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Research Article

Maker-Breaker total domination number

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Dedicated to Odile Favaron

Abstract: The Maker-Breaker total domination number, $\gamma_{\rm MBT}(G)$, of a graph G is introduced as the minimum number of moves of Dominator to win the Maker-Breaker total domination game, provided that he has a winning strategy and is the first to play. The Staller-start Maker-Breaker total domination number, $\gamma'_{\rm MBT}(G)$, is defined analogously for the game in which Staller starts. Upper and lower bounds on $\gamma_{\rm MBT}(G)$ and on $\gamma'_{\rm MBT}(G)$ are provided and demonstrated to be sharp. It is proved that for any pair of integers (k,ℓ) with $2 \le k \le \ell$, (i) there exists a connected graph G with $\gamma_{\rm MB}(G) = k$ and $\gamma_{\rm MBT}(G) = \ell$, (ii) there exists a connected graph G' with $\gamma'_{\rm MB}(G') = k$ and $\gamma'_{\rm MBT}(G') = \ell$, and (iii) there exists a connected graph G'' with $\gamma_{\rm MBT}(G'') = k$ and $\gamma'_{\rm MBT}(G'') = \ell$. Here, $\gamma_{\rm MB}$ and $\gamma'_{\rm MB}$ are corresponding invariants for the Maker-Breaker domination game. The (Staller-start) Maker-Breaker total domination number is determined for $2 \times n$ grid graphs. It is also demonstrated that the first player wins on the Cartesian product of P_3 by C_3, which gives a negative answer to a question from the seminal paper on the Maker-Breaker total domination game.

Keywords: Positional game, Maker–Breaker domination game, Maker–Breaker total domination game, Maker–Breaker total domination number.

AMS Subject Classification: 05C57, 05C69

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1. Introduction

Maker-Breaker game is a two-player game played on an arbitrary hypergraph by Maker and Breaker. The game belongs to the larger family of Maker-Breaker positional games introduced by Hales and Jewett [15], and later by Erdős and Selfridge [9]. The game has been the subject of extensive study, both in general settings and specific cases, as detailed in the book [16]. See also the recent paper [22] and references therein.

Duchêne, Gledel, Parreau, and Renault [8] introduced a variation of the Maker-Breaker game called the *Maker-Breaker domination game* (or *MBD game* for short). The *Maker-Breaker domination game* is played on a graph $G = (V(G), E(G))$ by two players named Dominator and Staller. These names align with the terminology used in the well-studied domination game [2, 3]. The players alternately select unplayed vertices of G. Dominator's goal is to occupy all vertices of some dominating set of G, while Staller prevents it from happening. Recent research on the game includes [1, 6].

The total domination game from [17] has received almost as much attention as the domination game, see for instance [19, 20, 23]. We would particularly like to highlight that recently Portier and Versteegen [21] proved the 3/4-conjecture posed in [18]. Motivated by the total domination game, the *Maker-Breaker total domination game* (*MBTD game* for short) was introduced in [13]. The game is played just like the Maker-Breaker domination game, except that Dominator aims to occupy all the vertices of a total dominating set of G. In [13], the outcome of the MBTD game was determined for cacti, and for Cartesian products of paths and cycles, while in [11] the focus was on cubic graphs. Let us also mention here a related game domination subdivision number studied by Favaron, Karami, and Sheikholeslami [10].

A D-game is a MBD game or a MBTD in which Dominator takes the first turn, otherwise we speak of an *S-game*. There are four graph invariants associated with the MBD game [4, 14]. The *Maker-Breaker domination number*, $\gamma_{\rm MB}(G)$, represents the minimum number of moves required for Dominator to win the D-game on G, assuming an optimal play by both players. Similarly, the *Staller-Maker-Breaker domination number*, $\gamma_{\rm SMB}(G)$, represents the minimum number of moves required for Staller to win the D-game. The parameters $\gamma'_{\rm MB}(G)$ and $\gamma'_{\rm SMB}(G)$ are defined for the S-game. The study in [4] focuses on $\gamma_{\rm SMB}(G)$ and $\gamma'_{\rm SMB}(G)$, providing, among other results, exact formulas for $\gamma_{\rm SMB}(G)$ when G is a path. In [5], trees T with $\gamma_{\rm SMB}(T) = k$ were characterized for every positive integer k, and exact formulas for $\gamma_{\rm SMB}(G)$ and $\gamma'_{\rm SMB}(G)$ were derived for caterpillars. The main result of [12] determined $\gamma_{\rm MB}$ and $\gamma'_{\rm MB}$ for Cartesian products of K_2 and a path. Additionally, in [7], the MBD game was further explored on Cartesian products of paths, stars, and complete bipartite graphs.

In this paper we introduce the Maker-Breaker total domination number $\gamma_{\rm MBT}(G)$ and the Staller-start Maker-Breaker total domination number $\gamma'_{\rm MBT}(G)$ of a graph G. In Section 2, we provide upper and lower bounds on $\gamma_{\rm MBT}$ and on $\gamma'_{\rm MBT}$, and demonstrate their sharpness. Then, in Section 3, we relate Maker-Breaker domination numbers with Maker-Breaker total domination numbers. We prove that for any pair of integers (k,ℓ) with $2 \le k \le \ell$, (i) there exists a connected graph G with $\gamma_{\rm MB}(G) = k$ and $\gamma_{\rm MBT}(G) = \ell$, (ii) there exists a connected graph G' with $\gamma'_{\rm MB}(G') = k$ and $\gamma'_{\rm MBT}(G') = \ell$, and (iii) there exists a connected graph G'' with $\gamma_{\rm MBT}(G'') = k$ and $\gamma'_{\rm MBT}(G'') = \ell$. In Section 4 we prove that $\gamma_{\rm MBT}(P_2 \,\square\, P_{2n}) = \gamma'_{\rm MBT}(P_2 \,\square\, P_{2n}) = 2n$ and that $\gamma_{\rm MBT}(P_2 \,\square\, P_{2n+1}) = \gamma'_{\rm MBT}(P_2 \,\square\, P_{2n+1}) = \infty$. This demonstrates a dichotomy with the Maker-Breaker domination game because parallel results for the latter game require a very long argument [12]. We also prove that $o(P_3 \,\square\, C_3) = \mathcal{N}$, thus providing a negative answer to the question from [13] whether it is true that $o(P_{2k+1} \,\square\, C_{2\ell+1}) = S$ for every $k, \ell \ge 1$.

