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Research Article

The geodesic-transversal problem on graphs of diameter at most three

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Abstract: In graph theory, finding the minimal set of vertices with specific covering properties is important. A *geodesic transversal* of a graph G is a set S of vertices such that every maximal geodesic of G includes at least one vertex from S. The minimum size of a geodesic transversal of G is called *geodesic transversal number*, denoted as $gt(G)$. It helps in understanding how graphs navigate, how efficiently they communicate, and how to monitor networks. This work finds $gt(G)$ for all graphs with a diameter of 2 and expands the analysis to various classes of graphs with a diameter of 3.

Keywords: geodesic transversal number, complementary prism, bipartite graph, split graphs.

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1. Introduction

Numerous classical problems in graph theory concentrate on identifying a minimum cardinality vertex collection that fulfills a particular coverage or structural property. Notable examples include the *Vertex Cover Problem* [20], which seeks the smallest set of vertices incident to all edges and the *Dominating Set Problem* [8] which requires the smallest set in which every vertex is either in the set or adjacent to a vertex in it. Another one is the *Metric Dimension Problem* [10, 31] which focuses on finding a

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minimum resolving set that can uniquely identify all vertices by their distance vectors. Next, the *Vertex Feedback Set Problem* [9, 20] aims to remove the fewest vertices to make the graph acyclic. Finally, we have the *Hitting Set Problem* [4, 20] which extends these ideas to hypergraphs by finding the smallest set that intersects every hyperedge. Each of these problems is NP-complete in general graphs, although polynomial-time algorithms can handle some restricted cases.

The *geodesic transversal problem* falls into a wider category of minimum vertex set problems in graph theory; it is an optimization problem that aims to find the smallest set of vertices that intersect every maximal geodesic of a graph G. The problem is NP-complete and was introduced independently in [15] and [25]. The geodesic transversal problem was then solved for trees and cactus graphs in [15], for the join and lexicographic product in [24], and for some networks in [17]. A *geodesic packing* of a graph G is the set of vertex-disjoint maximal geodesics. The maximum cardinality of vertex disjoint maximal geodesics in a graph is *geodesic packing number* of a graph. This concept was introduced in [16]. The geodesic packing number of trees and diagonal grids is solved in [16]. Very little work has been done on these topics. In this work, we determine the geodesic transversal number for all graphs of diameter two and for several classes of graphs with diameter three.

The *complementary prisms* of a graph is formed from the disjoint union of the graph and its complement by adding edges between the corresponding vertices of the graph and its complement [11]. For example, the complementary prism of C_5 is the Petersen graph. The topics studied on complimentary prisms include the chromatic index [39], domination [12, 29], cycle structure [19], complexity properties [7], spectral properties [5], convexity number [21], b-chromatic number [2] and general position problem [14, 18, 22, 23, 30, 38]. In this paper, we add to this list the geodesic transversal problem.

The *split graph* and the complement of *bipartite graphs* are examples of graphs with diameter less than or equal to 3. Many topics are studied on these topics which include general position problem [6, 22, 33–37], monophonic position problem [32], game domination number [13], dominating set [3], and many more. Within the extended settings of fuzzy graphs, researchers have recently examined domination [27], geodetic convexity [1], detour convexity [26], and geodetic spectral analysis [28].

The following Section 2 offers crucial definitions, observations, and notations that will be used in the rest of the paper. We establish a number of general results in Section 3 that apply to graphs with diameters of two and three. After that, we establish exact bounds for the problem within complementary prisms in Section 4. The problem is discussed for complement of bipartite graphs in Section 5, and the problem as it relates to split graphs is finally discussed in Section 6.

2. Notations and preliminaries

A *geodesic* in a graph G is the shortest path between two vertices in G. A geodesic is *maximal* if it is not a sub-path of a longer geodesic. A set S of vertices of G

is a *geodesic transversal set* of G, if every maximal geodesic of G contains at least one vertex of S. The *geodesic transversal number* of G, denoted by $gt(G)$, is the minimum cardinality of a geodesic transversal of G. A *geodesic packing* of a graph G is the set of vertex-disjoint maximal geodesics. The maximum cardinality of a geodesic packing is the *geodesic packing number* denoted by $gpack(G)$ and serves as a lower bound of $gt(G)$.

The graphs in this paper are finite, undirected, and simple. Let $G = (V(G), E(G))$ be a graph, and let $n(G)$ be the order of G. We will write $u \sim v$ if the vertices u and v are adjacent. A vertex set $X \subseteq V(G)$ is called *independent* if no two vertices of X are adjacent in G. $G[X]$ denotes he induced subgraph of G with vertex set $X \subseteq V(G)$. The maximum cardinality of an independent set in G is denoted by $\alpha(G)$. If an induced subgraph of graph G is complete, then it is called a *clique*. The maximum order of a clique in G is denoted by $\omega(G)$. The maximum order of disjoint union of cliques in graph G is denoted by $\omega_\alpha(G)$.

