

## On Harary-Euler Sombor index of a graph

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**Abstract:** Let  $G$  be an arbitrary undirected simple connected graph. In this paper, we introduce the modification of the Harary index of  $G$  in which the contribution of each edge  $uv$  is weighted by  $d_u^2 + d_v^2 + d_u d_v$  - a term inspired by the geometry of ellipses - rather than a constant unit weight. Then we compute the values of the Harary-Euler Sombor index of some familiar classes of graphs. Also, we establish mathematical relations between the Harary-Euler Sombor index and other classic indices. Moreover, we state an upper bound for the Harary-Euler Sombor index of bipartite graphs. In addition, we state an upper bound for the Harary-Euler Sombor index of  $G$  in terms of the order of  $G$  and the largest (smallest) eigenvalue of the Harary-Euler Sombor matrix of  $G$  and we introduce a family of graphs for which the given bound is sharp. Finally, we determine the extremum values of the Harary-Euler Sombor index of trees.

**Keywords:** bipartite graph, eigenvalue, Harary-Euler Sombor index, Harary-Euler Sombor matrix, tree.

**AMS Subject classification:** 05C05, 05C09, 05C12, 05C50

### 1. Introduction

Let  $G = (V(G), E(G))$  be an undirected simple connected graph where  $V(G)$  and  $E(G)$  are the set of vertices and edges of  $G$ , respectively. The *order* and the *size* of  $G$  refers to the number of vertices and edges of  $G$ , respectively. Throughout this paper,  $d_v$  denotes the degree of vertex  $v$  in  $G$  and  $d(u, v)$  represents the distance between vertices  $u$  and  $v$  in  $G$ . An automorphism is a permutation of the vertices such that edges are mapped to edges and non-edges are mapped to non-edges. A graph  $G$  is *vertex-transitive* if, given any two vertices  $u$  and  $v$  of  $G$ , there is an automorphism  $f$  such that  $f(u) = v$ .

The most well-known of distance-based indices is the Harary index which is defined as follows:

$$H(G) = \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ u \neq v}} \frac{1}{d(u, v)}.$$

In [4] Alizadeh et al. introduced another new index that is called the *additively weighted Harary index* as follows:

$$H_A(G) = \frac{1}{2} \sum_{\substack{u,v \in V(G) \\ u \neq v}} \frac{d_u + d_v}{d(u, v)}.$$

In [5], An and Xiong introduced another index called the *multiplicatively weighted Harary index* by replacing the additive weighting of vertices with multiplicative weighting.

$$H_M(G) = \frac{1}{2} \sum_{\substack{u,v \in V(G) \\ u \neq v}} \frac{d_u \cdot d_v}{d(u, v)}.$$

Next, the authors [11] introduced the Harary-Sombor index as follows:

$$HSO(G) = \frac{1}{2} \sum_{\substack{u,v \in V(G) \\ u \neq v}} \frac{\sqrt{d_u^2 + d_v^2}}{d(u, v)}.$$

The concept of the Sombor index was introduced by Gutman in [9] in the chemical graph theory. The Sombor index of a graph  $G$  is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}.$$

The Sombor index gained huge attention of researchers. Recently, using these expressions, Tang et al. proposed a novel Sombor-type topological index, namely the Euler Sombor index [19]. This index was defined as follows

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + d_u d_v}.$$

Some results for the stated indices of chemical graphs can be found in [1–3, 6–8, 10, 12, 15–20], and the references therein.

In this paper, we introduce the new topological index of a graph  $G$  that defined as

$$\mathcal{HES}(G) = \frac{1}{2} \sum_{\substack{u,v \in V(G) \\ u \neq v}} \frac{\sqrt{d_u^2 + d_v^2 + d_u d_v}}{d(u, v)}$$

and called it *Harary-Euler Sombor index*. Next, we calculate  $\mathcal{HES}(G)$ , for some famous classes of graphs. Also, we state an upper bound for the Harary-Euler Sombor index of  $G$  in terms of  $|V(G)|$  and the largest (smallest) eigenvalue of the Harary-Euler Sombor matrix of  $G$ . Furthermore, we introduce a family of graphs for which the given bound is sharp. Finally, we determine the extremum values of the Harary-Euler Sombor index of trees.

## 2. Results

**Lemma 1.** *Let  $G$  be a graph of order  $n$ . We have:*

(i)  $EU(G) \leq \mathcal{HES}(G) < H_A(G)$  and the first inequality becomes an equality if and only if  $G \cong K_n$ .

(ii)  $\sqrt{3}\delta(G)H(G) \leq \mathcal{HES}(G) \leq \sqrt{3}\Delta(G)H(G)$ , where  $\delta(G)$  and  $\Delta(G)$  are the minimum and the maximum degrees of  $G$ , respectively. Each equality holds if and only if  $G$  is regular.