In the rest of the section, additional definitions and concepts needed are given. For $k \in \mathbb{N}$ we will use the notation $[k] = \{1, \ldots, k\}$. Let $G = (V(G), E(G))$ be a graph. The order of G is represented as $n(G)$. A *dominating set* of a graph G is a subset $D \subseteq V(G)$ such that every vertex in $V(G) \setminus D$ has at least one neighbor in D, and D is a *total dominating set* if every vertex in $V(G)$ has a neighbor in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and the *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. A subset of pairs $\{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ of vertices in $V(G)$ is a *pairing total dominating set* if all the vertices are distinct and for any selection of vertices $x_i \in \{u_i, v_i\}$, $i \in [k]$, the set $\{x_1, \ldots, x_k\}$ is a total dominating set of G. This is closely related to the *pairing strategy* in general positional games, which asserts that if a partition of (a subset of) the board into pairs is possible such that each winning set contains one of the pairs, then a Breaker's win can be achieved easily, see [16, Remark 2.1.3].

The *outcome* $o(G)$ of the MBTD game played on G is

- $\mathcal{D}$, if Dominator has a winning strategy no matter who starts the game;
- $\mathcal{S}$, if Staller has a winning strategy no matter who starts the game;
- $\mathcal{N}$, if the first player has a winning strategy.

In an optimal strategy of Dominator (resp. Staller) in the MBTD game it is never an advantage for him (resp. for her) to skip a move. This fact is known as **No-Skip-Lemma** [13, Lemma 2.1].

The *Maker-Breaker total domination number*, $\gamma_{\rm MBT}(G)$, of a graph G is the minimum number of moves of Dominator to win the MBTD game, provided that he has a winning strategy and is the first to play. If Dominator has no winning strategy in the D-game, then set $\gamma_{\rm MBT}(G) = \infty$. The *Staller-start Maker-Breaker total domination number*, $\gamma'_{\rm MBT}(G)$, is defined analogously for the S-game.

2. Bounds on the MBTD numbers

In this section we demonstrate that straightforward upper and lower bounds on $\gamma_{\rm MBT}$ and on $\gamma'_{\rm MBT}$ are sharp. The result dealing with upper bounds reads as follows.

Theorem 1. *Let G be a graph.*

(i) If $\gamma_{\rm MBT}(G) < \infty$, then $\gamma_{\rm MBT}(G) \le \left\lceil \frac{n(G)}{2} \right\rceil$.

(ii) If $\gamma'_{\rm MBT}(G) < \infty$, then $\gamma'_{\rm MBT}(G) \le \left\lfloor \frac{n(G)}{2} \right\rfloor$.

Moreover, for any $k > 1$, there is a connected graph H with $\gamma_{\rm MBT}(H) = k = \left\lceil \frac{n(H)}{2} \right\rceil$, and there is a connected graph H' with $\gamma'_{\rm MBT}(H') = k = \left\lfloor \frac{n(H')}{2} \right\rfloor$.

Proof. Let G be a graph with $\gamma_{\rm MBT}(G) < \infty$. Since Dominator and Staller play alternately and Dominator starts the game, he can select at most $\left\lceil \frac{n(G)}{2} \right\rceil$ vertices, hence $\gamma_{\rm MBT}(G) \le \left\lceil \frac{n(G)}{2} \right\rceil$. Analogous argument applies to the inequality (ii).
To prove sharpness of the bounds we give four constructions based on the parity of k, and whether the D-game or the S-game is played.

Case 1: D-game, k even.
Let $k = 2\ell$ and let G_ℓ be the graph of order 4ℓ obtained from P_ℓ by respectively attaching a copy of C_4 to each of its vertices, see Fig. 1.

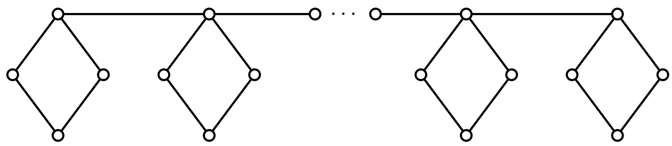

Figure 1. **Graph** G_ℓ

Since the vertex sets of the ℓ copies of C_4 partition $V(G_\ell)$ and since $o(C_4) = \mathcal{D}$, we can apply [13, Corollary 2.2(ii)] to conclude that $o(G_\ell) = \mathcal{D}$. It remains to verify that $\gamma_{\rm MBT}(G_\ell) = \left\lceil \frac{n(G_\ell)}{2} \right\rceil = 2\ell = k$. By the upper bound (i) we have $\gamma_{\rm MBT}(G_\ell) \le \left\lceil \frac{n(G)}{2} \right\rceil = 2\ell = k$. On the other hand, every total dominating set of G_ℓ necessarily contains at least two vertices from each of the ℓ 4-cycles, hence $\gamma_t(G_\ell) \ge 2\ell = k$. So $\gamma_t(G_\ell) = 2\ell = k$, hence $\gamma_{\rm MBT}(G_\ell) \ge k$, and we can conclude that $\gamma_{\rm MBT}(G_\ell) = k$.

