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Research Article

On maximizing private neighbors in graphs

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The authors dedicate this paper to Dr. Odile Favaron, in recognition of her impressive research in graph theory for so many years.

Abstract: Given a set $U \subset V$ of vertices in a graph $G = (V, E)$, a *private neighbor with respect to the set* U is any vertex $w \in V$ having precisely one neighbor, say v, in U. If $w \in V - U$, then w is called an *external private neighbor* of v with respect to U. If $w \in U$ then w is called an *internal private neighbor* of v with respect to U. We also add one special case: if $w \in U$ and $N(w) \cap U = \emptyset$, then we say that w is a *self private neighbor* with respect to U. By definition, a self private neighbor with respect to U is an isolated vertex in the subgraph of G induced by U. In this paper we consider the general problems of finding sets of vertices which maximize the number of private neighbors of specific types in a graph. In the process of doing this we define several new maximization parameters of graphs which generalize some known and well-studied parameters of graphs relating to vertex and edge independence, domination and irredundance in graphs.

Keywords: private neighbor, irredundance, domination.

AMS Subject Classification: 05C69

1. Introduction

Let $G = (V, E)$ be a finite, simple graph of order $n(G)$ and size $m(G)$. In general we follow the notation and definitions of [12]. We include here those used most often in this study. For $x \in V(G)$ the *open neighborhood* of x is the set $N_G(x)$ defined by $N_G(x) = \{w \in V : xw \in E\}$. Any $w \in N(x)$ is called a *neighbor* of x and we say that vertices w and x are *adjacent*. The *closed neighborhood* of x is the set $N_G[x]$ defined by $N_G[x] = N_G(x) \cup \{x\}$. Let $U \subseteq V$. The *open neighborhood* of U is the set $N_G(U) = \bigcup_{x \in U} N_G(x)$ and its *closed neighborhood* is the set $N_G[U] = N_G(U) \cup U$.

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We omit the subscripts from these notations when the graph G is clear from the context. The set U is a *dominating set* of G if $|N(v) \cap U| \geq 1$ for every vertex $v \in V - U$; equivalently, if $N[v] \cap U \neq \emptyset$ for every vertex $v \in V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G, while the *upper domination number*, $\Gamma(G)$, is the maximum cardinality of a minimal dominating set in G. A dominating set U is a *perfect* dominating set if $|N[x] \cap U| = 1$ for every $x \in V - U$ and is called an *efficient* dominating set if $|N[x] \cap U| = 1$ for every $x \in V$. If n is a positive integer, the set of positive integers not larger than n will be denoted $[n]$.
The Cartesian product, $G \,\square\, H$, of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ has vertex set $V_1 \times V_2$. Two vertices in $G \,\square\, H$ are adjacent if they are equal in one coordinate and adjacent in the other. To simplify notation, we let $G_{n,m} = P_n \,\square\, P_m$ denote the grid graph having m rows and n columns. Let p and q be positive integers. A *double star*, denoted $S(p,q)$, is the tree obtained from a path u, v of order 2 by adding p vertices adjacent to u and q vertices adjacent to v. The vertices u and v are called the *centers* of $S(p,q)$.

2. Private neighbors

In 1978, Cockayne, Hedetniemi and Miller [6] introduced the concept of private neighbors in graphs, as follows. Given a set $U \subset V$ of vertices in a graph G, a *private neighbor with respect to the set* U, is any vertex $w \in V$ having precisely one neighbor in U. That is, w is a private neighbor with respect to U if and only if $|N(w) \cap U| = 1$. If $w \in V - U$ and $N(w) \cap U = \{v\}$, then w is called an *external private neighbor* of v with respect to U. If $w \in U$ and $N(w) \cap U = \{v\}$, then w is called an *internal private neighbor* of v with respect to U. In other words, w is an internal private neighbor of $v \in U$ with respect to U if w is adjacent to v and w has degree 1 in the subgraph of G induced by U. In addition, we add one special case. If $w \in U$ and $N[w] \cap U = \{w\}$, then we say that w is a *self private neighbor* with respect to U. By definition, a self private neighbor with respect to U is an isolated vertex in the subgraph $G[U]$ induced by U. For each type of private neighbor defined here we will omit the phrase "with respect to U" if the set U is clear from the context.