The *distance* $d_G(u, v)$ between vertices u and v is the length of a shortest path u, v-path. An u, v-path of minimum length is also called an $u, v-$*geodesic*. The *eccentricity* of vertex u is $ecc_G(u) = max\{d_G(u, v) : v \in V(G)\}$. The *radius* and the *diameter* of G are $rad(G) = min\{ecc_G(v) : v \in V(G)\}$ and $diam(G) = max\{ecc_G(v) : v \in V(G)\}$ respectively. The *open neighbourhood* $N(u)$ of $u \in V(G)$ is $\{v \in V(G) : u \sim v\}$, while the *closed neigbourhood* $N[u]$ is defined by $N[u] = N(u) \cup \{u\}$. The two vertices u and v are called *true twins* if $N[u] = N[v]$ and *false twins* if $N(u) = N(v)$.

A subset X of the set of vertices of a graph G is called *geodesic transversal* if it has a non-empty intersection with all the maximal geodesics. The cardinality of the smallest geodesic transversal in a graph G is denoted by $gt(G)$ and is called the *geodesic transversal number* of G. Every geodesic transversal of G of cardinality $gt(G)$ is called a *gt(G)-set*. In [15] it is shown that $1 \leq gt(G) \leq n(G) - 1$ and the bounds are sharp.

A set $S \subseteq V(G)$ is *k-geodesic transversal* if every geodesic on k vertices contains a vertex from S. The minimum cardinality of k-geodesic transversal is *k-geodesic transversal number* of G denoted by $gt_k(G)$. Another variation of the problem we get by covering all maximal geodesics up to the length $k-2$ (i.e., on $k-1$ vertices) and all geodesics of order k. The minimum cardinality of such a set will be denoted by $gt_{\leq k}(G)$. Usually we denote by $gt_{\leq k}(G)$-set a set of cardinality $gt_{\leq k}(G)$ that covers all geodesics on k vertices and all maximal geodesics on at most $k-1$ vertices.

3. Graphs with diameter two

In this section, we will discuss the geodesic transversal problem in diameter 2 and diameter 3 graphs. Initially, we will introduce the necessary and sufficient conditions for various maximal geodesics of various sizes in a graph. Later, we prove a general result for diameter 2 graphs and conclude the section by giving an upper bound for diameter 3 graphs.

Let G be a connected graph with $diam(G) \leq 3$. If $n \geq 2$, then possible maximal geodesics of G are paths P_2, P_3 and P_4. So, $gt(G) = gt_{\leq 4}(G)$.

Lemma 1. *Let G be a connected graph and $u, v \in V(G)$, then the edge uv is a maximal geodesic in G if and only if u and v are true twins.*

Proof. Let uv be a maximal geodesic in G. We have to show that u and v are true twins, if not without loss of generality, there exist $x \in N_G[u]$ such that $x \notin N_G[v]$. Then $x - u - v$ is a geodesic in G and is longer than uv. That is not possible, as uv is a maximal geodesic in G. So u and v are true twins. Conversely, if $N_G[u] = N_G[v]$ then it is clear that uv cannot be extended to a longer geodesic, which proves the result. □

Lemma 2. *Let G be a connected graph with $diam(G) \leq 3$. Then a path $P_4 = u_1 - u_2 - u_3 - u_4$ is a maximal geodesic in G if and only if $N_G[u_1] \cap N_G[u_4] = \emptyset$.*

Proof. Let $P_4 = u_1 - u_2 - u_3 - u_4$ be a maximal geodesic in G and assume $x \in N_G[u_1] \cap N_G[u_4]$. Then $u_1 - x - u_4$ is a shorter path than P_4, which is not possible. So, no such vertex x exists. In contrast, let $N_G[u_1] \cap N_G[u_4] = \emptyset$, then $d(u_1, u_4) \geq 3$. But G is a connected graph with diameter at most 3, so $d(u_1, u_4) = 3$ and therefore there exists a path of length 3 in G between u_1 and u_4, say $u_1 - u_2 - u_3 - u_4$ for some $u_2, u_3 \in V(G)$. Clearly, the path cannot be extended, so it is maximal geodesic. □

From, Lemma 2 we can conclude the following result.

Lemma 3. *Let G be a connected graph with $diam(G) \leq 3$. Then a path $P_3 = u_1 - u_2 - u_3$ is a maximal geodesic in G if and only if $N_G[x] \cap N_G[u_3] \neq \emptyset$, for all $x \in N_G[u_1]$ and $N_G[y] \cap N_G[u_1] \neq \emptyset$, for all $y \in N_G[u_3]$.*

To proceed further, we will partition the vertex set $V(G)$ of the graph. We find that being true twins is an equivalence relation, so we can partition $V(G)$ into equivalence classes.

$$TW = \{W_1, W_2, \ldots, W_k\}$$

We select a representative from each equivalence class to construct an induced subgraph, called the twin-free subgraph of G, denoted as $TF(G)$, with order $t(G)$. Two vertices from the same equivalence class create a maximal geodesic, so all vertices in $V(G) \setminus V(TF(G))$ must be in geodesic transversal set of G. So, for a graph of order n, $gt(G) \geq n - t(G)$. In addition, we can observe that $diam(TF(G)) = diam(G)$.