Case 2: D-game, k odd.
Let $k = 2\ell + 1$, and let G'_ℓ be the graph constructed just as G_ℓ, except that at the first vertex of the base path the graph as shown in Fig. 2 is attached (and not C_4 as in the construction of G_ℓ).
Let Dominator start the game by selecting the vertex x, see Fig. 2. Then a simple analysis reveals that Dominator has a winning strategy. By (i),

$$\gamma_{\rm MBT}(G'_\ell) \le \left\lceil \frac{n(G'_\ell)}{2} \right\rceil = \left\lceil \frac{4\ell+2}{2} \right\rceil = 2\ell + 1 = k.$$

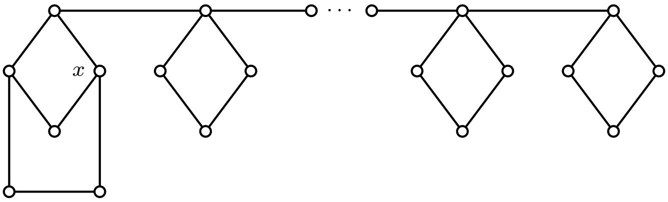

Figure 2. **Graph** G'_ℓ

On the other hand, $\gamma_t(G'_\ell) = 3 + 2(\ell - 1) = 2\ell + 1 = k$. Therefore, $\gamma_{\rm MBT}(G'_\ell) = k$.

Case 3: S-game, k even.
In this case, the graph G_ℓ from Case 1, where $k = 2\ell$, gives the required conclusion. The arguments are parallel, in particular, the bound (ii) is the same as the bound (i) because k is even.

Case 4: S-game, k odd.
Let $k = 2\ell + 1$, and let G''_ℓ be the graph constructed just as the graphs G_ℓ and G'_ℓ, except that at the first vertex of the base path the graph H as shown in Fig. 3 is attached.

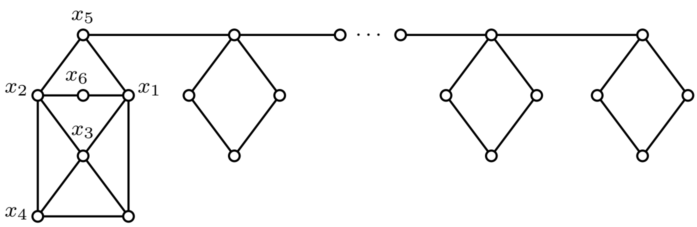

Figure 3. **Graph** G''_ℓ

To see that Dominator wins the S-game, note that as before, he has an easy control in the game in the 4-cycles. As for the subgraph H, assume that at some point of the game, possible the first move of it, Staller has selected the vertex x_1. Then Dominator is forced to play x_2. After that, Staller selects x_3 which in turn forces Dominator to select x_4. Now the only vertex in H which is not yet totally dominated is x_1. By selecting x_5 or x_6 in the next move, Dominator totally dominates H in three moves. We can argue similarly that Dominator can win if Staller selects any other vertex of H, notably x_3. Moreover, the above strategy of Staller implies that $\gamma'_{\rm MBT}(G''_\ell) \geq 3 + 2(\ell - 1)$, so that $\gamma'_{\rm MBT}(G''_\ell) \geq 2\ell + 1 = k$. On the other hand, (ii) yields

$$\gamma'_{\rm MBT}(G''_\ell) \leq \left\lfloor \frac{n(G''_\ell)}{2} \right\rfloor = \left\lfloor \frac{4\ell + 3}{2} \right\rfloor = 2\ell + 1 = k\,,$$

and we are done. □

We now turn our attention to lower bounds. Let G be a graph. A winning set for Dominator in a Maker-Breaker total domination game is a total dominating set, hence, $\gamma_t(G) \le \gamma_{\rm MBT}(G)$. Since a D-game can be viewed as an S-game in which Staller has skipped her first move, the No-Skip Lemma implies $\gamma_{\rm MBT}(G) \le \gamma'_{\rm MBT}(G)$. Moreover, since every total dominating set is a dominating set, we also infer that $\gamma_{\rm MB}(G) \le \gamma_{\rm MBT}(G)$. These observations can be summarized as follows:

$$\max\{\gamma_t(G), \gamma_{\rm MB}(G)\} \le \gamma_{\rm MBT}(G) \le \gamma'_{\rm MBT}(G)\,. \tag{2.1}$$

In view of (2.1), we now prove the following result which demonstrate that simultaneous equalities can hold in these inequalities.

Theorem 2. *If $k \ge 2$, then there exists a connected graph G with*

$$\gamma(G) = \gamma_t(G) = \gamma_{\rm MB}(G) = \gamma_{\rm MBT}(G) = \gamma'_{\rm MB}(G) = \gamma'_{\rm MBT}(G)\,.$$

Proof. (i) Let $G_{k,n}$, $k \ge 2$, $n \ge 4$, be the graph constructed in the following way. Take the disjoint union of K_n and $k-1$ copies of $K_n - e$. Then add two more vertices u and v, and add edges between each of u and v, and all the vertices of K_n, and between each of u and v, and each pair of non-adjacent vertices of each $K_n - e$. See Fig. 2 where $G_{k,4}$ is presented.