(iii)  $\frac{\sqrt{3}}{\Delta(G)}H_M(G) \leq \mathcal{HES}(G) \leq \frac{\sqrt{3}}{\delta(G)}H_M(G)$

*Proof.* (i) Clearly,  $\sqrt{d_u^2 + d_v^2 + d_u d_v} < d_u + d_v$  for any  $u, v \in V(G)$  and so  $\mathcal{HES}(G) < H_A(G)$ . Also,  $\mathcal{HES}(G) = EU(G) + \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ u \neq v, uv \notin E(G)}} \frac{\sqrt{d_u^2 + d_v^2 + d_u d_v}}{d(u, v)}$ .

Hence,  $\mathcal{HES}(G) \geq EU(G)$  and the equality holds if and only if  $G \cong K_n$ .

(ii) By considering  $\sqrt{3}\delta(G) \leq \sqrt{d_u^2 + d_v^2 + d_u d_v} \leq \sqrt{3}\Delta(G)$ , the proof of (ii) is straightforward.

(iii) Note that

$$\mathcal{HES}(G) = \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ u \neq v}} \frac{\sqrt{d_u^2 + d_v^2 + d_u d_v}}{d(u, v)} = \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ u \neq v}} \frac{d_u d_v}{d(u, v)} \sqrt{\frac{1}{d_v^2} + \frac{1}{d_u^2} + \frac{1}{d_u d_v}}$$

So, we obtain

$$\frac{\sqrt{3}}{\Delta(G)} \times \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ u \neq v}} \frac{d_u d_v}{d(u, v)} \leq \mathcal{HES}(G) \leq \frac{\sqrt{3}}{\delta(G)} \times \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ u \neq v}} \frac{d_u d_v}{d(u, v)}$$

and the proof is complete.  $\square$

Notice that if  $G$  is a graph of order  $n$  and size  $m$ , then

$$\mathcal{HES}(G) \leq EU(G) + \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ u \neq v, uv \notin E(G)}} \frac{\sqrt{3}\Delta(G)}{2} = EU(G) + \frac{\sqrt{3}}{2}\Delta(G)\left[\frac{1}{2}n(n-1) - m\right]$$

and equality holds if and only if either  $G \cong K_n$  or  $G$  is a regular graph with diameter 2. Also, we need to modify Lemma 1 of [11].

Recall that if  $G$  is a vertex-transitive graph, then  $G$  is regular. Also, for every vertices  $u$  and  $v$  of  $G$ , we have  $\mathcal{D}^{-1}(u) = \mathcal{D}^{-1}(v)$ , where

$$\mathcal{D}^{-1}(-) = \sum_{\substack{x \in V(G) \\ x \neq -}} \frac{1}{d(x, -)}.$$

Thus, we have the following.

**Lemma 2.** *Let  $G$  be a vertex-transitive graph  $G$  of size  $m$ . Then:*

$$\mathcal{HES}(G) = \sqrt{3}m\mathcal{D}^{-1}(a),$$

for any arbitrary vertex  $a \in V(G)$ .

*Proof.* Let  $a$  be an arbitrary vertex in  $V(G)$  and  $|V(G)| = n$ . If the valency of regularity of  $G$  is  $r$ , then

$$\begin{aligned} \mathcal{HES}(G) &= \frac{1}{2} \sum_{u \in V(G)} \sum_{\substack{x \in V(G) \\ x \neq u}} \frac{\sqrt{d_x^2 + d_u^2 + d_x d_u}}{d(x, u)} \\ &= \frac{1}{2} \sum_{u \in V(G)} \sqrt{3}r\mathcal{D}^{-1}(u) = \frac{\sqrt{3}}{2}rn\mathcal{D}^{-1}(a) \\ &= \sqrt{3}m\mathcal{D}^{-1}(a) \end{aligned}$$

and we are done. □

**Corollary 1.** *Let  $C_n$  be the cycle of order  $n$ . Then*

$$\mathcal{HES}(C_n) = \begin{cases} 2\sqrt{3}nH_{\frac{n-1}{2}} & n \text{ is odd} \\ 2\sqrt{3}(nH_{\frac{n}{2}} - 1) & n \text{ is even} \end{cases},$$

where the  $n$ -th harmonic number  $H_n$  is defined as the  $n$ -th partial sum of the harmonic series,  $\sum_{k=1}^n \frac{1}{k}$ .

*Proof.* Clearly,  $C_n$  is vertex-transitive. Let  $a \in V(C_n)$ . If  $n$  is odd, then we have  $\mathcal{D}^{-1}(a) = 2H_{\frac{n-1}{2}}$  by Lemma 2 and consequently  $\mathcal{HES}(C_n) = 2\sqrt{3}nH_{\frac{n-1}{2}}$ . If  $n$  is even, then  $\mathcal{D}^{-1}(a) = 2H_{\frac{n}{2}-1} + \frac{1}{\frac{n}{2}} = 2H_{\frac{n}{2}} - \frac{2}{n}$ . Hence  $\mathcal{HES}(C_n) = 2\sqrt{3}nH_{\frac{n}{2}} - 2\sqrt{3}$  and the proof is complete. □

Now, we compute the Harary-Euler Sombor index of complete bipartite graphs and state an upper bond for the Harary-Euler Sombor index of bipartite graphs.