Theorem 1. *Let G be a connected graph, then no edge in $TF(G)$ is a maximal geodesic in $TF(G)$.*

Proof. Let edge uv be a maximal geodesic in $TF(G)$, then $N_{TF(G)}[u] = N_{TF(G)}[v]$. But as $u, v \in V(TF(G))$ we have $N_G[u] \neq N_G[v]$, so there exists $x \in N_G[u]$ such that $x \notin N_G[v]$. Then there exists a vertex $x_1 \in V(TF(G))$ such that $N_G[x_1] = N_G[x]$, so $x_1 \in N_{TF(G)}[u]$ and hence $x_1 \in N_{TF(G)}[v]$. Therefore, $v \in N_G[x_1] = N_G[x]$, which concludes $x \in N_G[v]$, which is not possible. So, no edge in $TF(G)$ is a maximal geodesic. □

So, by Theorem 1 it is clear that, for any connected graph G, all the vertices in $TF(G)$ have a distinct closed neighbourhood in $TF(G)$. So, the name twin-free for $TF(G)$ fits.

Theorem 2. *Every maximal geodesic in $TF(G)$ is a maximal geodesic in G.*

Proof. Let P be a maximal geodesic in $TF(G)$ between the vertices u and v. First, we have to show that P is a geodesic in G. If this is not the case, there exists a shorter path P' in G between vertices u and v. Then representatives of all vertices of path P' with same closed neighbourhood are still present in $TF(G)$, which can produce a shorter uv path in $TF(G)$. This contradicts the fact that P is a uv geodesic in $TF(G)$. So P is still a geodesic in G. Using the same argument, we can justify that P is a maximal geodesic in G. □

Hence, by Theorem 2 it is clear that $TF(G)$ produces no new maximal geodesics. Thus, it suffices to determine $gt(TF(G))$ in order to obtain the final value of $gt(G)$, which leads to the following result.

Corollary 1. *For any connected graph G of order n, $gt(G) = n - t(G) + gt(TF(G))$.*

From Corollary 1 it is clear that to find the geodesic transversal number of a connected graph G, we only need to find the geodesic transversal number of its twin-free subgraph. The next theorem looks at this and provides an interesting result.

Theorem 3. *Let G a connected graph with diameter 2. $X \subseteq V(TF(G))$ is a geodesic transversal of $TF(G)$ if and only if the graph induced by $V(TF(G)) \setminus X$ is a union of disjoint cliques.*

Proof. From Theorem 1, we know that $TF(G)$ has only maximal geodesics of length 2. Also, it is clear that $gt(TF(G)) = gt_3(TF(G))$. Let $X \subseteq V(TF(G))$ be a geodesic transversal set of $TF(G)$ and let G_1 be a component of the graph induced by $V(TF(G)) \setminus X$. If G_1 is not complete, then there exists a geodesic of length 2 in G_1, say $u - v - w$. Since $diam(G) = 2$, $u - v - w$ is a maximal geodesic that does not hit X, which is a contradiction. So, G_1 is complete.
Conversely, let the graph induced by $V(TF(G)) \setminus X$ be a union of disjoint cliques. Let $u - v - w$ be a maximal geodesic in $TF(G)$, if $u, v, w \notin X$ then $V(TF(G)) \setminus X$

will not be a union of disjoint cliques. So, one among u, v, w must be in X and hence X is a geodesic transversal set of $TF(G)$. □

So, finally we can come to a conclusion from Corollary 1 and Theorem 3 about the geodesic transversal number of two-diameter graphs.

Corollary 2. *Let G be a connected graph with diameter 2, then $gt(G) = n(G) - \omega_\alpha(TF(G))$.*

In addition, we can see that the value of $gt(G)$ of two-diameter graphs in Corollary 2 serves as an upper bound for all connected graphs.

Theorem 4. *For any connected graph G, $gt(G) \leq n(G) - \omega_\alpha(TF(G))$.*

Proof. Let G be a connected graph, then $TF(G)$ has no maximal geodesic of length one. Let X be the set of vertices that induce a disjoint union of cliques in $TF(G)$, then there is no maximal geodesic of $TF(G)$ in $G[X]$. So $V(TF(G)) \setminus X$ covers all maximal geodesics of $TF(G)$. So, $gt(TF(G)) \leq t(G) - \omega_\alpha(TF(G))$ and therefore

$$\begin{aligned} gt(G) &= n(G) - t(G) + gt(TF(G)) \\ &\leq n(G) - t(G) + t(G) - \omega_\alpha(TF(G)) \\ &= n(G) - \omega_\alpha(TF(G)) \end{aligned}$$

□

4. Complementary prisms

We now look for the geodesic transversal number of some graphs with diameter three, and one among them is the complementary prisms. In this section, we discuss the geodesic transversal problem in complementary prisms. First, we introduce the necessary and sufficient conditions for different maximal geodesics to be in complementary prisms. Finally, we conclude the section by strict bounds of geodesic transversal number of complementary prisms.

