \alpha(CN(G\Box H)). \tag{3.1}$$

A natural question is to ask whether the following inequality is valid

$$\alpha(CN(G\Box H)) \geq \gamma(G)\gamma(H), \tag{3.2}$$

as both (3.1) and (3.2) together give a Vizing's conjecture (2.1). Unfortunately, (3.2) is not true for all graphs G and H. For instance, pairs for which (3.2) is not valid are $(G, H) \in \{(C_4, C_4), (C_5, C_4), (C_5, P)\}$, where P is the Peterson graph, but for all three mentioned pairs Vizing conjecture holds. In particular, we have

$$E(CN(G\Box H)) = E(G \times H) \cup E(CN(G)\Box CN(H)), \tag{3.3}$$

see [28] where the result is stated for open neighborhood graphs under the name Cartesian sum, but the same approach can be used for closed neighborhood graph. Therefore one can expect some problems with graphs of diameter two as they closed neighborhood graph is a complete graph. On the other hand it is easy to see that even for many graphs of diameter two, (3.2) still holds. For an example observe that $\alpha(CN(C_5 \square C_5)) = 5 > 4 = \gamma(C_5)\gamma(C_5)$ (notice that the set $\{v_{1,1}, v_{2,3}, v_{3,5}, v_{4,2}, v_{5,4}\}$ is an $\alpha(CN(C_5 \square C_5))$-set where $C_5 = v_1v_2v_3v_4v_5v_1$ and $v_{i,j} = (v_i, v_j)$ for $i, j \in \{1, 2, 3, 4, 5\}$).

Proposition 2. *If G and H are graphs, then $\alpha(CN(G \square H)) \geq \alpha(CN(G))\alpha(CN(H))$.*

Proof. Let A_G and A_H be maximum independent sets of $CN(G)$ and $CN(H)$, respectively. By (3.3) set $A_G \times A_H$ is independent in $CN(G \square H)$ and the result follows. □

The following result on Vising's conjecture is a special case of decomposable graphs in the meaning of Barcalkin and German [5], see also [8], and is well known. Nevertheless, we state it here because it nicely underline our approach.

Corollary 3. *If G and H are ECD-graphs, then $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.*

Proof. By (3.1) and Propositions 2 and 1 we have for ECD-graphs $\gamma(G \square H) \geq \alpha(CN(G \square H)) \geq \alpha(CN(G))\alpha(CN(H)) = \rho(G)\rho(H) = \gamma(G)\gamma(H)$. □

4. EOD-graphs

In this section we turn towards EOD-graphs and prove a characterisation similar to the one for ECD-graphs in Theorem 3. In [3], Acharya and Vartak characterized open neighborhood graphs as follows.

Theorem 4. *A connected graph G with vertices labeled $v_1, \ldots, v_n$ is an open neighborhood graph of some graph H if and only if G contains a collection of cliques $O_1, \ldots, O_n$ such that for all $i, j \in \{1, \ldots, n\}$*

(i) $v_i \notin O_i$,

(ii) $v_i \in O_j \Leftrightarrow v_j \in O_i$,

(iii) if $v_iv_j \in E(G)$, then there exits a O_k, $k \in \{1, \ldots, n\}$, containing v_iv_j.

From now on, we denote by $\mathcal{O}_G$ the collection of cliques described in Theorem 4. To underline Theorem 4 observe the graph house H and $G = ON(H)$ of Figure 1. The collection of cliques $\mathcal{O}_G$ consists of five cliques: $O_a = \{b, c\}$, $O_b = \{a, c, d\}$, $O_c = \{a, b, e\}$, $O_d = \{b, e\}$ and $O_e = \{c, d\}$. Notice that a clique $O_x \in \mathcal{O}_G$ equals to $N_H(x)$ for every $x \in V(H) = V(G)$. Notice that a maximum clique on vertices $\{a, b, c\}$ does not belong to $\mathcal{O}_G$. Observe also that $\tau(\mathcal{O}_G) = 2$ as for instance $\{b, c\}$ is a clique cover of $\mathcal{O}_G$.