Thus, with respect to any set $U \subset V$ in a graph G, we can associate three numbers $PN(U) = (S(U), I(U), E(U))$, where (i) $S(U)$ is the number of self private neighbors with respect to U, (ii) $I(U)$ is the number of internal private neighbors with respect to U, and (iii) $E(U)$ is the number of external private neighbors with respect to U. These three types of private neighbors, therefore, give rise to seven types of sets U depending on the types of private neighbors that every vertex in U must have. For sake of reference we will refer to these as being "irredundance-type" invariants.

1. Every vertex in U is a self private neighbor. Such sets are called *independent*, and the *vertex independence number*, $\alpha(G)$, is the maximum cardinality of an independent set in G.

2. Every vertex in U has an internal private neighbor. Such sets correspond to *strong matchings* in graphs, the induced subgraphs of which consist of disjoint unions of complete subgraphs of order 2. The *strong matching number*, $\alpha^*(G)$, equals the maximum cardinality of a strong matching in G.

3. Every vertex in U has an external private neighbor. Equivalently, $|N(v) - N[U - \{v\}]| \geq 1$ for every $v \in U$. Such sets are called *open irredundant sets*. The *upper open irredundance number*, $OIR(G)$, is the maximum cardinality of an open irredundant set in G.

4. Every vertex in U has an external private neighbor or is a self private neighbor. Equivalently, $|N[v] - N[U - \{v\}]| \geq 1$ for every $v \in U$. Such sets are called *irredundant*, and the *upper irredundance number*, $IR(G)$, is the maximum cardinality of an irredundant set in G.

5. Every vertex in U has an external private neighbor or an internal private neighbor. Equivalently, $|N(v) - N(U - \{v\})| \geq 1$ for every $v \in U$. Such sets are called *open-open irredundant sets*. The *upper open-open irredundance number*, $OOIR(G)$, equals the maximum cardinality of an open-open irredundant set in G.

6. Every vertex in U has an internal private neighbor or is a self private neighbor. Such sets are called 1-*dependent sets*, meaning that the maximum degree of the vertices in the subgraph induced by U is at most 1. The 1-*dependence number*, $\alpha_1(G)$, equals the maximum cardinality of a 1-dependent set in G.

7. Every vertex in U has at least one private neighbor of some kind, either external, internal or self. Equivalently, $|N[v] - N(U - \{v\})| \geq 1$ for every $v \in U$. Such sets are called *closed-open irredundant sets*. The *upper closed-open irredundance number*, $COIR(G)$, equals the maximum cardinality of a closed-open irredundant set in G.

The concept of irredundance is now well studied, there being more than 200 papers on this, a representative sample being the following [2, 4, 8, 10, 15, 17]. Cameron [3] first studied strong (or induced) matchings; Fink and Jacobson [11] introduced the concept of 1-dependence; Farley and Schacham [8] defined open irredundance, open-open irredundance and closed-open irredundance. See the papers by Fellows, Fricke, Hedetniemi and Jacobs [9] and Cockayne [4] that present all of these types of irredundance as they relate to each other.

3. Maximizing private neighbors

In this section we define graphical invariants arising from the three types of private neighbors defined in Section 2. Instead of focusing on the subsets of vertices each of which has some kind of private neighbor, in this new context we are interested in the set of private neighbors themselves. For each of the seven types of private neighbor combinations there correspondents a maximization parameter.

1. $SPN(G)$, the maximum number of self private neighbors with respect to a set $U \subseteq V$. Such a set U is called an $SPN(G)$-set. This is equivalent to the maximum number of vertices of degree zero in a subgraph $G[U]$ induced by a vertex set $U \subseteq V$.

2. $IPN(G)$, the maximum number of internal private neighbors with respect to a set $U \subseteq V$. Such a set U is called an $IPN(G)$-set. This is equivalent to the maximum number of vertices of degree one in the subgraph $G[U]$ induced by a vertex set $U \subseteq V$.

3. $EPN(G)$, the maximum number of external private neighbors with respect to a set $U \subseteq V$. Such a set U is called an $EPN(G)$-set.

4. $ESPN(G)$, the maximum number of external and self private neighbors with respect to a set $U \subseteq V$. Such a set U is called an $ESPN(G)$-set.

5. $EIPN(G)$, the maximum number of external and internal private neighbors with respect to a set $U \subseteq V$. Such a set U is called an $EIPN(G)$-set.

6. $ISPN(G)$, the maximum number of internal and self private neighbors with respect to a set $U \subseteq V$. Such a set U is called an $ISPN(G)$-set.