Let G be a graph, then we denote $G\overline{G}$ as its complementary prism. We will consider $V(G\overline{G})$ as the disjoint union of $V(G)$ and $V(\overline{G})$. We will use the convention that if $u \in V(G)$, then its unique neighbour in $V(G\overline{G}) \cap V(\overline{G})$ will be denoted by $\overline{u}$ and is called the *partner* of u in $\overline{G}$. If $S \subseteq V(G)$, we define $\overline{S} = \{\overline{u} \in V(\overline{G}) : u \in S\}$.

For all graphs G, $G\overline{G}$ is connected and $diam(G\overline{G}) \leq 3$. If $|V(G)| = 1$, then $G\overline{G} \cong P_2$ and its geodesic transversal number is one. But if $diam(G) = 2$, then $diam(G\overline{G}) = 2$. If $|V(G)| \geq 2$, then no edge in $G\overline{G}$ is a maximal geodesic. So, $G\overline{G}$ contains only maximal geodesics with 3 or 4 vertices. They are $M_1 : u - w - v$, $M_2 : \overline{u} - \overline{w} - \overline{v}$, $M_3 : u - v - \overline{v}$, $M_4 : u - \overline{u} - \overline{v}$, $M_5 : u - w - x - v$, $M_6 : \overline{u} - \overline{x} - \overline{w} - \overline{v}$, $M_7 : u - \overline{u} - \overline{v} - v$

and $M_8 : \overline{u} - u - v - \overline{v}$, where $u, v, w, x \in V(G)$. Moreover, all geodesics of length 3 in G and $\overline{G}$ are maximal in $G\overline{G}$.

Theorem 5. *Let G be a graph with $|V(G)| \geq 2$, then maximal geodesic of type M_5 or M_7 between vertices $u, v \in V(G)$ is in $G\overline{G}$ if and only if $N_G[u] \cap N_G[v] = \emptyset$.*

Proof. Let u, v be vertices of G and there be a maximal geodesic of type $M_5 : u - w - x - v$ between them in $G\overline{G}$, for some $w, x \in V(G)$. Then by Lemma 2, $N_{G\overline{G}}[u] \cap N_{G\overline{G}}[v] = \emptyset$. We can observe that $N_{G\overline{G}}[u] = N_G[u] \cup \{\overline{u}\}$ and $N_{G\overline{G}}[v] = N_G[v] \cup \{\overline{v}\}$. So, $N_G[u] \cap N_G[u] = \emptyset$. The same conclusion occurs if $G\overline{G}$ has a maximal geodesic of type $M_7 : u - \overline{u} - \overline{v} - v$. Now for the converse, let $N_G[u] \cap N_G[v] = \emptyset$, if there exist two adjacent vertices $w \in N_G[u]$ and $x \in N_G[v]$, then the geodesic $u - w - x - v$ is in $G\overline{G}$. If not, then the geodesic $u - \overline{u} - \overline{v} - v$ is in $G\overline{G}$. As $diam(G\overline{G}) \leq 3$ both geodesics are maximal and hence the result follows. □

Theorem 6. *Let G be a graph with $|V(G)| \geq 2$, then maximal geodesic of type M_6 or M_8 between vertices $\overline{u}, \overline{v} \in V(\overline{G})$ is in $G\overline{G}$ if and only if $N_G(u) \cup N_G(v) = V(G)$.*

Proof. Let u, v be vertices of G and there be a maximal geodesic of type $M_6 : \overline{u} - \overline{x} - \overline{w} - \overline{v}$ or $M_8 : \overline{u} - u - v - \overline{v}$ between $\overline{u}$ and $\overline{v}$ in $G\overline{G}$, for some $w, x \in V(G)$. We can observe that for the existence of geodesic M_6 or M_8, u and v must be adjacent in G. Clearly, $N_G(u) \cup N_G(v) \subseteq V(G)$. Let $x \in V(G)$ be arbitrary. If $x \notin N_G(u) \cup N_G(v)$, then $\overline{u} - \overline{x} - \overline{v}$ is a shorter path than M_6 and M_8, which is a contradiction since both are geodesics. Hence, $x \in N_G(u) \cup N_G(v)$ and so $V(G) = N_G(u) \cup N_G(v)$. Conversely, let $N_G(u) \cup N_G(v) = V(G)$, if there exist non-adjacent vertices $w \in N_G(u) \setminus N_G(v)$ and $x \in N_G(v) \setminus N_G(u)$, then $\overline{u} - \overline{x} - \overline{w} - \overline{v}$ is a geodesic in $G\overline{G}$. If not, $\overline{u} - u - v - \overline{v}$ is a geodesic in $G\overline{G}$. As $diam(G\overline{G}) \leq 3$ both geodesics are maximal and hence the result follows. □

We can observe that, non existence of maximal geodesics discussed in Theorem 5 and Theorem 6 will result in the existence of maximal geodesics of type, M_1 and M_2. Combining Theorem 5 and Theorem 6, we derive the subsequent results.