**Theorem 1.** (i) *Let  $K_{p,q}$  be the complete bipartite graph of order  $n = p + q$ . Then*

$$\mathcal{HES}(K_{p,q}) = \frac{\sqrt{3}}{4}pq \left( \sqrt{\frac{16}{3}(p^2 + q^2 + pq) + n - 2} \right).$$

In particular  $\mathcal{HES}(S_n) = (n-1)\sqrt{n^2 - n + 1} + \frac{\sqrt{3}}{4}(n-1)(n-2)$ , where  $S_n$  is the star graph of order  $n$ .

(ii) If  $G$  is a bipartite graph of order  $n$ , then

$$\mathcal{HES}(G) \leq \begin{cases} \frac{\sqrt{3}}{16}n^2(3n-2) & n \text{ is even} \\ \frac{\sqrt{3}}{16}(n^2-1)(n-2+2\sqrt{n^2+\frac{1}{3}}) & n \text{ is odd} \end{cases},$$

with equality if and only if  $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ .

*Proof.* (i) The result can be obtained by direct calculation.

(ii) Let  $G = G(X, Y)$  be a bipartite graph of order  $n$  such that  $|X| \geq |Y|$ . If  $|X| = p$  and  $|Y| = q$ , then

$$\mathcal{HES}(G) \leq \mathcal{HES}(K_{p,q}) = \frac{\sqrt{3}}{4}p(n-p) \left( \sqrt{\frac{16}{3}(p^2-np+n^2)} + n-2 \right).$$

Suppose

$$f(x) := (nx - x^2) \left( n-2 + \sqrt{\frac{16}{3}(x^2-nx+n^2)} \right) \quad \text{for } \left[ \left\lceil \frac{n}{2} \right\rceil, n-1 \right].$$

Therefore

$$f'(x) \sqrt{\frac{16}{3}(x^2-nx+n^2)} = f_1(x) + f_2(x), \quad (*)$$

where

$$f_1(x) = (n-2)(n-2x) \sqrt{\frac{16}{3}(x^2-nx+n^2)}, \quad f_2(x) = -\frac{48}{3}x^3 + \frac{72}{3}nx^2 - \frac{56}{3}n^2x + \frac{16}{3}n^3.$$

We have

$$f'_1(x) \sqrt{\frac{16}{3}(x^2-nx+n^2)} = -2(n-2) \left( \frac{32}{3}x^2 - \frac{32}{3}nx + \frac{20}{3}n^2 \right) < 0.$$

It concludes that  $f'_1(x) < 0$  and consequently  $f_1(x)$  is decreasing. Also,

$$f'_2(x) = -48x^2 + 48nx - \frac{56}{3}n^2 < 0$$

and hence  $f_2(x)$  is decreasing. Hence  $f'(x) \sqrt{\frac{16}{3}(x^2-nx+n^2)}$  is decreasing, by Eq. (\*). On the other hand,  $\sqrt{\frac{16}{3}(x^2-nx+n^2)}$  is increasing on  $[\frac{n}{2}, \infty]$ . These imply that  $f'(x)$  are decreasing on  $[\lceil \frac{n}{2} \rceil, n-1]$ . Furthermore,  $f_1(\frac{n}{2}) = f_2(\frac{n}{2}) = 0$  and therefore

$f'(\frac{n}{2}) = 0$ , by Eq. (\*). It concludes that  $f'(x) < 0$  for any  $x \in (\lceil \frac{n}{2} \rceil, n-1]$  and so  $f(x)$  is decreasing on  $[\lceil \frac{n}{2} \rceil, n-1]$ . Thus maximum value of  $f(x)$  occurs at  $\lceil \frac{n}{2} \rceil$ . If  $n$  is even, then

$$\mathcal{HES}(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) = \frac{\sqrt{3}}{4} \frac{n}{2} (n - \frac{n}{2}) \left( \sqrt{\frac{16}{3} (\frac{n^2}{4} - \frac{n^2}{2} + n^2)} + n - 2 \right) = \frac{\sqrt{3}}{16} n^2 (3n - 2)$$

If  $n$  is odd, then

$$\begin{aligned} \mathcal{HES}(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) &= \frac{\sqrt{3}}{4} \frac{n^2 - 1}{4} \left( \sqrt{\frac{16}{3} (\frac{(n+1)^2}{4} - n \frac{n+1}{2} + n^2)} + n - 2 \right) \\ &= \frac{\sqrt{3}}{16} (n^2 - 1) (2\sqrt{n^2 + \frac{1}{3}} + n - 2) \end{aligned}$$

and we are done.  $\square$

Now, we define the *Harary-Euler Sombor matrix*  $A_{\mathcal{HES}(G)} = [h_{ij}]$  of a graph  $G$  with  $V(G) = \{v_1, \dots, v_n\}$  as an  $n \times n$  matrix with

$$h_{ij} = \frac{\sqrt{d_{v_i}^2 + d_{v_j}^2 + d_{v_i} d_{v_j}}}{d(v_i, v_j)}$$

if  $i \neq j$  and  $h_{ii} = 0$  for all  $i = 1, \dots, n$ . We denote its eigenvalues by  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ .

**Theorem 2 (Rayleigh-Ritz variational principle).** [13, 14] *Let  $X$  be a non-zero real  $n$ -vector and  $\mathcal{M}$  be a nonzero symmetric matrix. Then*

$$\mu_n \leq \frac{X^T \mathcal{M} X}{X^T X} \leq \mu_1,$$

where  $\mu_1$  and  $\mu_n$  are the largest and the smallest eigenvalues of  $\mathcal{M}$ , respectively. These inequalities become equalities if and only if  $X$  is an eigenvector corresponding to  $\mu_n$  and  $\mu_1$ , respectively.

**Theorem 3.** *Let  $G$  be a graph of order  $n$ . Then*

(i)  $\mu_1 \geq \frac{2}{n^2 - n} \mathcal{HES}(G)$ . Also, the equality holds if and only if  $G \cong K_2$  (the complete graph of order 2). Then  $\mu_1 = \mathcal{HES}(G) = \sqrt{3}$ .

(ii)  $\mu_n \leq \frac{2}{n - n^2} \mathcal{HES}(G)$ . Moreover, the equality holds if and only if  $G$  is the complete graph  $K_n$ . Then  $\mu_n = \sqrt{3}(1 - n)$  and  $\mathcal{HES}(G) = \frac{\sqrt{3}}{2} n(n-1)^2$ .

*Proof.* Let  $1 \leq r, s \leq n$  be arbitrary elements. Suppose  $X_{r,s}^T = [x_\ell]$  is an  $n$ -dimensional  $(0,1)$ -row-vector, where  $x_r = x_s = 1$  and  $x_\ell = 0$ , otherwise. By the Rayleigh–Ritz variational principle, we obtain

$$\mu_1 \geq \frac{X_{r,s}^T A_{\mathcal{HES}(G)} X_{r,s}}{X_{r,s}^T X_{r,s}}$$

and therefore

$$\mu_1 \geq h_{rs} = \frac{\sqrt{d_{v_r}^2 + d_{v_s}^2 + d_{v_r} d_{v_s}}}{d(v_r, v_s)}.$$

It concludes that

$$\binom{n}{2} \mu_1 \geq \mathcal{HES}(G) = \frac{1}{2} \sum_{\substack{v_i, v_j \in V(G) \\ v_i \neq v_j}} \frac{\sqrt{d_{v_i}^2 + d_{v_j}^2 + d_{v_i} d_{v_j}}}{d(v_i, v_j)}.$$

Also, equality will hold if each  $X_{r,s}$  is the eigenvector of  $A_{\mathcal{HES}(G)}$  corresponding to the eigenvalue  $\mu_1$ . Then by the Perron–Frobenius theorem, all components of  $X_{r,s}$  must be positive-valued. Therefore,  $X_{r,s}$  must be of dimension  $n = 2$ , and therefore it must be  $G \cong K_2$ . In this case,  $\mathcal{HES}(G) = \mu_1 = \sqrt{3}$ . Next, let  $Y_{r,s} = [y_\ell]$  is an  $n$ -dimensional row-vector with entries in  $\{-1, 0, 1\}$ , where  $y_r = -1$ ,  $y_s = 1$  and  $y_\ell = 0$ , otherwise. By the Rayleigh–Ritz variational principle, we obtain

$$\mu_n \leq \frac{Y_{r,s}^T A_{\mathcal{HES}(G)} Y_{r,s}}{Y_{r,s}^T Y_{r,s}}$$

and consequently  $\mu_n \leq -h_{rs}$ . This implies that

$$\binom{n}{2} \mu_n \leq -\mathcal{HES}(G) = -\frac{1}{2} \sum_{\substack{v_i, v_j \in V(G) \\ v_i \neq v_j}} \frac{\sqrt{d_{v_i}^2 + d_{v_j}^2 + d_{v_i} d_{v_j}}}{d(v_i, v_j)}.$$

Furthermore, equality will holds if and only if every  $Y_{i,j}$  is the eigenvector of  $A_{\mathcal{HES}(G)}$  corresponding to the eigenvalue  $\mu_n$ . It concludes that  $A_{\mathcal{HES}(G)} Y_{i,j} = \mu_n Y_{i,j}$  and consequently  $h_{ij} = -\mu_n$  for all  $i, j$  with  $i \neq j$ . Let  $u$  be an arbitrary vertex of  $G$  and  $v_1, v_2$  are two neighbors of  $u$ . So, we have  $\sqrt{d_u^2 + d_{v_1}^2 + d_u d_{v_1}} = \sqrt{d_u^2 + d_{v_2}^2 + d_u d_{v_2}}$  and hence  $d_{v_1}^2 + d_u d_{v_1} = d_{v_2}^2 + d_u d_{v_2}$ . If  $d_{v_1} \neq d_{v_2}$ , then  $d_{v_1} + d_{v_2} = -d_u$ , a contradiction. Thus  $d_{v_1} = d_{v_2}$ . Therefore, all neighbors of any arbitrary vertex of  $G$  are of the same degree. The connectivity of  $G$  causes this sameness to extend to all vertices of the graph. So,  $G$  is regular. Now, it is not hard to check that  $d(u, v) = 1$  for all  $u, v \in V(G)$ . These imply that  $G$  is a complete graph  $K_n$  and the proof is complete.  $\square$

Recall that, according to Theorem 1, we have

$$\mathcal{HES}(S_n) = (n-1)\sqrt{n^2 - n + 1} + \frac{\sqrt{3}}{4}(n-1)(n-2).$$

In the following, first we compute the Harary-Euler Sombor index of the path with  $n$  vertices  $P_n$ . Next, we prove that  $P_n$  ( $S_n$ ) has the minimum (maximum) value of the Harary-Euler Sombor index among all trees of order  $n$ .