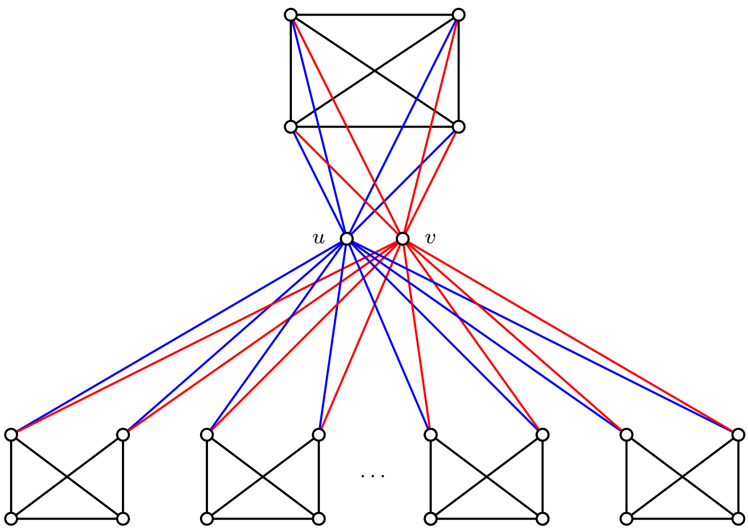

Figure 4. **Graph** $G_{k,4}$ **with** $\gamma(G_{k,4}) = \gamma_t(G_{k,4}) = \gamma_{\rm MB}(G_{k,4}) = \gamma_{\rm MBT}(G_{k,4}) = \gamma'_{\rm MB}(G_{k,4}) = \gamma'_{\rm MBT}(G_{k,4}) = k$

It is straightforward to see that $\gamma(G_{k,n}) = \gamma_t(G_{k,n}) = k$. In particular, the vertex u together with exactly one vertex of each of $K_n - e$ adjacent to u form a γ_t-set. Let

u_i, v_i, $i \in [k-1]$, be the two non-adjacent vertices of the i^{th} copy of $K_n - e$. Then no matter who starts the game, Dominator can follow the pairing strategy on the set of pairs $\{\{u,v\},\{u_1,v_1\},\ldots,\{u_{k-1},v_{k-1}\}\}$ to win the game in k moves. Since $\gamma_t(G_{k,n}) = k$, we get $\gamma_{\rm MBT}(G_{k,n}) = \gamma'_{\rm MBT}(G_{k,n}) = k$. But this in turn also implies that $\gamma_{\rm MB}(G_{k,n}) = \gamma'_{\rm MB}(G_{k,n}) = k$. □

3. Realization of parameters

In this section we relate Maker-Breaker domination numbers with Maker-Breaker total domination numbers and demonstrate that in all non-trivial cases the difference can be arbitrarily large.

If $\gamma_{\rm MB}(G) = 1$, then $\gamma(G) = 1$. Hence, if $n(G) \geq 3$, then $\gamma_{\rm MBT}(G) = 2$. Similarly, if $\gamma'_{\rm MB}(G) = 1$, then G contains at least two vertices of degree $n(G)-1$. Therefore, if $n(G) \geq 4$, then $\gamma'_{\rm MBT}(G) = 2$. In this section, we build on these observations and show that for any pair of integers (k,ℓ) with $2 \leq k \leq \ell$, there exists a connected graph G with $\gamma_{\rm MB}(G) = k$ and $\gamma_{\rm MBT}(G) = \ell$, as well as a connected graph H with $\gamma'_{\rm MB}(G) = k$ and $\gamma'_{\rm MBT}(G) = \ell$. Several different constructions are needed for this task because the case $k = 2$ behaves special.

Theorem 3. *For any two integers k and ℓ with $2 \leq k \leq \ell$ the following hold.*

(i) There exists a connected graph G with $\gamma_{\rm MB}(G) = k$ and $\gamma_{\rm MBT}(G) = \ell$.

(ii) There exists a connected graph H with $\gamma'_{\rm MB}(H) = k$ and $\gamma'_{\rm MBT}(H) = \ell$.

Proof. (i) Clearly, $\gamma(C_4) = \gamma_{\rm MB}(C_4) = 2$ and $\gamma_{\rm MBT}(C_4) = 2$. Assume in the rest that $\ell \geq 3$. We consider two cases.

Case 1: $k = 2$.
Let $G_{2,\ell}$, $\ell \geq 3$, be the graph constructed from the disjoint union of $\ell - 1$ triangles as follows. Let $v_{i,1}, v_{i,2}, v_{i,3}$ be the vertices of the i^{th} triangle, $i \in [\ell-1]$. Take two additional vertices u and v both adjacent to $v_{1,1}$. In addition, u is adjacent to $v_{i,j}$ for $i \in \{2,3,\ldots,\ell-1\}$ and $j \in [3]$. Also, v is adjacent to the vertices $v_{i,1}, v_{i,2}$ for $i \in \{2,3,\ldots,\ell-1\}$. See Fig. 5.

Consider a Maker-Breaker domination game on $G_{2,\ell}$. The sets $\{u, v_{1,1}\}$ and $\{u, v_{1,2}\}$ are dominating sets of $G_{2,\ell}$. Dominator first selects u, and then he can select either $v_{1,1}$ or $v_{1,2}$ as his second move. Therefore, Dominator can finish the game in two moves and hence $\gamma_{\rm MB}(G_{2,\ell}) = 2$ because $G_{2,\ell}$ has no vertex adjacent to all the other vertices.