Corollary 3. *Let G be a graph with $|V(G)| \geq 2$, then maximal geodesic of type M_1 between $u, v \in V(G)$ is in $G\overline{G}$ if and only if $N_G[u] \cap N_G[x] \neq \emptyset$ and $N_G[v] \cap N_G[y] \neq \emptyset$ for all $x \in N_G[v]$ and $y \in N_G[u]$.*

Corollary 4. *Let G be a graph with $|V(G)| \geq 2$, then maximal geodesic of type M_2 between $\overline{u}, \overline{v} \in V(\overline{G})$ is in $G\overline{G}$ if and only if $N_G(u) \cup N_G(x) \neq |V(G)|$ and $N_G(v) \cup N_G(y) \neq V(G)$ for all $x \in N_G[v]^c$ and $y \in N_G[u]^c$.*

In addition, we can also infer about maximal geodesics of type M_3 and M_4 from Theorem 5 and Theorem 6 through the following results.

Corollary 5. *Let G be a graph with $|V(G)| \geq 2$, then maximal geodesic of type M_3 between $u \in V(G)$ and $\overline{v} \in V(\overline{G})$ is in $G\overline{G}$ if and only if $uv \in E(G)$ and $N_G(u) \cup N_G(v) \neq V(G)$.*

Proof. The geodesic, $M_3 : u - v - \overline{v}$ can only be extended to $\overline{u}$ from u to obtain a longer geodesic $\overline{u} - u - v - \overline{v}$. So, M_3 is a maximal geodesic if and only if $\overline{u} - u - v - \overline{v}$ is not a maximal geodesic, that is, if and only if $N_G(u) \cup N_G(v) \neq V(G)$ by Theorem 6. □

Corollary 6. *Let G be a graph with $|V(G)| \geq 2$, then maximal geodesic of type M_4 between $u \in V(G)$ and $\overline{v} \in V(\overline{G})$ is in $G\overline{G}$ if and only if $uv \notin E(G)$ and $N_G[u] \cap N_G[v] \neq \emptyset$.*

Proof. The geodesic, $M_4 : u - \overline{u} - \overline{v}$ can only be extended to v from $\overline{v}$ to obtain a longer geodesic $u - \overline{u} - \overline{v} - v$. So, M_4 is a maximal geodesic if and only if $u - \overline{u} - \overline{v} - v$ is not a maximal geodesic, that is, if and only if $N_G[u] \cap N_G[v] \neq \emptyset$ by Theorem 5. □

At this stage, we are equipped to establish the values of $gt(G\overline{G})$.

Theorem 7. *Let G be a graph of order n, then $gt(G\overline{G}) = n$ or $gt(G\overline{G}) = n - 1$.*