**Example 1.** Let  $P_n = v_1 \cdots v_n$ . We have:

$$\begin{aligned} \mathcal{HES}(P_n) &= \frac{\sqrt{3}}{n-1} + \sqrt{7} \sum_{k=2}^{n-1} \frac{1}{d(v_1, v_k)} + \sqrt{7} \sum_{k=2}^{n-1} \frac{1}{d(v_n, v_k)} + 2\sqrt{3} \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} \frac{1}{d(v_i, v_j)} \\ &= \frac{\sqrt{3}}{n-1} + 2\sqrt{7}H_{n-2} + 2\sqrt{3} \sum_{k=1}^{n-3} H_k \\ &= \frac{\sqrt{3}}{n-1} + 2\sqrt{7}H_{n-2} + 2\sqrt{3}[(n-2)H_{n-3} - (n-3)] \\ &= \frac{\sqrt{3}}{n-1} + 2\sqrt{7}H_{n-2} + 2\sqrt{3}(n-2)H_{n-3} - 2\sqrt{3}(n-3). \end{aligned}$$

**Theorem 4.** Let  $T$  be an arbitrary tree of order  $n$ . Then  $\mathcal{HES}(T) \geq \mathcal{HES}(P_n)$  and the equality holds if and only if  $T = P_n$ .

*Proof.* We start the proof by showing that if  $a \geq 1$  is a real number, then

$$f(z) = \sqrt{z^2 + az + a^2} - \sqrt{(z-1)^2 + a(z-1) + a^2}$$

is a strictly increasing function on  $[0, \infty)$ . We have

$$f'(z) = \frac{2z + a}{2\sqrt{z^2 + az + a^2}} - \frac{2z - 2 + a}{2\sqrt{(z-1)^2 + a(z-1) + a^2}}.$$

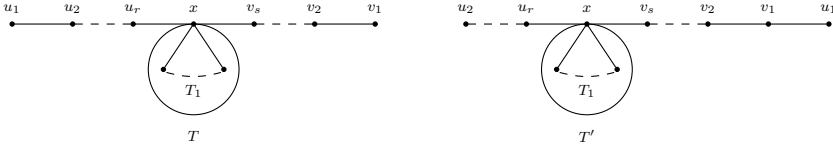
On the other hand, we have the following recursive inequalities:

$$\begin{aligned} (2z + a)\sqrt{(z-1)^2 + a(z-1) + a^2} &> (2z - 2 + a)\sqrt{z^2 + az + a^2} \\ \Leftrightarrow (2z + a)^2(z^2 + (a-2)z + (a^2 - a + 1)) &> (2z - 2 + a)^2(z^2 + az + a^2) \\ \Leftrightarrow -2a^2z + a^4 - a^3 + a^2 &> -8a^2z + a^4 - 4a^3 + 4a^2 \\ \Leftrightarrow 6a^2z + 3a^3 - 3a^2 &> 0 \end{aligned}$$

Thus  $f'(z) > 0$  and  $f$  is strictly increasing. It concludes that for any  $z \geq 3$

$$f(z) = \sqrt{z^2 + az + a^2} - \sqrt{(z-1)^2 + a(z-1) + a^2} > f(2) = \sqrt{a^2 + 2a + 4} - \sqrt{a^2 + a + 1}. \quad (\dagger)$$

Now, let  $T \neq P_n$  be a tree of order  $n$ . Using the symbols used in the proof of Theorem 2 of [11], as seen in Figure (1), there exists a vertex  $x \in V(T)$  with  $d_x \geq 3$  such that at least two components of  $T - x$  are paths. Suppose these paths are at lengths  $r - 1$  and  $s - 1$ . without loss of generality, one can assume that  $r \leq s$ . In fact  $T - x = (T_1 - x) \cup P_r \cup P_s$ .



**Figure 1.** The graphs  $T$  and  $T'$ .

First let  $r \geq 2$ . Due to the differences of the summands related to  $u_1, V(T_1)$ ;  $u_2, V(T_1)$ ; and  $v_1, V(T_1)$  in the calculation of  $\mathcal{HES}(T)$  and  $\mathcal{HES}(T')$  we have:

$$\begin{aligned} \mathcal{HES}(T') &= \mathcal{HES}(T) - \sum_{y \in V(T_1)} \frac{\sqrt{d_y^2 + d_y + 1}}{r + d(x, y)} + \sum_{y \in V(T_1)} \frac{\sqrt{d_y^2 + d_y + 1}}{s + 1 + d(x, y)} \\ &\quad - \sum_{y \in V(T_1)} \frac{\sqrt{d_y^2 + 2d_y + 4}}{r - 1 + d(x, y)} + \sum_{y \in V(T_1)} \frac{\sqrt{d_y^2 + d_y + 1}}{r - 1 + d(x, y)} \\ &\quad - \sum_{y \in V(T_1)} \frac{\sqrt{d_y^2 + d_y + 1}}{s + d(x, y)} + \sum_{y \in V(T_1)} \frac{\sqrt{d_y^2 + 2d_y + 4}}{s + d(x, y)} \end{aligned}$$