Theorem 5. *Let $G = ON(H)$ for a connected graph H. Then H is an EOD-graph with an EOD-set D if and only if D is an $\alpha(G)$-set and D covers every clique from $\mathcal{O}_G$.*

Proof. Let H be an EOD-graph with an EOD-set D and let $G = ON(H)$. If $V(H) = \{v_1, \dots, v_n\}$, then $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{v_iv_j : d_H(v_i, v_j) = 2$ or v_i, v_j belong to a common triangle in $H\}$.
If there exist $v_i, v_j \in D$ such that $v_iv_j \in E(G)$, then either $d_H(v_i, v_j) = 2$ or v_i, v_j belong to a common triangle in H, and both possibilities yield a contradiction with D being an EOD-set in H. Therefore $d_G(v_i, v_j) \geq 2$ and D is an independent set in G. Let us prove now that D is an $\alpha(G)$-set. Assume there exists an independent set P in G of cardinality $|P| > |D|$.
Suppose first that $D \cap P = \emptyset$. Since D is an EOD-set and $|P| > |D|$, there exist at least two vertices $p_i, p_j \in P$ which are neighbors of the same vertex $v_i \in D$ in H. Therefore $p_ip_j \in E(G)$ and this contradicts the fact that P is an independent set in G. Suppose now that $D \subset P$. Since D is an EOD-set in H, then for every $p \in P - D$ there exist $v_i, v_j \in D$ such that $v_i \in N_H(v_j) \cap N_H(p)$. Thus, $pv_j \in E(G)$ and this contradicts the fact that P is an independent set in G.
Let now $D \cap P \neq \emptyset$ and $D \not\subset P$. Since D is an EOD-set in H, every vertex of H contains a unique neighbor in D. So, for every $p \in P - D$ there exist its unique neighbor $v_j \in D$ and further the unique neighbor $v_i \in D$ of v_j. Suppose first that $v_j \in D \cap P$. Since P is an independent set of G, then $v_i \notin D \cap P$. Note that, for each vertex in $P - D$ which has a neighbor in $D \cap P$, there exists exactly one vertex in $D - P$. Now, assume that $v_j \in D - P$. Since P is an independent set of G, we have $v_i \in D - P$ and either $N_H(v_i) \cap (P \cup D) = \{v_j\}$ or $N_H(v_i) \cap (P \cup D) = \{v_j, p'\}$, where $p' \in P - D$ and $p' \neq p$. In this way, we have either a bijection between $P - D$ and $D - P$, or we assign two vertices in $D - P$ to a vertex in $P - D$. Thus, we conclude that $|P| \leq |D|$ in any case, which is a contradiction.
Finally, note that for every clique $O_i \in \mathcal{O}_G$, we have $V(O_i) = N_H(v_i)$, with $v_i \in V(H)$. Since D is an EOD-set in H and D is independent set in G, every vertex v_i in H is adjacent to exactly one vertex in D. Therefore, each clique in $\mathcal{O}_G$ contains exactly one vertex of D and thus, D covers $\mathcal{O}_G$.
Let us prove the second implication. Let D be an $\alpha(G)$-set, such that D covers every clique from $\mathcal{O}_G$. Assume that D is not an EOD-set in H. If there exits $v_k \in V(H)$ such that $v_k \in N_H(v_i) \cap N_H(v_j)$ for some $v_i, v_j \in D$, $v_i \neq v_j$, then either $d_H(v_i, v_j) = 2$ or v_i, v_j belong to a common triangle in H. Hence, $v_iv_j \in E(G)$, which yields a contradiction with D being an $\alpha(G)$-set. Therefore, $N_H(v_i) \cap N_H(v_j) = \emptyset$ for any $v_i, v_j \in D$, $v_i \neq v_j$. On the other hand, if there exits $x \in V(H)$ such that $x \notin \cup_{v_i \in D} N_H(v_i)$, then this contradicts the fact that D covers all cliques in $\mathcal{O}_G$. Thus, we conclude D is an EOD-set in H. □

Recall graphs H and $ON(H)$ from Figure 1. We have $\alpha(ON(H)) = 2$ and $\tau(\mathcal{C}_G) = 2$ where $D = \{b, d\}$ is a desired $\alpha(ON(H))$-set that is also a cover of $\mathcal{O}_{ON(H)}$. Hence, by Theorem 5 D is an EOD-set of H and H is an EOD-graph. The following remark is also clear since a set D from Theorem 5 is both an $\alpha(ON(H))$-set and a cover of $\mathcal{O}_{ON(H)}$.

Remark 1. A set D from Theorem 5 covers every clique from $\mathcal{O}_{ON(H)}$ exactly once.

We use Theorem 5 to prove Conjecture 1. We start with the following proposition, which may be of independent interest, for which we need

$$ON(G\Box H) = (ON(G)\Box ON(H)) \cup (G \times H)$$

that was shown in [28], see Theorem 1 (with a different notation).