Proof. If $n = 1$ then $G\overline{G}$ is an edge, and $gt(G\overline{G}) = 1 = n$. Now we let $n \geq 2$. First, we prove that $gt(G\overline{G}) \geq n - 1$. We can divide the proof into two cases. For that, let $K \subseteq V(G\overline{G})$ be a geodesic transversal set of $G\overline{G}$ and $|K| \leq n - 2$, then there exists a pair of vertices $u, v \in V(G)$ such that $u, v, \overline{u}, \overline{v} \notin K$.
Case 1: Let $uv \in E(G)$, then $N_G(u) \cup N_G(v) = V(G)$ or $N_G(u) \cup N_G(v) \neq V(G)$. If $N_G(u) \cup N_G(v) = V(G)$, then $\overline{u} - u - v - \overline{v}$ is a maximal geodesic in $G\overline{G}$. If $N_G(u) \cup N_G(v) \neq V(G)$, then $u - v - \overline{v}$ is a maximal geodesic in $G\overline{G}$. However, both maximal geodesics do not hit K, which is a contradiction.
Case 2: Let $uv \notin E(G)$, then $N_G[u] \cap N_G[v] = \emptyset$ or $N_G[u] \cap N_G[v] \neq \emptyset$. If $N_G[u] \cap N_G[v] = \emptyset$, then $u - \overline{u} - \overline{v} - v$ is a maximal geodesic in $G\overline{G}$. If $N_G[u] \cap N_G[v] \neq \emptyset$, then $u - \overline{u} - \overline{v}$ is a maximal geodesic in $G\overline{G}$. However, both maximal geodesics do not hit K, which is a contradiction.
Hence K cannot be a geodesic transversal set of $G\overline{G}$, so $gt(G\overline{G}) \geq n - 1$.
On the other hand, let $u, v \in V(G)$ be arbitrary. Here we are committed to prove that $gt(G\overline{G}) \leq n$. Again, we divide the proof into two cases based on the adjacency of vertices u and v in G.
Case 1: Let $uv \in E(G)$, then $N_G(u) \cup N_G(v) = V(G)$ or $N_G(u) \cup N_G(v) \neq V(G)$. If $N_G(u) \cup N_G(v) = V(G)$, then $\overline{u} - u - v - \overline{v}$ and $\overline{u} - \overline{x} - \overline{w} - \overline{v}$, where $x, w \in V(G)$, are the only possible maximal geodesics on $G\overline{G}$. If $N_G(u) \cup N_G(v) \neq V(G)$, then $u - v - \overline{v}$, $\overline{u} - \overline{w} - \overline{v}$, and $v - u - \overline{u}$ are the only possible maximal geodesics on $G\overline{G}$.
Case 2: Let $uv \notin E(G)$, then $N_G[u] \cap N_G[v] = \emptyset$ or $N_G[u] \cap N_G[v] \neq \emptyset$. If $N_G[u] \cap N_G[v] = \emptyset$, then $u - \overline{u} - \overline{v} - v$ and $u - w - x - v$, where $x, w \in V(G)$, are the only possible maximal geodesics on $G\overline{G}$. If $N_G[u] \cap N_G[v] \neq \emptyset$, then $u - \overline{u} - \overline{v}$, $\overline{u} - \overline{v} - v$, and $u - w - v$ are the only possible maximal geodesics on $G\overline{G}$.

In both cases, we minimally require the set $\{u, \overline{v}\}$ to cover all maximal geodesics only involving the vertices $\{u, v, \overline{u}, \overline{v}\}$. Hence $gt(G\overline{G}) \leq n$. So, $gt(G\overline{G})$ takes only two values n and $n-1$. □

The Peterson graph is isomorphic to $C_5\overline{C}_5$ and so from [15], $gt(C_5\overline{C}_5) = 4$, achieving the lower bound. Moreover, $gt(C_7\overline{C}_7) = 7$ achieving the upper bound. So, the bounds are sharp, which concludes the result. Finally, from the above results it will be interesting to categorise a graph G of order n, based on the value of $gt(G\overline{G})$.

5. Complement of bipartite graphs

In this section, we examine the problem in the context of the complement of a bipartite graph. We begin by characterising all maximal geodesics in the complement and establishing the necessary and sufficient conditions for their existence. Subsequently, we conclude with a general formula for determining its geodesic transversal number. Let $G = (V(G), E(G))$ be a bipartite graph with partition $V(G) = P \cup Q$, then we write G as a triple $(P, Q, E(G))$. We take $|P| = p$ and $|Q| = q$, so $p + q = n(G)$. For a complete bipartite graph K_{pq}, the complement is isomorphic to $K_p \cup K_q$ and hence $gt(\overline{K}_{pq}) = p + q - 2$. The complement of a bipartite graph of order n with no edges is isomorphic to K_n, so its geodesic transversal number is $n - 1$.
Let B_{pq} be a non-complete bipartite graph of order $n = p+q$ with at least one edge and partite set $P = \{u_1, u_2, \ldots, u_p\}$ and $Q = \{v_1, v_2, \ldots, v_q\}$. In $\overline{B}_{pq}$ the partite sets P and Q cliques and $diam(\overline{B}_{pq}) \leq 3$. If $\overline{B}_{pq}$ is connected, the possible maximal geodesics are of the form $M_1 : u_i - u_j$, $M_2 : v_i - v_j$, $M_3 : u_i - v_j$, $M_4 : u_i - u_j - v_k$, $M_5 : v_i - v_j - u_k$ and $M_6 : u_i - u_j - v_k - v_l$.

Theorem 8. *Let $B_{pq} = (P, Q, E(B_{pq}))$ be a non-complete bipartite graph with at least one edge, then $G = \overline{B}_{pq}$ has a maximal geodesic of type M_6 between $u_i \in P$ and $v_l \in Q$ if and only if $N_G[u_i] = P$ and $N_G[v_l] = Q$.*

Proof. $G = \overline{B}_{pq}$ is a graph with $diam(G) = 3$, so by Lemma 2, $M_6 : u_i - u_j - v_k - v_l$, where $u_j \in P$ and $v_k \in Q$, is maximal geodesic in G if and only if $N_G[u_i] \cap N_G[v_l] = \emptyset$. But this happens when $N_G[u_i] = P$ and $N_G[v_l] = Q$. Hence, the result. □

From, Theorem 8 we deduce the following result.