It concludes that

$$\begin{aligned} \mathcal{HES}(T') &< \mathcal{HES}(T) - \left( \frac{1}{r - 1 + d(x, y)} - \frac{1}{s + d(x, y)} \right) \times \sum_{y \in V(T_1)} (\sqrt{d_y^2 + 2d_y + 4} - \sqrt{d_y^2 + d_y + 1}) \\ &< \mathcal{HES}(T) \end{aligned}$$

If  $r = 1$ , then due to the differences of the summands related to  $u_1, V(T_1 - x)$ ;  $v_1, V(T_1 - x)$ ;  $x, V(T_1 - x)$ ;  $x, v_i$ 's ( $i = 2, \dots, s$ );  $x, u_1$ ; and  $x, v_1$  in the calculation of  $\mathcal{HES}(T)$  and  $\mathcal{HES}(T')$  we have:

$$\begin{aligned}
\mathcal{HES}(T') &= \mathcal{HES}(T) - \sum_{\substack{y \in V(T_1) \\ y \neq x}} \frac{\sqrt{d_y^2 + d_y + 1}}{1 + d(x, y)} + \sum_{\substack{y \in V(T_1) \\ y \neq x}} \frac{\sqrt{d_y^2 + d_y + 1}}{s + 1 + d(x, y)} \\
&\quad - \sum_{\substack{y \in V(T_1) \\ y \neq x}} \frac{\sqrt{d_y^2 + d_y + 1}}{s + d(x, y)} + \sum_{\substack{y \in V(T_1) \\ y \neq x}} \frac{\sqrt{d_y^2 + 2d_y + 4}}{s + d(x, y)} \\
&\quad - \sum_{\substack{y \in V(T_1) \\ y \neq x}} \frac{\sqrt{d_x^2 + d_y^2 + d_x d_y}}{d(x, y)} + \sum_{\substack{y \in V(T_1) \\ y \neq x}} \frac{\sqrt{(d_x - 1)^2 + d_y^2 + (d_x - 1)d_y}}{d(x, y)} \\
&\quad - \sum_{i=2}^s \frac{\sqrt{d_x^2 + 2d_x + 4}}{d(x, v_i)} + \sum_{i=2}^s \frac{\sqrt{(d_x - 1)^2 + 2(d_x - 1) + 4}}{d(x, v_i)} \\
&\quad - \sqrt{d_x^2 + d_x + 1} + \frac{\sqrt{(d_x - 1)^2 + (d_x - 1) + 1}}{s + 1} \\
&\quad - \frac{\sqrt{d_x^2 + d_x + 1}}{s} + \frac{\sqrt{(d_x - 1)^2 + 2(d_x - 1) + 4}}{s}.
\end{aligned}$$

On the other hand, for any  $y \in V(T_1 - x)$  we obtain:

$$\sqrt{d_x^2 + d_y^2 + d_x d_y} - \sqrt{(d_x - 1)^2 + d_y^2 + (d_x - 1)d_y} > \sqrt{d_y^2 + 2d_y + 4} - \sqrt{d_y^2 + d_y + 1},$$

(put  $z = d_x$  and  $a = d_y$  in Inequality (†))

and consequently

$$\frac{\sqrt{d_x^2 + d_y^2 + d_x d_y}}{d(x, y)} - \frac{\sqrt{(d_x - 1)^2 + d_y^2 + (d_x - 1)d_y}}{d(x, y)} > \frac{\sqrt{d_y^2 + 2d_y + 4}}{s + d(x, y)} - \frac{\sqrt{d_y^2 + d_y + 1}}{s + d(x, y)}.$$

Also,

$$\frac{\sqrt{d_x^2 + d_x + 1}}{s} - \frac{\sqrt{(d_x - 1)^2 + 2(d_x - 1) + 4}}{s} = \frac{\sqrt{d_x^2 + d_x + 1}}{s} - \frac{\sqrt{d_x^2 + 3}}{s} > 0.$$

These imply that  $\mathcal{HES}(T') < \mathcal{HES}(T)$ . Therefore, given that  $P_n$  can be achieved by repeating the above transformation, we have  $\mathcal{HES}(T) > \mathcal{HES}(P_n)$  and the proof is complete.  $\square$

**Theorem 5.** *Let  $T$  be an arbitrary tree of order  $n$ . Then  $\mathcal{HES}(T) \leq \mathcal{HES}(S_n)$  and the equality holds if and only if  $T = S_n$ .*

*Proof.* To prove the theorem we need the following lemmas.