Proposition 3. *If G and H are connected bipartite graphs, then $ON(G\Box H)$ is disconnected graph with exactly two connected components.*

Proof. Let G and H be connected bipartite graphs with bi-partitions $V(G) = A \cup B$ and $V(H) = C \cup D$. By the definition of $ON(G)$, if two vertices are adjacent in $ON(G)$, then they both belong either to A or to B. Similar, if two vertices are adjacent in $ON(H)$, then they both belong either to C or to D. Since G and H are connected, sets A, B, C and D yields connected components of $ON(G)$ and $ON(H)$, respectively. Hence $ON(G)\Box ON(H)$ has four components induced by $A \times C, A \times D, B \times C$ and $B \times D$.
On the other hand, it is well known that the direct product of two connected bipartite graphs G and H is dis-connected with exactly two components induced by $A \times C \cup B \times D$ and by $B \times C \cup A \times D$, see [31] or the book [14] (Theorem 5.9 pp. 55). Hence $ON(G\Box H)$ contains the same connected components as $G \times H$ and we are done. □

It is clear that $ON(C_k) \cong C_k$ whenever $k \geq 3$ is an odd integer and that $ON(C_k) \cong 2C_{k/2}$ whenever $k \geq 6$ is an even integer and that $ON(C_4) \cong 2K_2$. Hence the next corollary follows directly from Proposition 3.

Corollary 4. *If r and t are even integers greater than 3, then $ON(C_r\Box C_t)$ is disconnected graph with two isomorphic connected components.*

In what follows we denote $V(C_r\Box C_t) = V(ON(C_r\Box C_t)) = \{v_{i,j} : 0 \leq i \leq r-1, 0 \leq j \leq t-1\}$ and in the computation on subscripts we use (mod r) for i and (mod t) for j. Since $C_r\Box C_t$ is 4-regular, every vertex $v_{i,j}$ lies in four different open neighborhoods and yields therefore a (maximum) 4-clique of $\mathcal{O}_{C_r\Box C_t}$ in $ON(C_r\Box C_t)$ denoted by $Q_{i,j}$. Clearly $Q_{i,j} = \{v_{i-1,j}, v_{i+1,j}, v_{i,j-1}, v_{i,j+1}\}$ and every vertex $v_{i,j}$ is contained in exactly four different cliques $Q_{i-1,j}, Q_{i+1,j}, Q_{i,j-1}$ and $Q_{i,j+1}$. Also $|V(C_r\Box C_t)| = |\mathcal{O}_{C_r\Box C_t}|$.

Theorem 6. *A Cartesian product $C_r\Box C_t$ is an EOD-graph if and only if $r, t \equiv 0 \pmod 4$.*

Proof. Let $G \cong C_r \Box C_t$. If $r, t \equiv 0 \pmod 4$, then G is an EOD-graph by Corollary 5 from [11]. So we may assume that at least one from r and t is not a multiple of 4. We will show that there exists no independent set in $ON(G)$ that covers all cliques from $\mathcal{O}_G$ and therefore, by Theorem 5, G is not an EOD-graph. Suppose first that both r and t are odd. Clearly $|V(ON(G))| = rt$ is an odd number. Since every vertex $v_{i,j}$ covers exactly four cliques from $\mathcal{O}_G$, there exists no independent set that covers all cliques from $\mathcal{O}_G$ exactly once as rt is not a multiple of four. The same argumentation holds if one, say r, is an odd number and $t \equiv 2 \pmod 4$. Let now $r, t \equiv 2 \pmod 4$. Now rt is a multiple of four, but by Corollary 4 $ON(G)$ is not connected and both components contains $rt/2$ vertices, which is not a multiple of four. Again, by the same argument there exists no independent set that covers all cliques from $\mathcal{O}_G$ exactly once. So, we may assume that exactly one, say r, is a multiple of four and t is not. In order to gain a contradiction we assume that D is an $\alpha(ON(G))$-set that covers all cliques from $\mathcal{O}_G$ exactly once. We claim that every C_t-layer contains a vertex from D. Observe first that any three consecutive C_t-layers, say $C_t^{v_i}, C_t^{v_{i+1}}, C_t^{v_{i+2}}$, must contain at least one vertex from D, because otherwise $Q_{i+1,j}$ is not covered by D for any $0 \le j \le t-1$. Suppose next that there exists two consecutive C_t-layers, say $C_t^{v_i}$ and $C_t^{v_{i+1}}$ without a vertex in D. A clique $Q_{i+1,j}$ is covered by D and therefore $v_{i+2,j} \in D$ for every $0 \le j \le t-1$, a contradiction with D being an $\alpha(ON(G))$-set. To end the claim assume conversely that $C_t^{v_i}$ poses no vertex from D. To cover $Q_{i,j}$ either $v_{i-1,j} \in D$ or $v_{i+1,j} \in D$, say $v_{i-1,j} \in D$. With this $v_{i-1,j-2} \notin D$ because it is adjacent to $v_{i-1,j}$ in $ON(G)$. But then $v_{i+1,j-2} \in D$ to cover $Q_{i,j-2}$. Now, $v_{i+1,j-4} \notin D$ because it is adjacent to $v_{i+1,j-2}$ in $ON(G)$ and $v_{i-1,j-4} \in D$ to cover $Q_{i,j-4}$. By repeating this argument we get $v_{i-1,j-4k} \in D$ for every integer $k \ge 0$. In any case we get $v_{i-1,j-2} \in D$ or $v_{i-1,j+2} \in D$, a contradiction because $v_{i-1,j-2}v_{i-1,j}, v_{i-1,j+2}v_{i-1,j} \in E(ON(G))$. Therefore every C_t-layer contains at least one vertex from D.