Corollary 7. *Let $B_{pq} = (P, Q, E(B_{pq}))$ be a non-complete bipartite graph with at least one edge and $G = \overline{B}_{pq}$, then for $u_i, u_j \in P$ and $v_k \in Q$, geodesic $u_i - u_j - v_k$ is maximal if and only if $N_G[u_i] \neq P$ or $N_G[v] \neq Q$ for every $v \in Q$.*

A similar result can be obtained for the maximal geodesic of type M_5 in $\overline{B}_{pq}$.
The twin-free graph, $TF(\overline{B}_{pq})$, will not have maximal geodesics of types M_1, M_2, and M_3. But $TF(\overline{B}_{pq})$ will still be a complement of a non-complete bipartite graph

with at least one edge and partite set $P_t \subseteq P$ and $Q_t \subseteq Q$. Now we only need to find $gt(TF(\overline{B}_{pq}))$.

Theorem 9. *Let $B_{pq} = (P, Q, E(B_{pq}))$ be a non-complete bipartite graph with at least one edge and $G = TF(\overline{B}_{pq})$. Then $X \subseteq V(G)$ is a geodesic transversal set of G if and only if $V(G) \setminus X$ induces a disjoint union of one or two cliques.*

Proof. As G is the complement of a non-complete bipartite graph with at least one edge, it is clear that its induced subgraph can only have at most two cliques. We can divide this proof into two cases based on the existence of a maximal geodesic of type M_6 in $G = TF(\overline{B}_{pq})$.
Case 1: If a maximal geodesic of type M_6 does not exist in G, then G is a graph of diameter two. So, by Theorem 3, $X \subseteq V(G)$ is a geodesic transversal of G if and only if $V(G) \setminus X$ induces a disjoint union of one or two cliques.
Case 2: If a geodesic of type $M_6 : u - u_i - v_j - v$, where $u, u_i \in P_t$ and $v, v_j \in Q_t$, is in G, then by Theorem 8 $N_G[u] = P_t$ and $N_G[v] = Q_t$. Since G is twin-free, no vertices in G will have the same neighbourhood as u and v. We remove u from G to obtain the subgraph G_1 where $diam(G_1) = 2$. Then by Theorem 3 $X \subseteq V(G_1)$ is a geodesic transversal of G if and only if $V(G_1) \setminus X$ is a disjoint union of cliques. As $u_i - v_j - v$ is a maximal geodesic in G_1, X contains one among the vertices u_i, v_j or v. So, X hits M_6 as well, hence is a geodesic transversal of G and $V(G) \setminus X$ is still a disjoint union of cliques. □

So, from Corollary 1 and Theorem 9 we have the next result.

Corollary 8. *Let B_{pq} be a non-complete bipartite graph with at least one edge and $G = \overline{B}_{pq}$. Then $gt(G) = n(G) - \omega_\alpha(TF(G))$.*

Thus, the property of a graph G having diameter two constitutes a necessary condition for the geodesic transversal number to be $gt(G) = n(G) - \omega_\alpha(TF(G))$. However, this condition alone is not sufficient since the same value of the geodesic transversal number can also be achieved by graphs whose diameters exceed two. In particular, a notable example arises in the case of the complement of a bipartite graph, which demonstrates that graphs with larger diameters may also satisfy $gt(G) = n(G) - \omega_\alpha(TF(G))$.

6. Split graph

In the present section, our focus is on the study of the geodesic transversal problem in another important class of diameter three graphs, namely the split graphs. Following the approach used in the earlier sections, we begin our analysis by examining and characterising the maximal geodesics that arise in split graphs. This characterisation provides the basis for formulating a necessary and sufficient condition for their existence. Building on this result, we then establish a concluding property of the

geodesic transversal set in split graphs, which contributes to a more comprehensive understanding of the problem within this class of graphs.
A graph $G = (V(G), E(G))$ is a split graph if the vertex set $V(G)$ can be partitioned into two sets of vertices P and Q, where $G[P] \cong K_p$ and $G[Q] \cong \overline{K}_q$. The pair (P, Q) is called the *split partition* of G. For notational convenience, let $P = \{u_1, u_2, \dots, u_p\}$ and $Q = \{v_1, v_2, \dots, v_q\}$. Now, possible maximal geodesics of a connected split graph S_{pq} are of forms: $M_1 : u_i - u_j, M_2 : u_i - v_j, M_3 : u_i - u_j - v_k, M_4 : v_i - u_j - v_k$ and $M_5 : v_i - u_j - u_k - v_l$. We can observe that $TF(S_{pq})$ will still be a connected split graph with split partition (P_t, Q_t) and will not have maximal geodesics of the form M_1 and M_2.
On the basis of Lemma 2 and Lemma 3, the following two results can be established.