Now we prove $\gamma_{\rm MBT}(G_{2,\ell}) = \ell$. To show that $\gamma_{\rm MBT}(G_{2,\ell}) \leq \ell$, we need to describe a strategy for Dominator in which he can finish the game in at most ℓ moves. Dominator starts by selecting $v_{1,1}$. Then Staller replies by selecting u, for otherwise Dominator can finish the game in two moves by playing u as his second move. Then Dominator selects v as his second move. The only remaining undominated vertices are

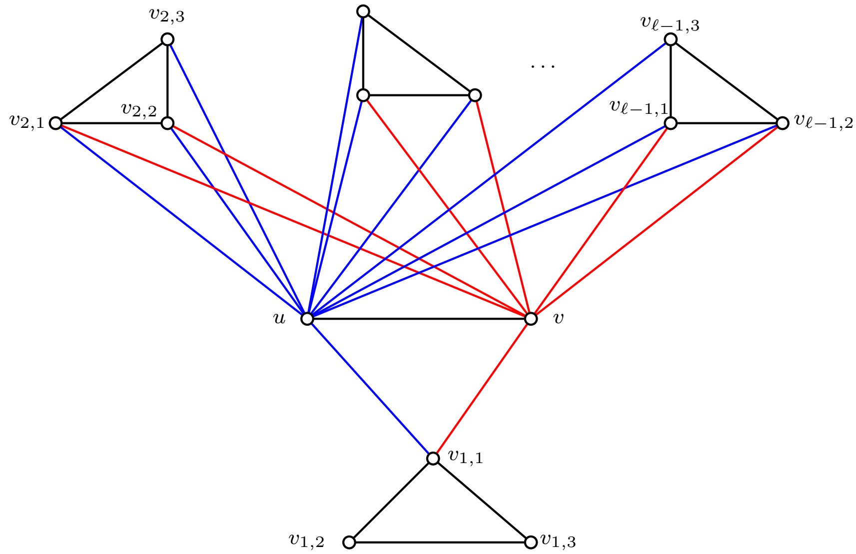

Figure 5. **Graph $G_{2,\ell}$ with $\gamma_{\rm MB}(G_{2,\ell}) = 2$ and $\gamma_{\rm MBT}(G_{2,\ell}) = \ell$**

$v_{2,3}, v_{3,3}, \ldots, v_{\ell-1,3}$. So in the continuation of the game, Dominator selects either $v_{i,1}$ or $v_{i,2}$ for $i \in \{2, 3, \ldots, \ell-1\}$ which he can clearly achieve. Hence Dominator can finish the game in at most $2 + \ell - 2 = \ell$ moves and so $\gamma_{\rm MBT}(G) \le \ell$.
To prove that $\gamma_{\rm MBT}(G_{2,\ell}) \ge \ell$, we describe a strategy of Staller. Suppose that Dominator selects a vertex other than $v_{1,j}$, $j \in [3]$. In this case, Staller selects $v_{1,1}$ as her response and wins the game by playing as her next move $v_{1,2}$ or $v_{1,3}$. Hence Dominator's first move is a vertex of the first triangle. If he selects $v_{1,2}$ or $v_{1,3}$, say $v_{1,2}$, then Staller selects $v_{1,1}$. In this case, Dominator is forced next to select $v_{1,3}$. Now Staller selects u as her second move which forces Dominator to play v, and at least $\ell - 2$ more moves. Finally, if Dominator first plays $v_{1,1}$, then Staller replies by selecting u as her first move, which forces Dominator to play at least $\ell-1$ more moves to finish the game (where v is his second move). Hence $\gamma_{\rm MBT}(G_{2,\ell}) \ge \ell$ and we get $\gamma_{\rm MBT}(G_{2,\ell}) = \ell$.

Case 2: $k \ge 3$.
Construct the graph $G_{k,\ell}$, $k \ge 3$, $\ell \ge 3$, as follows. Take $k+\ell-2$ triangles, where the $i^{\rm th}$ triangle, $i \in [k+\ell-2]$, has vertices $v_{i,j}$, $j \in [3]$. Then add two more vertices, u and v. The vertex u is adjacent to $v_{i,1}$, $i \in [k-1]$, and to $v_{i,j}$, $i \in \{k, k+1, \ldots, k+\ell-2\}$, $j \in [3]$. The vertex v is adjacent to $v_{i,j}$, $i \in [k-1]$, $j \in [3]$, and to $v_{i,j}$, $i \in \{k, k+1, \ldots, k+\ell-2\}$, $j \in [2]$. See Fig. 6.
Consider an MBD-game on $G_{k,\ell}$. First, we prove that Dominator has a strategy to finish the D-game on $G_{k,\ell}$ by at most k moves. Dominator selects the vertex u as his first move. If Staller selects a vertex other than v, then Dominator can finish the

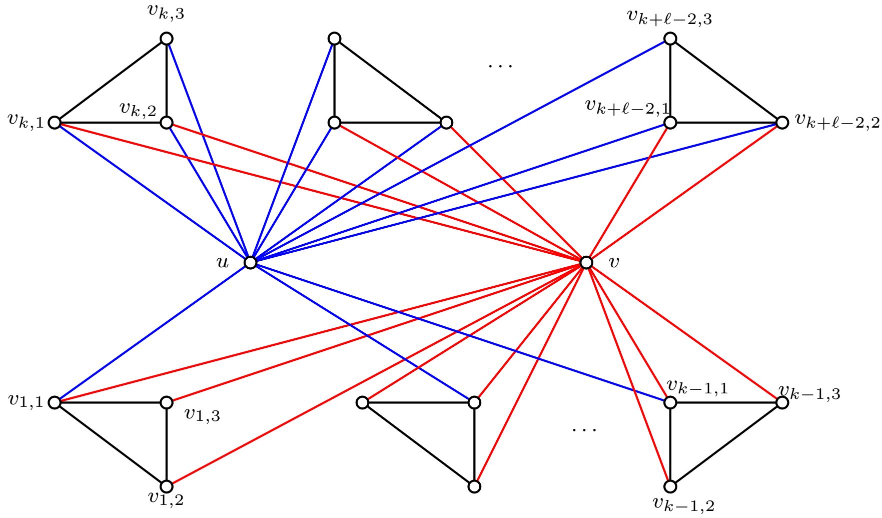

Figure 6. **Graph $G_{k,\ell}$ with $\gamma_{\mathrm{MB}}(G_{k,\ell}) = k$ and $\gamma_{\mathrm{MBT}}(G_{k,\ell}) = \ell$**