Let $v_{i,j}, v_{i+1,j-k} \in D$ for some odd k be such vertices that $v_{i,j-2\ell}, v_{i+1,j-2\ell+1} \notin D$ for every $1 \le \ell < k/2$. Clearly, $k \ne 1$ because $v_{i,j}v_{i+1,j-1} \in E(ON(G))$. We would like to show that $k = 3$. If $k \ne 3$, then $v_{i+2,j-2} \in D$ to cover $Q_{i+1,j-2}$. Further, to cover $Q_{i+1,j-4}$, either $v_{i,j-4} \in D$ or $v_{i+1,j-5} \in D$. If $v_{i,j-4} \in D$, then we have a contradiction by the choice of $v_{i,j}$ and $v_{i+1,j-k}$. So, $v_{i+1,j-5} \in D$ and we have two consecutive layers $C_t^{v_{i+2}}$ and $C_t^{v_{i+1}}$ two vertices $v_{i+2,j-2}, v_{i+1,j-5} \in D$ where $v_{i+2,j-3}, v_{i+1,j-4} \notin D$. By the change of notation we have the desired consecutive C_t-layers and vertices $v_{i,j}, v_{i+1,j-3} \in D$ where $v_{i,j-2}, v_{i+1,j-1} \notin D$. Moreover, by the symmetry of G we may assume that $i = 0$ and $j = 3$ and so $v_{0,3}, v_{1,0} \in D$.

Next we show by induction on k that $v_{2k+1,2k}, v_{2k,2k+3} \in D$. For $k = 1$ we have $v_{3,2} \in D$ to cover $Q_{2,2}$ and with this also $v_{2,5} \in D$ to cover $Q_{2,4}$ and the basis is complete. Let $k > 1$ and $v_{2k+1,2k}, v_{2k,2k+3} \in D$ by induction hypothesis. Only vertex $v_{2k+3,2k+2}$ from $Q_{2k+2,2k+2}$ is not adjacent to $v_{2k+1,2k}, v_{2k,2k+3} \in D$ in $ON(G)$ and must therefore be in D. Similar, only vertex $v_{2k+2,2k+5}$ from $Q_{2k+2,2k+4}$ is not adjacent to $v_{2k+3,2k+2}, v_{2k,2k+3} \in D$ and must also be in D and the induction is completed.

We continue by the following diagonals

$$D_i = \{v_{j,i+j} : j \in \{0, 1, \ldots, t-1\}\}$$

for $i \in \{0, 1, \ldots, t-1\}$. In particular notice that we have shown that every second vertex starting from $v_{1,0}$ belongs to $D \cap D_{t-1}$ and every second vertex starting from $v_{0,3}$ belongs to $D \cap D_3$. To cover $Q_{1,7}$ either $v_{0,7}$ or $v_{1,8}$ belongs to $D \cap D_7$ and later every second vertex from D_7 belongs to D as well. By the same argument we get that for every $j > 1$ either $v_{0,4j-1}$ or $v_{1,4j}$ belongs to $D \cap D_{4j-1}$ and later every second vertex from D_{4j-1} belongs to D as well. In particular, if $v_{0,4j-1} \in D$, then also $v_{2,4j+1} \in D$. We end the proof by considering three cases depending on different non-zero remainders of $t \pmod 4$.