Lemma 4. *Let G be a connected split graph with split partition (P, Q), then G has a maximal geodesic of type M_5 between vertices $v_i, v_l \in Q$ if and only if $N_G[v_i] \cap N_G[v_l] = \emptyset$.*

Lemma 5. *Let G be a connected split graph with split partition (P, Q), then G has a maximal geodesic of type M_3 between vertices $u_i \in P$ and $v_k \in Q$ if and only if $N_G[x] \cap N_G[v_k] \neq \emptyset$ for every $x \in N_G[u_i] \cap Q$.*

In split graphs, M_3 and M_4 are types of maximal geodesics with 3 vertices. But maximal geodesics of type M_4 stands out from M_3 because of its properties mentioned in the next Lemma.

Lemma 6. *Let G be a connected split graph with split partition (P, Q). Then the following are equivalent:*
1) *G has a maximal geodesic of type M_4 between vertices $v_i, v_k \in Q$.*
2) *There exist vertices $v_i, v_k \in Q$ such that $N_G[v_i] \cap N_G[v_k] \neq \emptyset$.*
3) *There exist vertex $u_j \in P$ such that $|N_G[u_j] \cap Q| \geq 2$.*

Proof. First, we assume that $P : v_i - u_j - v_k$ is a maximal geodesic of type M_4, where $v_i, v_k \in Q$ and $u_j \in P$. Then clearly, $u_j \in N_G[v_i] \cap N_G[v_k] \neq \emptyset$. Next, let $v_i, v_k \in Q$ be vertices such that $N_G[v_i] \cap N_G[v_k] \neq \emptyset$. It is evident that neighbourhood of vertices from the set Q has only vertices from the set P. So, let $u_j \in N_G[v_i] \cap N_G[v_k]$, then $v_i, v_k \in N_G[u_j] \cap Q$ and hence $|N_G[u_j] \cap Q| \geq 2$. Finally, let $u_j \in P$ be such that $|N_G[u_j] \cap Q| \geq 2$. So, there exist distinct vertices $v_i, v_k \in N_G[u_j] \cap Q$ that form a geodesic $P : v_i - u_j - v_k$. The geodesic P can only be extended to a vertex in the set P, but it will not result in a longer geodesic. Hence, geodesic P is maximal and is of the type M_4. □

In order to establish the final result, we will introduce a new definition. Let K_n denote the complete graph with n vertices. Add a new vertex v in K_n and draw edges between v and any s, $0 \leq s \leq n$, number of vertices in K_n to obtain the graph called the $(s, n)-$*complete graph* denoted by K_n^s. Here, v is called the $s-$vertex. If $s = 0$, then $K_n^s \cong K_n \cup K_1$ and if $s = n$, then $K_n^s \cong K_{n+1}$.

Theorem 10. *Let S_{pq} be a connected split graph with split partition, (P, Q) and $G = TF(S_{pq})$. Let $X \subseteq V(G)$ be a geodesic transversal set of G, then the induced subgraph of X^c is a disjoint union of $(s, n)-$complete graphs. Moreover, for any $(s, n)-$ complete graph component in $G[X^c]$ if $1 \leq s \leq n-1$, then for every vertex u in the $(s, n)-$complete graph that is not adjacent to $s-$vertex v, there exists a vertex $x \in N_G[u] \cap Q_t$ such that $N_G[x] \subseteq N_G[v]^c$.*

Proof. Let G_1 be a connected component of $G[X^c]$. If G_1 is complete, we are done. If not, then G_1 contains the vertices of P_t and Q_t. If G_1 has more than one vertex from Q_t, then there will be a maximal geodesic of type M_4 or M_5 in $G[X^c]$, which is not possible. So, G_1 has only one vertex from Q_t, and therefore G_1 is a $(s, n)-$complete graph for some $n \geq 1$ and for some s, $0 \leq s \leq n$.
Now, let G_1 be a $(s, n)-$ complete graph component in $G[X^c]$ with $1 \leq s \leq n-1$. Let $v \in Q_t$ be the $s-$vertex in G_1 and $u \in P_t$ be a vertex in G_1 that is not adjacent to v in G_1. Then there exists a vertex u_1 in G_1 such that $u - u_1 - v$ is a geodesic. Since X is a geodesic transversal set of G this geodesic is not maximal and can be extended to some $x \in N_G[u] \cap Q_t$. For that, $N_G[x] \cap N_G[v] = \emptyset$, which implies $N_G[x] \subseteq N_G[v]^c$. □

Conclusion

From Corollary 2, we established a general result on the geodesic transversal problem for diameter two graphs. Subsequently, as observed in Corollary 8, certain diameter three graphs exhibit geodesic transversal numbers comparable to those of diameter two graphs, while others display distinct values, which is evident from Theorem 7 and Theorem 10. These findings naturally lead to an interest in determining a comprehensive generalisation for graphs of diameter three.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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