game by selecting v. Thus, Staller selects v as her optimal response. Then the only undominated vertices are in the first $k-1$ triangles. So Dominator can select one vertex from each of the $k-1$ triangles. Hence $\gamma_{\mathrm{MB}}(G_{k,\ell}) \le 1+k-1 = k$. Second, we prove that there exists a strategy for Staller which forces Dominator to play at least k moves. If Dominator selects a vertex other than u as his first move, then Staller selects u as her first response. In this case, Dominator needs at least $\ell-1$ more moves to dominate all the vertices of $G_{k,\ell}$, and we have $k \le \ell$. And if Dominator selects u as his first move, then Staller selects v as her response. So, Dominator must select at least one vertex from each of the first $k-1$ triangles. Hence, $\gamma_{\mathrm{MB}}(G_{k,\ell}) \ge 1+k-1 = k$. Therefore, $\gamma_{\mathrm{MB}}(G_{k,\ell}) = k$.

We next verify that $\gamma_{\mathrm{MBT}}(G_{k,\ell}) = \ell$. Consider the following strategy of Dominator. He selects v as his first move, and then the only undominated vertices are $v_{k,3}, v_{k+1,3}, \ldots, v_{k+\ell-2,3}$. Therefore, Dominator can finish the game by selecting either $v_{i,1}$ or $v_{i,2}$ for $i \in \{k, k+1, \ldots, k+\ell-2\}$. Clearly, the above set of vertices selected by Dominator becomes a total dominating set. Thus, $\gamma_{\mathrm{MBT}}(G_{k,\ell}) \le \ell$. On the other hand, if Dominator first selects a vertex other than u, then Staller selects u as her reply. Since any total dominating set without u contains at least ℓ vertices, we have $\gamma_{\mathrm{MBT}}(G_{k,\ell}) \ge \ell$. And if Dominator selects u as his first move, then Staller replies by the move v. If Dominator selects a vertex in the j^{th} triangle for some $j \in [k+\ell-2]$ as his second move, then Staller selects an unplayed vertex $v_{i,1}$ for $i \ne j$ and $i \in [k-1]$. Note that this is possible because $k \ge 3$. But then Staller wins the game by selecting either $v_{i,2}$ or $v_{i,3}$ after the next move of Dominator. We can conclude that $\gamma_{\mathrm{MBT}}(G_{k,\ell}) \ge \ell$ and therefore $\gamma_{\mathrm{MBT}}(G_{k,\ell}) = \ell$.

(ii) For given k and ℓ, let $G_{k,\ell}$ be the graph constructed in (i). Let $H_{k,\ell}$ be the graph obtained from $G_{k,\ell}$ by adding a universal vertex w to it. In the S-game, it is clearly optimal for Staller to select w in the first move, and then the game becomes the one from (i). □

Our second realization result reads as follows.

Theorem 4. *For any two integers k and ℓ with $2 \leq k \leq \ell$, there exists a connected graph G with $\gamma_{\rm MBT}(G) = k$ and $\gamma'_{\rm MBT}(G) = \ell$.*

Proof. Let $F_{k,\ell}$, $\ell \geq k \geq 2$, be the graph constructed in the following way. First, take $k+\ell-2$ disjoint triangles, where $v_{i,j}$, $j \in [3]$, are the vertices of the $i^{\rm th}$ triangle, $i \in [k+\ell-2]$. Next, take two more vertices u and v. The vertex u is adjacent to $v_{i,j}$, $i \in [k+\ell-2]$, $j \in [2]$, and to the vertices $v_{i,3}$, $i \in \{k, k+1, \ldots, k+\ell-2\}$. The vertex v is adjacent to $v_{i,j}$, $i \in [k+\ell-2]$, $j \in [2]$, and to $v_{i,3}$, $i \in [k-1]$. See Fig. 7.

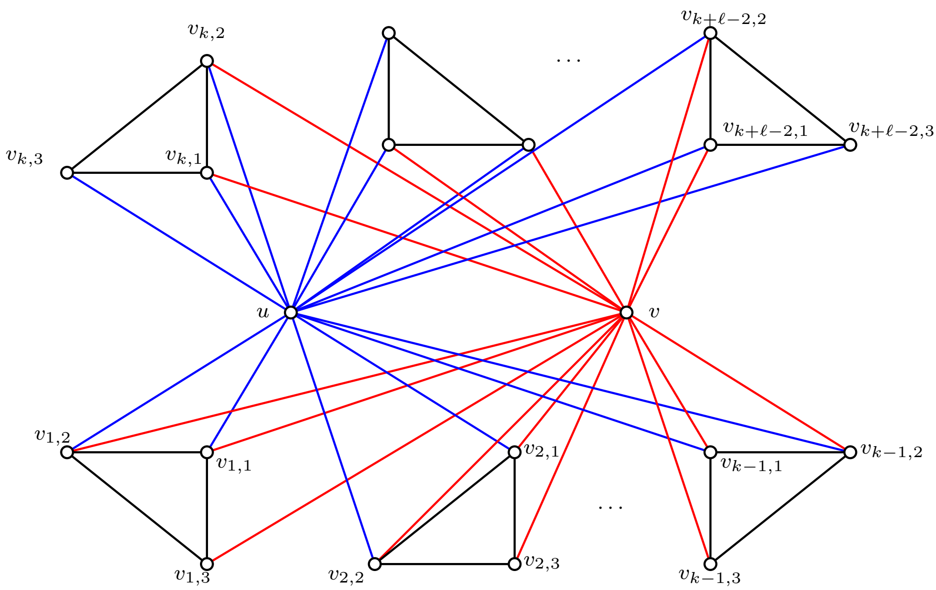

Figure 7. **Graph $F_{k,\ell}$ with $\gamma_{\rm MBT}(F_{k,\ell}) = k$ and $\gamma'_{\rm MBT}(F_{k,\ell}) = \ell$**