7. $EISPN(G)$, the maximum number of external, internal, and self private neighbors with respect to a set $U \subseteq V$. Such a set U is called an $EISPN(G)$-set.

It is easy to see from the definitions above that $SPN(G) = \alpha(G)$ and $ISPN(G) \geq \alpha_1(G)$. To our knowledge, the other five invariants have not been studied. These maximum private neighbors invariants, with the exception of SPN, are illustrated on $P_8 \square P_3$ in Figure 1. The black vertices in each case form a set that realizes the labeled parameter value.
By their very definitions these seven private neighbor parameters are naturally related to one another. The following inequalities follow directly from the definitions.

Proposition 1. *If G is any graph, then*

1. $SPN(G) \leq ESPN(G) \leq EISPN(G)$,
2. $SPN(G) \leq ISPN(G) \leq EISPN(G)$,
3. $IPN(G) \leq ISPN(G) \leq EISPN(G)$,
4. $IPN(G) \leq EIPN(G) \leq EISPN(G)$,
5. $EPN(G) \leq ESPN(G) \leq EISPN(G)$, *and*
6. $EPN(G) \leq EIPN(G) \leq EISPN(G)$.

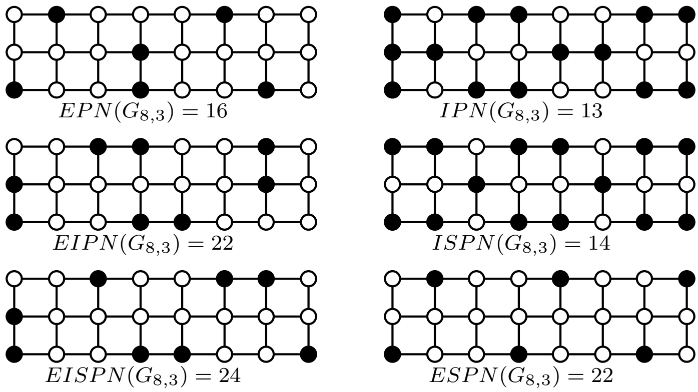

Figure 1. Maximizing private neighbors on $P_8 \,\square\, P_3$

The private neighbor maximization parameters we have defined on a graph G measure the largest subset of vertices that are the corresponding type(s) of private neighbors. The irredundance-type invariants are the maximum size of a subset of vertices, each of which has the specified type(s) of private neighbors. For example, if D is an open-open irredundant set in G with $|D| = OOIR(G)$ and $v \in D$, then v has an external private neighbor or an internal private neighbor with respect to D. It follows that $OOIR(G) = |D| \leq EIPN(G)$, which is inequality (5) in the following proposition. Justification of the other six inequalities is similar.

Proposition 2. *If G is any graph, then*

1. $\alpha(G) = SPN(G)$,
2. $2\alpha^*(G) \leq IPN(G)$,
3. $OIR(G) \leq EPN(G)$,
4. $IR(G) \leq ESPN(G)$,
5. $OOIR(G) \leq EIPN(G)$,
6. $\alpha_1(G) \leq ISPN(G)$,
7. $COIR(G) \leq EISPN(G)$.

We note that it is not the case that in every graph G there exists a strong matching M such that $2|M| = IPN(G)$. For example, the graph G in Figure 2 has $IPN(G) = 6$ and $2\alpha^*(G) = 4$. The white vertices form an $IPN(G)$-set, and it is easy to see that $\alpha^*(G) = 2$.

On the other hand, each of the other six maximum private neighbor invariants can be realized in any graph using a set of vertices that has the corresponding irredundance-type property expressed in Proposition 2. We hasten to add, however, that the set having the corresponding irredundance-type property may not be a maximum such

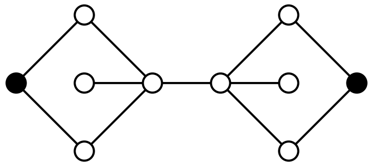

Figure 2. $IPN(G) = 6$ **and** $2\alpha^*(G) = 4$

set. For example, consider the relationship expressed in Proposition 4 below. It can be shown that $ESPN(K_{4,7}) = 9$ and $IR(K_{4,7}) = 7$, but the only irredundant sets U that have $E(U) + S(U) = 9$ have cardinality 2.