**Claim 1.** If  $a \geq 1$  is a real number, then

$$f(z) = \sqrt{(z+a)^2 + (z+a) + 1} - \frac{1}{2}\sqrt{z^2 + z + 1}$$

is a strictly increasing function on  $[0, \infty)$ . In particular,

$$\sqrt{(z_0+a)^2 + (z_0+a) + 1} - \frac{1}{2}\sqrt{z_0^2 + z_0 + 1} > \sqrt{(1+a)^2 + (1+a) + 1} - \frac{\sqrt{3}}{2},$$

for any  $z_0 \geq 2$ .

**Proof of Claim 1.** We have

$$f'(z) = \frac{2z + 2a + 1}{2\sqrt{(z+a)^2 + (z+a) + 1}} - \frac{2z + 1}{4\sqrt{z^2 + z + 1}}.$$

Also, we have the following recursive inequalities:

$$\begin{aligned} (4z + 4a + 2)\sqrt{z^2 + z + 1} &> (2z + 1)\sqrt{(z+a)^2 + (z+a) + 1} \\ \Leftrightarrow (4z + 4a + 2)^2(z^2 + z + 1) &> (2z + 1)^2(z^2 + (2a + 1)z + (a^2 + a + 1)) \\ \Leftrightarrow 12z^4 + 24(a + 1)z^3 + 3(4a^2 + 12a + 9)z^2 &+ 3(4a^2 + 14a + 5)z + 3(5a^2 + 5a + 1) > 0 \end{aligned}$$

Thus  $f'(z) > 0$  and  $f$  is strictly increasing. As a result, for any  $z_0 \geq 2$  we have

$$f(z_0) = \sqrt{(z_0+a)^2 + (z_0+a) + 1} - \frac{1}{2}\sqrt{z_0^2 + z_0 + 1} > \sqrt{(1+a)^2 + (1+a) + 1} - \frac{\sqrt{3}}{2} = f(1),$$

and the claim is proved.

**Claim 2.** If  $a, b, c \geq 1$  are real numbers, then

$$f(z) = \frac{1}{c}\sqrt{(z+b)^2 + a(z+b) + a^2} - \frac{1}{c+1}\sqrt{z^2 + az + a^2}$$

is a strictly increasing function on  $[0, \infty)$ . In particular,

$$\begin{aligned} f(z_0) &= \frac{1}{c}\sqrt{(z_0+b)^2 + a(z_0+b) + a^2} - \frac{1}{c+1}\sqrt{z_0^2 + az_0 + a^2} \\ &> \frac{1}{c}\sqrt{(1+b)^2 + a(1+b) + a^2} - \frac{1}{c+1}\sqrt{1+a+a^2} = f(1), \end{aligned}$$

for any  $z_0 \geq 2$ .

**Proof of Claim 2.** The proof is similar to the proof of Claim 1.

**Claim 3.** If  $a, b \geq 1$  are real numbers, then

$$f(z) = \sqrt{(z+b)^2 + a(z+b) + a^2} - \sqrt{z^2 + az + a^2}$$

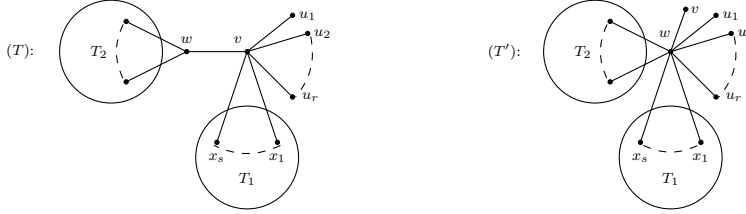
is a strictly increasing function on  $[0, \infty)$ . In particular,

$$\begin{aligned} f(z_0) &= \sqrt{(z_0 + b)^2 + a(z_0 + b) + a^2} - \sqrt{z_0^2 + az_0 + a^2} \\ &> \sqrt{(1 + b)^2 + a(1 + b) + a^2} - \sqrt{1 + a + a^2} = f(1), \end{aligned}$$

for any  $z_0 \geq 2$ .

**Proof of Claim 3.** The proof is similar to the proof of Claim 1.

Now, let  $T \neq S_n$  be a tree of order  $n$ . Using the symbols used in the proof of



**Figure 2.** The graphs  $T$  and  $T'$ .

Theorem 3 of [11], as seen in Figure (2), due to the differences of the summands related to  $v, u_i$ 's;  $w, u_i$ 's;  $v, w$ ;  $v, V(T_1)$ ;  $w, V(T_1)$ ;  $V(T_2), u_i$ 's;  $v, V(T_2)$ ;  $w, V(T_2)$ ; and  $V(T_1), V(T_2)$  in the calculation of  $\mathcal{HES}(T)$  and  $\mathcal{HES}(T')$  we have:

$$\begin{aligned} \mathcal{HES}(T') &= \mathcal{HES}(T) - r \left[ \sqrt{(r+s+1)^2 + (r+s+1) + 1} - \frac{\sqrt{3}}{2} \right] \\ &\quad - r \left[ \frac{\sqrt{d_w^2 + d_w + 1}}{2} - \sqrt{(d_w + r + s)^2 + (d_w + r + s) + 1} \right] \\ &\quad - \sqrt{d_w^2 + (r+s+1)d_w + (r+s+1)^2} + \sqrt{(d_w + r + s)^2 + (d_w + r + s) + 1} \\ &\quad - \sum_{x \in V(T_1)} \frac{\sqrt{d_x^2 + (r+s+1)d_x + (r+s+1)^2}}{d(x, v)} + \sum_{x \in V(T_1)} \frac{\sqrt{d_x^2 + d_x + 1}}{d(x, v) + 1} \\ &\quad - \sum_{x \in V(T_1)} \frac{\sqrt{d_x^2 + d_w^2 + d_x d_w}}{d(x, v) + 1} + \sum_{x \in V(T_1)} \frac{\sqrt{d_x^2 + (d_w + r + s)d_x + (d_w + r + s)^2}}{d(x, v)} \\ &\quad - r \left[ \sum_{y \in V(T_2)} \frac{\sqrt{d_y^2 + d_y + 1}}{d(y, w) + 2} - \sum_{y \in V(T_2)} \frac{\sqrt{d_y^2 + d_y + 1}}{d(y, w) + 1} \right] \\ &\quad - \sum_{y \in V(T_2)} \frac{\sqrt{d_y^2 + (r+s+1)d_y + (r+s+1)^2}}{d(y, w) + 1} + \sum_{y \in V(T_2)} \frac{\sqrt{d_y^2 + d_y + 1}}{d(y, w) + 1} \\ &\quad - \sum_{y \in V(T_2)} \frac{\sqrt{d_y^2 + d_w^2 + d_y d_w}}{d(y, w)} + \sum_{y \in V(T_2)} \frac{\sqrt{d_y^2 + (d_w + r + s)d_y + (d_w + r + s)^2}}{d(y, w)} \end{aligned}$$

$$- \sum_{x \in V(T_1), y \in V(T_2)} \frac{\sqrt{d_x^2 + d_y^2 + d_x d_y}}{d(x, v) + d(y, w) + 1} + \sum_{x \in V(T_1), y \in V(T_2)} \frac{\sqrt{d_x^2 + d_y^2 + d_x d_y}}{d(x, v) + d(y, w)}$$

According to Claim 1 (put  $z_0 = d_w$  and  $a = r + s$ ), we have

$$\sqrt{(d_w + r + s)^2 + (d_w + r + s) + 1} - \frac{\sqrt{d_w^2 + d_w + 1}}{2} > \sqrt{(r + s + 1)^2 + (r + s + 1) + 1} - \frac{\sqrt{3}}{2}.$$

Also,

$$\begin{aligned} & \sqrt{(d_w + r + s)^2 + (d_w + r + s) + 1} - \sqrt{d_w^2 + (r + s + 1)d_w + (r + s + 1)^2} \\ &= \sqrt{d_w^2 + (2r + 2s + 1)d_w + (r + s)^2 + (r + s + 1)} - \sqrt{d_w^2 + (r + s + 1)d_w + (r + s + 1)^2} > 0 \end{aligned}$$

For any  $x \in V(T_1)$ , according to Claim 2 (put  $z_0 = d_w$ ,  $a = d_x$ ,  $b = r + s$  and  $c = d(x, v)$ ), we have

$$\begin{aligned} & \frac{\sqrt{d_x^2 + (d_w + r + s)d_x + (d_w + r + s)^2}}{d(x, v)} - \frac{\sqrt{d_x^2 + d_w^2 + d_x d_w}}{d(x, v) + 1} \\ &> \frac{\sqrt{d_x^2 + (r + s + 1)d_x + (r + s + 1)^2}}{d(x, v)} - \frac{\sqrt{d_x^2 + d_x + 1}}{d(x, v) + 1} \end{aligned}$$

Finally, for any  $y \in V(T_2)$ , according to Claim 3 (put  $z_0 = d_w$ ,  $a = d_y$  and  $b = r + s$ ), we have

$$\begin{aligned} & \sqrt{d_y^2 + (d_w + r + s)d_y + (d_w + r + s)^2} - \sqrt{d_y^2 + d_w^2 + d_y d_w} \\ &> \sqrt{d_y^2 + (r + s + 1)d_y + (r + s + 1)^2} - \sqrt{d_y^2 + d_y + 1} \end{aligned}$$

and consequently

$$\begin{aligned} & \frac{\sqrt{d_y^2 + (d_w + r + s)d_y + (d_w + r + s)^2}}{d(y, w)} - \frac{\sqrt{d_y^2 + d_w^2 + d_y d_w}}{d(y, w)} \\ &> \frac{\sqrt{d_y^2 + (r + s + 1)d_y + (r + s + 1)^2}}{d(y, w) + 1} - \frac{\sqrt{d_y^2 + d_y + 1}}{d(y, w) + 1} \end{aligned}$$

These imply that  $\mathcal{HES}(T') > \mathcal{HES}(T)$ . Therefore, given that  $S_n$  can be achieved by repeating the above transformation, we have  $\mathcal{HES}(T) < \mathcal{HES}(S_n)$  and the result follows.  $\square$

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## Declarations

**Ethical Approval.** Not Applicable.

**Competing interests.** The author declare that there are no conflict of interests in the manuscript.

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