Case 1. $t = 4\ell + 2$ for some positive integer ℓ. By our choice of D we have either $v_{0,4\ell-1}, v_{2,4\ell+1} \in D_{t-3} \cap D$ or $v_{1,4\ell} \in D_{t-3} \cap D$. In both options we have a contradiction with D being an $\alpha(ON(G))$-set since $v_{2,4\ell+1}$ and $v_{1,4\ell}$ are both adjacent to $v_{1,0} \in D$ in $ON(G)$.

Case 2. $t = 4\ell+1$ for some positive integer ℓ. By our choice of D we have $D_{t-2} \cap D = D_{4\ell-1} \cap D$. After one additional step we obtain that $D_2 \cap D \neq \emptyset$ and after additional $\ell - 1$ steps we have either $v_{0,4\ell-2}, v_{2,4\ell} \in D_{t-3} \cap D$ or $v_{1,4\ell-1} \in D_{t-3} \cap D$. In both options we have a contradiction with D being an $\alpha(ON(G))$-set since $v_{2,4\ell}$ and $v_{1,4\ell-1}$ are both adjacent to $v_{1,0} \in D$ in $ON(G)$.

Case 3. $t = 4\ell+3$ for some positive integer ℓ. By our choice of D we have $D_{t-3} \cap D = D_{4\ell} \cap D$. After one additional step we obtain that $D_0 \cap D \neq \emptyset$ and after additional $\ell - 1$ steps we have either $v_{0,4\ell}, v_{2,4\ell+2} \in D_{t-3} \cap D$ or $v_{1,4\ell+1} \in D_{t-3} \cap D$. In both options we have a contradiction with D being an $\alpha(ON(G))$-set since $v_{2,4\ell+2}$ and $v_{1,4\ell+1}$ are both adjacent to $v_{1,0} \in D$ in $ON(G)$.

In all possibilities we obtain a contradiction with D being an $\alpha(ON(G))$-set and by Theorem 5 $C_t \Box C_r$ is not an EOD-graph when (at least) one factor is not a multiple of four. □

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References

[1] G. Abay-Asmerom, R.H. Hammack, and D.T. Taylor, *Total perfect codes in tensor products of graphs*, Ars Combin. **88** (2008), 129–134.

[2] ______, *Perfect r-codes in strong products of graphs*, Bull. Inst. Combin. Appl. **55** (2009), 66–72.

[3] B.D. Acharya and M.N. Vartak, *Open neighbourhood graphs*, Indian Institute of Technology Department of Mathematics Research Report **6** (1973).

[4] D.W. Bange, A.E. Barkauskas, and P.J. Slater, *Efficient dominating sets in graphs*, Appl. Discrete Math. **189** (1988), 189–199.

[5] A.M. Barcalkin and L.F. German, *The external stability number of the cartesian product of graphs*, Bul. Akad. Stiinte RSS Moldoven **1** (1979), no. 94, 5–8.

[6] N. Biggs, *Perfect codes in graphs*, J. Combin. Theory Ser. B **15** (1973), no. 3, 289–296.
https://doi.org/10.1016/0095-8956(73)90042-7.

[7] A. Brandstädt, *Efficient domination and efficient edge domination: A brief survey*, Conference on Algorithms and Discrete Applied Mathematics, vol. 10743, Springer, 2018, pp. 1–14.
https://doi.org/10.1007/978-3-319-74180-2_1.

[8] B. Brešar, P. Dorbec, W. Goddard, B.L. Hartnell, M.A. Henning, S. Klavžar, and D.F. Rall, *Vizing's conjecture: a survey and recent results*, J. Graph Theory **69** (2012), no. 1, 46–76.
https://doi.org/10.1002/jgt.20565.

[9] T.T. Chelvam and S. Mutharasu, *Efficient open domination in cayley graphs*, Appl. Math. Lett. **25** (2012), no. 10, 1560–1564.
https://doi.org/10.1016/j.aml.2011.12.036.

[10] R. Cowen, S.H. Hechler, J.W. Kennedy, and A. Steinberg, *Odd neighborhood transversals on grid graphs*, Discrete Math. **307** (2007), no. 17-18, 2200–2208.
https://doi.org/10.1016/j.disc.2006.11.006.