Proposition 3. *If A is an $EPN(G)$-set, then there exists an $EPN(G)$-set B, such that B is open irredundant and $B \subseteq A$.*

Proof. Suppose A is not open irredundant and $E(A) = EPN(G)$. Let $a \in A$ such that a does not have an external private neighbor with respect to A and let $A' = A - \{a\}$. It is clear that $E(A) \leq E(A')$, and since $E(A) = EPN(G)$ we have $E(A) = E(A')$. Therefore, by repeating this process, we conclude that there exists an $EPN(G)$-set B that is open irredundant and a subset of A. □

Proposition 4. *If A is an $ESPN(G)$-set, then there exists an $ESPN(G)$-set B, such that B is irredundant and $B \subseteq A$.*

Proof. Suppose A is not an irredundant set in G. Let $a \in A$ be such that $N[a] - N[A - \{a\}] = \emptyset$ (or equivalently $N[a] \subseteq N[A - \{a\}]$). It follows that a is not a self private neighbor with respect to A. Let $A' = A - \{a\}$. We claim that A' is an $ESPN(G)$-set. It is clear that $S(A') \geq S(A)$. Let w be an external private neighbor with respect to A. Then, there exists a unique $x \in A$ such that $\{x\} = N[w] \cap A$. Since $N[a] \subseteq N[A - \{a\}]$, it follows that $x \neq a$ and hence w is an external private neighbor with respect to A'. Therefore,

$$ESPN(G) = S(A) + E(A) \leq S(A') + E(A') \leq ESPN(G)\,.$$

By repeating this process, we conclude that there exists a $ESPN(G)$-set that is irredundant and a subset of A. □

Proposition 5. *If A is a $EIPN(G)$-set, then there exists a $EIPN(G)$-set B, such that B is open-open irredundant and $B \subseteq A$.*

Proof. Suppose that A is an $EIPN(G)$-set that is not open-open irredundant. That is, there exists $v \in A$ such that $N(v) \subseteq N(A - \{v\})$. Thus, if $v' \in N(v) \cap A$, then $|N(v') \cap A| \geq 2$. Similarly, if $v'' \in N(v) \cap (V(G) - A)$, then $|N(v'') \cap A| \geq 2$. Let

$A' = A - \{v\}$. Suppose x is an external private neighbor of A. That is, $x \notin A$ and $|N(x) \cap A| = 1$. Since v does not have an external private neighbor with respect to A, it follows that $|N(x) \cap A'| = 1$. Now, suppose that y is an internal private neighbor of A. This means that there exists $z \in A$ such that $N(y) \cap A = \{z\}$, which implies that $z \neq v$ since every neighbor of v in A has at least two neighbors in A. It follows that $N(y) \cap A' = \{z\}$ and hence y is an internal private neighbor with respect to A'. We infer that

$$EIPN(G) = E(A) + I(A) \leq E(A') + I(A') \leq EIPN(G)\,,$$

and it follows that A' is an $EIPN(G)$-set. Therefore, by repeating this process, we conclude that there exists an $EIPN(G)$-set B that is open-open irredundant and a subset of A. □

Proposition 6. *If A is a $ISPN(G)$-set, then there exists a $ISPN(G)$-set B, such that $\alpha_1(G) = |B|$ and $B \subseteq A$.*

Proof. Let A be an $ISPN(G)$-set that is not a 1-dependent set in G. That is, $\Delta(G[A]) \geq 2$. Let $a \in A$ such that a has at least two neighbors in $G[A]$, and let $A' = A - \{a\}$. If w is a self private neighbor with respect to A, then $w \neq a$ since $\deg_{G[A]}(a) \geq 2$. It follows that w is a self private neighbor with respect to A'. Now, suppose that w is an internal private neighbor with respect to A. That is, there exists $y \in A$ such that $N(w) \cap A = \{y\}$. If $y = a$, then w is a self private neighbor with respect to A'. On the other hand, if $y \neq a$, then w is an internal private neighbor with respect to A'. We infer that A and A' have the same number of self and internal private neighbors. Therefore, by repeating the process of removing vertices that have degree at least 2 in the subgraph induced by the $ISPN(G)$-set we arrive at a 1-dependent set that is also an $ISPN(G)$-set. □

Proposition 7. *If A is a $EISPN(G)$-set, then there exists a $EISPN(G)$-set B, such that B is closed-open irredundant and $B \subseteq A$.*