To show that $\gamma_{\rm MBT}(F_{k,\ell}) \leq k$, assume that Dominator selects u as his first move. If Staller selects a vertex other than v, then Dominator can finish the game by selecting v, and subsequently by playing an unplayed vertex $v_{i,1}$ for some i. Therefore, Staller is forced to select v first. Then Dominator can finish the game by selecting one of the vertices from $\{v_{i,1}, v_{i,2}\}$ for $i \in [k-1]$. Hence, Dominator can finish the game in at most $1 + k - 1 = k$ moves, hence $\gamma_{\rm MBT}(F_{k,\ell}) \leq k$.

To prove that $\gamma_{\rm MBT}(F_{k,\ell}) \geq k$, observe that any total dominating set of $F_{k,\ell}$ without the vertices u and v contains at least $2(k+\ell-2)$ vertices. Since $k \leq \ell$ and $k \geq 2$, we

have $2(k+\ell-2) > k$. Therefore, Dominator starts the game by playing either v or u. In the first case Staller's reply is u. Then the vertices $v_{i,3}$, $i \in \{k, k+1, \dots, k+\ell-2\}$, are undominated. Therefore, Dominator needs at least $\ell - 1$ more moves to finish the game. And if Dominator selects u as his first move, then Staller selects v as her response. In this case, the vertices $v_{i,3}$, $i \in [k-1]$, remain undominated, and Dominator needs at least $k-1$ more moves to finish the game. Hence $\gamma_{\rm MBT}(F_{k,\ell}) \geq 1 + k - 1 = k$ which implies that $\gamma_{\rm MBT}(F_{k,\ell}) = k$.
It remains to prove that $\gamma'_{\rm MBT}(F_{k,\ell}) = \ell$. Assume that Staller starts the game by selecting u. Note that any total dominating set without u and v contains at least $2(k+\ell-2)$ vertices. Since $k \geq 2$ and $k \leq \ell$, we have $2(k+\ell-2) > \ell$. Thus, Dominator must select v as his first move, for otherwise he needs at least $2(k+\ell-2) \geq \ell$ moves to obtain a total dominating set. Now, Dominator can win the game by selecting one vertex from $\{v_{i,1}, v_{i,2}\}$ for $i \in \{k, k+1, \dots, k+\ell-2\}$. Hence Dominator needs at least $\ell - 1$ more moves to get a total dominating set. Thus $\gamma'_{\rm MBT}(F_{k,\ell}) \geq \ell$. Finally, consider the following strategy of Dominator. Assume first that Staller selects a vertex from the $i^{\rm th}$ triangle, where $i \in [k-1]$. Then Dominator replies by the move v. This enables him to win the game by selecting one of $v_{i,1}$ and $v_{i,2}$ for $i \in \{k, k+1, \dots, k+l-2\}$. Assume second that Staller first selects a vertex from the $i^{\rm th}$ triangle, where $k \leq i \leq k+\ell-2$. In this case, Dominator selects u as his first move, and he can win the game by selecting one of $v_{i,1}$ and $v_{i,2}$ for $i \in [k-1]$. Thus, the game has at most $1 + k - 1 = k \leq \ell$ moves. Assume finally that Staller selects u or v as her first move. In this case Dominator selects the other of these two vertices in his first move. In both cases he can finish the game in at most ℓ moves. Thus $\gamma'_{\rm MBT}(F_{k,\ell}) \leq \ell$ and we can conclude that $\gamma'_{\rm MBT}(F_{k,\ell}) = \ell$. □

4. On the MBTD number of grids

In this section we consider the MBTD game played on grids, that is, Cartesian products of paths. Recall that if G and H are graphs, then their Cartesian product $G \,\square\, H$ has the vertex set $V(G) \times V(H)$, and a vertex (g,h) is adjacent to a vertex (g',h') if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. If $h \in V(H)$, then the subgraph of $G \,\square\, H$ induced by the vertices (g,h), $g \in V(G)$, is called a *G-layer* and denoted by G^h. A corresponding *H-layer* with respect to $g \in V(G)$ is denoted by ${}^gH$. In [13, Theorem 4.2] it is proved that if $n, m \geq 2$, then the outcome of the MBTD game played on $P_m \,\square\, P_n$ is $\mathcal{D}$ if both n and m are even, and it is $\mathcal{S}$ otherwise. If both n and m are even, then by partitioning $V(P_n \,\square\, P_m)$ into $\frac{mn}{4}$ sets of disjoint 4-cycles, Dominator can select in each of these cycles two adjacent vertices independent of the fact that who starts the game. Thus we have the following.

Proposition 1. *Let $n, m \geq 2$. If m and n are both even, then $\gamma_{\rm MBT}(P_n \,\square\, P_m) \leq \gamma'_{\rm MBT}(P_n \,\square\, P_m) \leq \frac{mn}{2}$. Otherwise, $\gamma_{\rm MBT}(P_n \,\square\, P_m) = \gamma'_{\rm MBT}(P_n \,\square\, P_m) = \infty$.*

The next result demonstrates that the upper bound of Proposition 1 is sharp.