[11] I.J. Dejter, *Perfect domination in regular grid graphs*, Australas. J. Combin. **42** (2008), 99–114.

[12] M.R. Fellows and M.N. Hoover, *Perfect domination*, Australas. J. Combin. **3** (1991), 141–150.

[13] H. Gavlas, K. Schultz, and P. Slater, *Efficient open domination in graphs*, Sci. Ser. A Math. Sci **6** (2003), 77–84.

[14] R.H. Hammack, W. Imrich, and S. Klavžar, *Handbook of Product Graphs*, vol. 2, CRC press Boca Raton, 2011.

[15] S. Klavžar, I. Peterin, and I.G. Yero, *Graphs that are simultaneously efficient open domination and efficient closed domination graphs*, Discrete Appl. Math. **217** (2017), 613–621.
https://doi.org/10.1016/j.dam.2016.09.027.

[16] S. Klavžar, S. Špacapan, and J. Žerovnik, *An almost complete description of perfect codes in direct products of cycles*, Adv. Appl. Math. **37** (2006), no. 1, 2–18.
https://doi.org/10.1016/j.aam.2005.10.002.

[17] W.F. Klostermeyer and E.M. Eschen, *Perfect codes and independent dominating sets*, Congr. Numer. **142** (2000), 7–28.

[18] W.F. Klostermeyer and J.L. Goldwasser, *Total perfect codes in grid graphs*, Bull. Inst. Combin. Appl. **46** (2006), 61–68.

[19] J. Kratochvíl and M. Křivánek, *On the computational complexity of codes in*

graphs, International Symposium on Mathematical Foundations of Computer Science, vol. 324, Springer, 1988, pp. 396–404.
https://doi.org/10.1007/BFb0017162.

[20] J. Kratochvíl, P.D. Manuel, and M. Miller, *Generalized domination in chordal graphs*, Nordic J. Comput. **2** (1995), no. 1, 41–50.

[21] D. Kuziak, I. Peterin, and I.G. Yero, *Efficient open domination in graph products*, Discrete Math. Theor. Comput. Sci. **16** (2014), no. 1, 105–120.
https://doi.org/10.46298/dmtcs.1267.

[22] I.H. Ladinek and J. Žerovnik, *Perfect codes in direct graph bundles*, Inform. Process. Lett. **115** (2015), no. 9, 707–711.
https://doi.org/10.1016/j.ipl.2015.03.010.

[23] A.A. McRae, *Generalizing NP-Completeness Proofs for Bipartite Graphs and Chordal Graphs*, Clemson University, 1994.

[24] M. Milanič, *Hereditary efficiently dominatable graphs*, J. Graph Theory **73** (2013), no. 4, 400–424.
https://doi.org/10.1002/jgt.21685.

[25] M. Mollard, *On perfect codes in cartesian products of graphs*, European J. Combin. **32** (2011), no. 3, 398–403.
https://doi.org/10.1016/j.ejc.2010.11.007.

[26] A. Mukhopadhyay, *The square root of a graph*, J. Combin. Theory **2** (1967), no. 3, 290–295.
https://doi.org/10.1016/S0021-9800(67)80030-9.

[27] C.B. Smart and P.J. Slater, *Complexity results for closed neighborhood order parameters*, Congr. Numer. **112** (1995), 83–96.

[28] M. Sonntag and H.-M. Teichert, *Neighborhood structures and products of undirected graphs*, Discrete Math. **313** (2013), no. 4, 563–574.
https://doi.org/10.1016/j.disc.2012.11.028.

[29] T.K. Šumenjak, I. Peterin, D.F. Rall, and A. Tepeh, *Partitioning the vertex set of G to make $G\Box H$ an efficient open domination graph*, Discrete Math. Theor. Comput. Sci. **18** (2016), no. 3, #10.
https://doi.org/10.46298/dmtcs.1277.

[30] D.T. Taylor, *Perfect r-codes in lexicographic products of graphs*, Ars Combin. **93** (2009), 215–223.

[31] P.M. Weichsel, *The Kronecker product of graphs*, Proceedings of the American mathematical society **13** (1962), no. 1, 47–52.

[32] J. Žerovnik, *Perfect codes in direct products of cycles–A complete characterization*, Adv. Appl. Math. **41** (2008), no. 2, 197–205.
https://doi.org/10.1016/j.aam.2007.04.006.