Proof. Suppose that A is an $EISPN(G)$-set that is not closed-open irredundant. This means that there exists $v \in A$ such that $N[v] \subseteq N(A - \{v\})$. It follows that v is not isolated in the subgraph $G[A]$ and $N(v) \subseteq N(A - \{v\})$. Let $A' = A - \{v\}$. As in the proof of Proposition 5 we see that $E(A) \leq E(A')$ and $I(A) \leq I(A')$. Also, since v is not isolated in $G[A]$, we have $S(A) \leq S(A')$. Therefore,

$$EISPN(G) = E(A) + I(A) + S(A) \leq E(A') + I(A') + S(A') \leq EISPN(G)\,.$$

It follows that A' is an $EISPN(G)$-set. Therefore, by repeating this process, we conclude that there exists an $EISPN(G)$-set B that is closed-open irredundant and a subset of A. □

4. Maximizing private neighbors in graph classes

In this section we consider some of the well-known classes of graphs and determine the maximum number of private neighbors they possess. The proofs of the following closed form formulas for paths, cycles and complete bipartite graphs are straightforward, although the proofs are often case studies. Since $SPN(G) = \alpha(G)$, the values for maximizing self private neighbors are omitted.

Proposition 8. *If $n \geq 2$ is a positive integer, then*

1. $IPN(P_n) = 2\lceil \frac{n-1}{3} \rceil$.
2. $EPN(P_n) = \begin{cases} 2\lfloor \frac{n}{3} \rfloor, & \text{if } n \not\equiv 2 \pmod 3; \\ 2\lfloor \frac{n}{3} \rfloor + 1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$
3. $EIPN(P_n) = \begin{cases} 4\lfloor \frac{n}{4} \rfloor, & \text{if } n \pmod 4 \in \{0,1\}; \\ n, & \text{if } n \pmod 4 \in \{2,3\}. \end{cases}$
4. $ESPN(P_n) = EISPN(P_n) = n$.
5. $ISPN(P_n) = \begin{cases} 2\lceil \frac{n-1}{3} \rceil + 1, & \text{if } n \equiv 1 \pmod 3; \\ 2\lceil \frac{n-1}{3} \rceil, & \text{if } n \not\equiv 1 \pmod 3. \end{cases}$

Proposition 9. *If $n \geq 3$ is a positive integer, then*

1. $IPN(C_n) = EPN(C_n) = 2\lfloor \frac{n}{3} \rfloor$.
2. $EIPN(C_n) = \begin{cases} 4\lceil \frac{n-2}{4} \rceil, & \text{if } n \not\equiv 3 \pmod 4; \\ n-1, & \text{if } n \equiv 3 \pmod 4. \end{cases}$
3. $ESPN(C_n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 3; \\ n-1, & \text{if } n \not\equiv 0 \pmod 3. \end{cases}$
4. $ISPN(C_n) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2 \pmod 3; \\ 2\lfloor \frac{n}{3} \rfloor, & \text{if } n \not\equiv 2 \pmod 3. \end{cases}$
5. $EISPN(C_n) = \begin{cases} 4, & \text{if } n = 5; \\ n, & \text{if } n \neq 5. \end{cases}$

Proposition 10. *If p and q are positive integers such that $p \leq q$, then*

- $IPN(K_{p,q}) = ISPN(K_{p,q}) = q$.
- $EPN(K_{p,q}) = \begin{cases} q, & \text{if } p = 1; \\ p+q-2, & \text{if } p \geq 2. \end{cases}$
- $EIPN(K_{p,q}) = EISPN(K_{p,q}) = p + q$.

- $ESPN(K_{p,q}) = \begin{cases} q+1, & \text{if } p = 1; \\ \max\{1+q, p+q-2\}, & \text{if } p \geq 2. \end{cases}$

The class of grid graphs are well-studied for some of the irredundance-type invariants. In the following tables we include computed values for the maximum number of private neighbors of some small grids. The resulting numbers suggest several conjectures or open problems, some of which we list in Section 8. We then prove the exact values of the maximum number of external or self private neighbors in $n \times 2$ grids.