Theorem 5. *If $n \geq 1$, then $\gamma_{\rm MBT}(P_2 \,\square\, P_{2n}) = \gamma'_{\rm MBT}(P_2 \,\square\, P_{2n}) = 2n$.*

Proof. Let $n \geq 2$, and set $G = P_2 \,\square\, P_{2n}$ for the rest of the proof. Setting also $V(P_k) = [k]$, $k \geq 2$, we thus have $V(G) = \{(i,j) : \ i \in [2], j \in [2n]\}$.
In view of Proposition 1 it suffices to prove that $\gamma_{\rm MBT}(G) \geq 2n$. Consider a D-game played G. Partition $V(G)$ into 4-cycles C_4^i, $i \in [n]$, induced by the vertices $(1, 2i-1)$, $(1, 2i)$, $(2, 2i-1)$, and $(2, 2i)$.
Assume that Dominator first selects the vertex $(1, 2i-1)$ in C_4^i. If $i = 1$, then we move to the next stage of the game as described in the next paragraph. Otherwise, Staller selects the vertex $(1,2)$. This move forces Dominator to play the vertex $(2,1)$ as his second move. Afterwards Staller selects $(2,2)$ which in turn forces Dominator to play the vertex $(1,1)$. In a similar manner, Staller and Dominator play in $C_4^2, \ldots, C_4^{i-1}$. That is, the vertices $(1,2j)$ and $(2,2j)$ of C_4^j, where $j \in [i-1]$, are selected by Staller, while the vertices $(1, 2j-1)$ and $(2, 2j-1)$ of C_4^j are selected by Dominator. Note that Dominator is the last player up to this stage.
After the above phase of the game is finished and if $i < n$, then Staller selects the vertex $(1, 2n-1)$ of C_4^n. This forces Dominator to select the vertex $(2, 2n)$ as his next move. Then Staller selects $(2, 2n-1)$ which forces Dominator to select $(1, 2n)$. Staller and Dominator play in the same pattern in each $C_4^n, \ldots C_4^{i+1}$. That is, the vertices $(1, 2j-1)$ and $(2, 2j-1)$ are selected by Staller and the vertices $(1,2j)$ and $(2,2j)$ by Dominator, where $j \in \{i+1, \ldots, n\}$.
When the above second phase of the game is finished, Dominator has selected exactly two vertices from each C_4^j, where $j \neq i$, and one vertex from C_4^i. Since the vertex $(1, 2i-1)$ is not yet totally dominated, Dominator needs at least one more move to finish the game on G. So, in total, he played $2n$ vertices.
The strategy of Staller for the cases when Dominator first selects some other vertex but $(1, 2i-1)$ in C_4^i is parallel. Hence we can conclude that $\gamma_{\rm MBT}(G) \geq 2n$. □

With respect to Theorem 5 we want to emphasize that the $\gamma_{\rm MB}(P_2 \,\square\, P_n)$ and $\gamma'_{\rm MB}(P_2 \,\square\, P_n)$ has been determined in [12]. In fact, these values are the main result of [12] which comprises over thirty pages.
In [13] it was posed as open problem whether it is true that $o(P_{2k+1} \,\square\, C_{2\ell+1}) = S$ for every $k, \ell \geq 1$. In the next result we prove that the answer is negative for $k = \ell = 1$.

Theorem 6. *$o(P_3 \,\square\, C_3) = \mathcal{N}$ and $\gamma_{\rm MBT}(P_3 \,\square\, C_3) = 4$.*

Proof. Set $X = P_3 \,\square\, C_3$ for the rest of the proof and let $V(P_3) = V(C_3) = [3]$, so that $V(X) = \{(i,j) : \ i, j \in [3]\}$.
Consider a D-game played on X and let Dominator's first move be the vertex $(2,2)$. If Staller selects $(1,1)$ or $(1,3)$ then Dominator selects $(1,2)$. After that move, all the vertices in layers ${}^1C_3$ and ${}^2C_3$ are totally dominated. Then Staller selects $(3,2)$, for otherwise Dominator wins the game by selecting $(3,2)$ at his third move. Now Dominator selects $(3,1)$ and wins the game by selecting either $(2,1)$ or $(3,3)$ as his fourth move.

Assume second that Staller first selects $(1,2)$ after the Dominator's move $(2,2)$. Dominator replies to this move by selecting the vertex $(2,1)$. Then Staller selects $(2,3)$, for otherwise Dominator wins the game by playing $(2,3)$ as his third move. Dominator replies by selecting $(1,1)$, and wins the game by playing either $(3,1)$ or $(3,2)$ as his fourth move.
By the symmetry, the only case left to be consider is when Staller selects $(2,1)$ after the Dominator's move $(2,2)$. In this case Dominator selects $(2,3)$ as his second move and can afterwards win the game by selecting one vertex from $\{(1,2),(1,3)\}$ and one vertex from $\{(3,2),(3,3)\}$.
We have thus proved that Dominator wins the D-game. Since it is proved in [13, Theorem 4.4] that Staller wins the S-game, we can conclude that $o(P_3 \,\Box\, C_3) = \mathcal{N}$.
From the above arguments we also infer that $\gamma_{\rm MBT}(P_3 \,\Box\, C_3) \le 4$. To establish the second assertion of the theorem it thus suffices to prove that Staller has a strategy which requires at least four moves of Dominator to win the game. For this sake note that $\gamma_t(X) = 3$ and that the only minimum total dominating sets of X are $V(P_3^1)$, $V(P_3^2)$, $V((P_3^3)$, and $V(^2C_3)$. Now, if Deminator selects two vertices from any of these four layers, then Staller can select the remaining vertex of the layer in her second move (provided she did not select it already as her first move). In this way, Staller forces Dominator to select at least four vertices. □

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