Table 1. $IPN(P_n \,\square\, P_m)$

	2	3	4	5	6	7	8	9
2	2	4	4	6	6	8	8	10
3		4	7	8	10	12	13	15
4			8	10	12	14	16	18

Table 2. $EPN(P_n \,\square\, P_m)$

	2	3	4	5	6	7	8	9
2	2	4	5	7	8	10	11	13
3		6	8	10	12	15	16	19
4			12	14	17	20		

Table 3. $EIPN(P_n \,\square\, P_m)$

	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3		8	10	14	16	20	22	25
4			16	18	24	28		

Table 4. $ESPN(P_n \,\square\, P_m)$

	2	3	4	5	6	7	8	9
2	3	6	7	10	11	14	15	18
3		8	11	14	16	19	22	25
4			16	18	23	27		

Proposition 11. *If n is a positive integer such that $n \geq 2$, then*

$$ESPN(P_n \,\square\, P_2) = \begin{cases} 2n, & \text{if } n \text{ is odd;} \\ 2n-1, & \text{if } n \text{ is even.} \end{cases}$$

Table 5. $ISPN(P_n \square P_m)$

	2	3	4	5	6	7	8	9
2	2	4	4	6	6	8	8	10
3		5	7	9	10	12	14	15
4			8	11	12	15		

Table 6. $EISPN(P_n \square P_m)$

	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3		8	12	14	18	21	24	27
4			16	20	24	28		

Proof. Recall that $G_{n,2} = P_n \square P_2$. Let $V(P_n) = [n]$ with $E(P_n) = \{ij : i \in [n-1] \text{ and } j = i+1\}$, let $V(P_2) = [2]$, and let

$$U = \{(k,1) : k \equiv 1 \pmod 4 \text{ and } k \le n\} \cup \{(k,2) : k \equiv 3 \pmod 4 \text{ and } k \le n\}.$$

If n is odd, a straightforward check shows that the set U is an efficient dominating set of $G_{n,2}$. Therefore, $ESPN(P_n \square P_2) = 2n$. Now, assume that n is even, say $n = 2r$. It is known that $\gamma(G_{2r,2}) = r+1$; see Theorem 17.1 in [12]. One can easily check that $S(U) = r$ and $E(U) = 3r - 1$, which implies that $ESPN(G_{n,2}) \ge 2n - 1$. We claim that $P_n \square P_2$ does not have an efficient dominating set. Suppose to the contrary that $D = \{v_1, \dots, v_k\}$ is such a dominating set. It follows that $4r = p_1 + p_2 + \cdots + p_k$, where $p_i = |N[v_i]| \in \{3,4\}$ for each $i \in [k]$. Since $G_{n,2}$ has only four vertices of degree 2, analysis shows that $4r = p_1 + p_2 + \cdots + p_k$ is possible only if $|\{i \in [k] : p_i = 3\}| = 4$. This implies that $\{(1,1),(1,2),(n,1),(n,2)\} \subseteq D$, which is a contradiction. □

5. Generalized efficient graphs

Suppose a graph G has an efficient dominating set D. In other contexts an efficient dominating set is called a *perfect code*. The graph G is then said to be *efficient*. By definition, each vertex in D is a self private neighbor with respect to D (that is, $S(D) = |D|$), and every vertex in $V - D$ is an external private neighbor with respect to D (that is, $E(D) = n(G) - |D|$). Therefore, $ESPN(G) = S(D) + E(D) = n(G)$. Let us say that such a graph G is *ES-efficient*. The converse is also true.

Proposition 12. *A graph G is efficient if and only if $ESPN(G) = n(G)$.*

Suppose G is an efficient graph and let $D = \{v_1, v_2, \ldots, v_k\} \subset V(G)$ be an efficient dominating set in G. It follows that $\pi = \{N[v_1], N[v_2], \ldots, N[v_k]\}$ is a partition of $V(G)$. Not every graph G is ES-efficient. Two simple examples are the cycles C_4 and C_5, although it is easy to see that all cycles and paths having order congruent to zero modulo 3 are ES-efficient.

In [13] Hedetniemi, et al. defined two other types of efficient dominating sets. Using our current terminology they said a dominating set D of G is a $1,2$-*efficient* dominating set if every vertex in $V - D$ is an external private neighbor with respect to D (that is, $E(D) = n(G) - |D|$) and every vertex in D is a self private or an internal private neighbor (that is, $|D| = S(D) + I(D)$) with respect to D. That is, $EISPN(G) = E(D) + S(D) + I(D) = n(G)$. We will say that such a graph G is EIS-*efficient*. Klostermeyer and Goldwasser [16] showed that $G_{n,2}$ is EIS-efficient for every n. All paths and all cycles except C_5 were shown to be EIS-efficient in [13]. Furthermore, for $3 \leq m \leq 5$, all grid graphs $G_{n,m}$ (with the exception of $G_{3,3}, G_{5,3}, G_{3,5}$ and $G_{9,5}$) were shown to be EIS-efficient in [13]. They also proved that each $G_{r,s}$ for $r + s$ odd and each $G_{n,6}$ for $n \pmod 7 \in \{0, 1, 2, 4, 6\}$ are EIS-efficient and left the determination for the two cases $n \pmod 7 \in \{3, 5\}$ as open problems.

In addition, they defined a dominating set D to be a *total efficient* dominating set if every vertex in $V - D$ is an external private neighbor with respect to D (that is, $E(D) = n(G) - |D|$) and every vertex in D has an internal private neighbor with respect to D (that is, $|D| = I(D)$). Therefore, $EIPN(G) = E(D) + I(D) = n(G)$. We will say that such a graph is EI-*efficient*. It is easy to see that a path is EI-efficient if and only if its order is not congruent to 1 modulo 4. Klostermeyer and Goldwasser characterized the EI-efficient grid graphs. See [16].

Every connected EI-efficient graph G can be constructed as follows. Let $\mathcal{F}_1$ be the family of all double stars and let $\mathcal{F}_2$ the family of all stars with at least two leaves. For any positive integer k, select $H_1, \ldots, H_k$ from $\mathcal{F}_1 \cup \mathcal{F}_2$. For each selection from $\mathcal{F}_2$, consider the center and one leaf as being marked, and for each double star chosen from $\mathcal{F}_1$ consider the two centers as being marked. The vertex set of G is $V(H_1) \cup \cdots \cup V(H_k)$. The edges of G are the edges of the selected graphs together with any collection of edges joining the unmarked vertices (that is, the leaves) so that the resulting graph is connected. Note that if u_i and v_i are the marked vertices of H_i, then $\{N[\{u_1, v_1\}], \ldots, N[\{u_k, v_k\}]\}$ is a partition of $V(G)$. Although the above is a constructive characterization of the class of EI-efficient graphs, determining whether a given connected graph is EI-efficient is most likely a difficult problem.

In an analogous way, the class of connected EIS-efficient graphs can be constucted. Let $\mathcal{F}_3$ be the family of all stars of order at least 2. For any positive integer k, select $H_1, \ldots, H_k$ from $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. For each selection from $\mathcal{F}_2$, consider the center and one leaf as being marked; for each double star chosen from $\mathcal{F}_1$ consider the two centers as being marked; for each star chosen from $\mathcal{F}_3$ mark the center. Proceed as in the case of EI-efficient graphs and add any set of edges joining leaves in the disjoint union $V(H_1) \cup \cdots \cup V(H_k)$ so long as the resulting graph is connected. For example, $K_{r,s}$, where $2 \leq r \leq s$, is EIS-efficient and can be constructed as shown by starting with the single double star $S(r - 1, s - 1)$.

In a similar vein there are three additional kinds of efficient graphs, although their structure is rather trivial. A graph G is *I-efficient* (respectively, *S-efficient*, *IS-efficient*) if there is a set $U \subseteq V(G)$ such that every vertex of G is an internal (respectively, a self, an internal or self) private neighbor with respect to U. It is clear that the only I-efficient graphs are disjoint unions of complete subgraphs of order 2; the only S-efficient graphs are sets of isolated vertices; the only IS-efficient graphs are disjoint unions of isolated vertices and complete subgraphs of order 2.

6. Perfect dominating sets in graphs

In 1993, Cockayne, Hartnell, Hedetniemi and Laskar [5] defined a set $S \subset V$ to be a *perfect dominating set* if for every vertex $u \in V - S$, $|N(u) \cap S| = 1$. Notice in this definition that no conditions are placed on the vertices in S; the subgraph $G[S]$ induced by S can be any graph. They defined the *perfect domination number* $\gamma_p(G)$ to equal the minimum cardinality of a perfect dominating set, and the *upper perfect domination number* $\Gamma_p(G)$ to equal the maximum cardinality of a perfect dominating set. They pointed out that for some graphs the only perfect dominating set is vacuously the entire set $V(G)$.

Also in 1993, Bernhard, Hedetniemi and Jacobs [1] defined, for any set $U \subset V$, $ED(U) = \{v : v \in V - U \text{ and } |N(v) \cap U| = 1\}$, that is, $ED(U)$ equals the number of vertices in $V - U$ which are *efficiently* dominated by vertices in U. This is equivalent to saying that $ED(U)$ equals the number of vertices in $V - U$ which are external private neighbors with respect to the set U. They went on to define the *efficient domination number* to equal the minimum cardinality of a dominating set U for which $ED(U) = V - U$. Thus, the efficient domination number in [1] is the same thing as the perfect domination number in [5]. We should point out that the [1] definition of an efficient dominating set is not the same as the definition we have given of an efficient dominating set, the difference being that in the [1] definition no conditions are placed on the vertices in U.

The following inequality is clear from the definitions.

Proposition 13. *For any graph G of order n, $n - \gamma_p(G) \leq EPN(G)$.*

As defined by Dunbar et al. [7], a set $U \subset V$ is called a *total perfect dominating set* if for every vertex $v \in V(G)$, $|N(v) \cap U| = 1$. Notice that in a total perfect dominating set U, the subgraph $G[U]$ induced by the vertices in U is, again, a disjoint union of K_2 subgraphs. Define the *total perfect domination number* $\gamma_{tp}(G)$ and the *upper total perfect domination number* $\Gamma_{tp}(G)$ to equal the minimum and maximum cardinality of a total perfect dominating set of G, if such a set exists. Notice that if a total perfect dominating set U exists, then every vertex $v \in V$ is either an external private neighbor with respect to U or an internal private neighbor with respect to U and the following must be true.

Proposition 14. *If a graph G of order n has a total perfect dominating set U, then $E(U) + I(U) = EIPN(G) = n$.*

7. Upper private domination

In 1979 Bollobás and Cockayne [2] showed that every graph G without isolated vertices has at least one minimum dominating set U such that every vertex in U has at least one external private neighbor with respect to U. Based on this result, in 1990 Hedetniemi, Hedetniemi and Jacobs [14] defined a dominating set U to be a *private dominating set* if every vertex $v \in U$ has at least one external private neighbor. From our previous definitions, such a set could also be called an *open irredundant dominating set*. The minimum cardinality of a private dominating set in a graph G is the *private domination number*, denoted by $\gamma_{pvt}(G)$, while the *upper private domination number* $\Gamma_{pvt}(G)$ equals the maximum cardinality of a minimal private dominating set in a graph. The next results follow from [2] and the definitions.

Proposition 15. *For any graph without isolated vertices,*

$$\gamma(G) = \gamma_{pvt}(G) \leq \Gamma_{pvt}(G) \leq EPN(G).$$

8. Open problems

We note that $ESPN(G) \geq IR(G) = \alpha(G) \geq \frac{1}{2}n(G)$ for any bipartite graph G. In particular, $ESPN(T) \geq \frac{1}{2}n(T)$ for every tree. We suspect the lower bound in terms of the order of a tree is larger. The following class of trees shows that a sharp lower bound could be $\frac{4}{5}n(T)$. For each positive integer $k \geq 2$, let T_k be the tree constructed from P_k by attaching two leaves adjacent to each vertex of P_k. In addition, to each vertex of the original path identify a leaf from a path of order 3. It can be shown that $ESPN(T_k) = \frac{4}{5}n(T_k)$. For example, the black vertices in Figure 3 illustrate an $ESPN(T_4)$-set and $ESPN(T_4) = 16$.

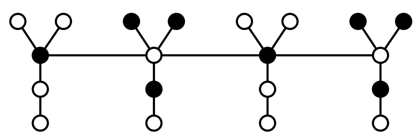

Figure 3. **The tree T_4.**

Problem 1. Find the largest positive constant c such that $ESPN(T) \geq c \cdot n(T)$ for every tree T.

Based on the data presented in Section 4 we pose the following conjectures.

Conjecture 1. If $m \geq 2$, then $EIPN(P_2 \,\Box\, P_m) = 2m$.

The truth of Conjecture 1 would imply that $EISPN(P_2 \,\Box\, P_m) = 2m$. In addition, we pose the following:

Conjecture 2. If $k \geq 1$, then $EISPN(P_3 \,\Box\, P_{2k}) = 6k$. If $m \geq 2$, then $EISPN(P_4 \,\Box\, P_m) = 4m$.

Problem 2. Construct algorithms to compute each of $IPN(T)$, $EPN(T)$, $ESPN(T)$, $EIPN(T)$, $ISPN(T)$, and $EISPN(T)$ for a tree T.

Problem 3. Determine the complexity of computing each of $IPN(G)$, $EPN(G)$, $ESPN(G)$, $EIPN(G)$, $ISPN(G)$, and $EISPN(G)$ for a graph G.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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