

On eccentric Euler Sombor index of a graph

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Abstract: A novel vertex-degree-based topological index, namely the Euler Sombor index was defined as $EU(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + d_u d_v}$. Based on this index, here we initiate the distance-based graph index as $\mathcal{E}_{ES}(G) = \sum_{uv \in E(G)} \sqrt{e^2(u) + e^2(v) + e(u)e(v)}$ and call it the eccentric Euler Sombor index of a (chemical) graph $G = (V(G), E(G))$, where $e(u)$ and $e(v)$ are the eccentricity of u and v in $V(G)$, respectively. We establish basic mathematical properties of this new index. Also, we state some bounds for $\mathcal{E}_{ES}(G)$ in terms of order, size, degrees, radius, and diameter of G . We also determine trees (connected graphs, respectively) of a given order that have the minimum and maximum value of this index. Furthermore, we pose a conjecture about the maximum value of $\mathcal{E}_{ES}(G)$ when G is a connected graph of fixed order.

Keywords: eccentric Euler Sombor index, tree, graph invariants, eccentricity, distance.

AMS Subject Classification: 05C09, 05C12

1. Introduction

Let $G = (V(G), E(G))$ be an undirected simple graph, where $V(G)$ and $E(G)$ are the set of vertices and edges of G , respectively. The *order* and the *size* of G refer

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to the number of vertices n and edges m in G , respectively. For an arbitrary vertex $u \in V(G)$, d_u denotes the degree of u . The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Also, $d(u, v)$ denotes the distance between vertices u and v in G .

The *eccentricity* $e(u)$ is defined as the distance to the farthest vertex from u . The minimum of the eccentricities across all vertices is referred to as the *radius*, denoted by $\text{rad}(G)$, while the maximum is called the *diameter*, denoted by $\text{diam}(G)$. A graph is said to be self-centered if all vertices have the same eccentricity. In a tree (a connected graph with no cycle) T , a *leaf* is a vertex of degree 1. Let P_n , S_n , $DS_{p,q}$, C_n , K_n , \overline{K}_n and $K_{p,q}$ ($p + q = n$) denote the path, the star, the double star, the cycle, the complete, the empty and the complete bipartite graph of order n , respectively. Recall that the *double star* $DS_{p,q}$ is a tree obtained from S_p and S_q by connecting the center of S_p with that of S_q .

The join $G = G_1 + G_2$ of graphs G_1 and G_2 is a graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1) \text{ and } y \in V(G_2)\}$. In this work, we use the symbol W_n to denote the wheel graph $C_n + K_1$. Also, a fan graph $F_{m,n}$ is defined as the graph join $\overline{K}_m + P_n$. The case $m = 1$ corresponds to the usual fan graphs, while $m = 2$ corresponds to the double fan, etc. For other graph-theoretical terminology and notation, we refer the readers to the books [1, 2].

Topological indices are important tools in mathematical chemistry, nanomaterials, pharmaceutical engineering, etc [13, 28]. They are widely used to analyze and predict the physicochemical properties of compounds and have good prospects for applications. Various topological indices have been introduced in the mathematical chemistry literature to study quantitative structure-activity relationships (QSAR) and quantitative structure-property relationships (QSPR) for modeling different activities and properties of chemical compounds [14, 23, 24].

The concept of a Sombor index was introduced by Ivan Gutman in [7] in the chemical graph theory. The Sombor index of a graph G is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}.$$

The Sombor index gained huge attention of researchers. Therefore, many properties of SO have been investigated along with its applications, and also many Sombor-type indices have been introduced [6, 9, 12, 17–21, 25, 29].

In 2024, in [8] it was demonstrated that the length distances of the semi-minor and semi-major axes in an ellipse are equal. Euler approximated the perimeter of an ellipse as

$$\pi \sqrt{2(d_u^2 + d_v^2)(d_u + d_v)^2}.$$

Recently, using these expressions, Tang, Li, and Deng proposed a novel Sombor-type topological index, namely the Euler Sombor index [27]. This index was defined as follows

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + d_u d_v}.$$

Even though it is defined quite recently, there are already many results on EU , see for example [8, 10, 15, 22].

In 1997, Sharma et al. [26] introduced the eccentric-based graph index. After that in 2021, Kulli [16] initiated the eccentric Sombor index (fourth Sombor index) of a graph G , which is defined as

$$\mathcal{E}_{SO}(G) = \sum_{uv \in E(G)} \sqrt{e^2(u) + e^2(v)}.$$

In this paper, we introduce new eccentricity-based graph index called an eccentric Euler Sombor index of a graph G , and it is defined as

$$\mathcal{E}_{EU}(G) = \sum_{uv \in E(G)} \sqrt{e^2(u) + e^2(v) + e(u)e(v)}.$$

There are several eccentricity-based graph indices available in the literature, see for example [3, 4].

2. Eccentric Euler Sombor index of certain graphs

Clearly, if G is a self-centered graph of size m , then $\mathcal{E}_{EU}(G) = \sqrt{3}me(u)$ for any vertex u . So, we start this section by the the following example.

Example 1. (i) $\mathcal{E}_{EU}(K_n) = \frac{\sqrt{3}n(n-1)}{2}$.

(ii) $\mathcal{E}_{EU}(C_n) = \begin{cases} \frac{\sqrt{3}}{2}n(n-1) & n \text{ is odd} \\ \frac{\sqrt{3}}{2}n^2 & n \text{ is even.} \end{cases}$

(iii) If $p, q \geq 2$, then $\mathcal{E}_{EU}(K_{p,q}) = 2\sqrt{3}pq$.

(iv) If $m \geq 2$ and $n \geq 3$, then $\mathcal{E}_{EU}(F_{m,n}) = 2\sqrt{3}(mn + n - 1)$.

Clearly, for a connected graph of size m , we have $\mathcal{E}_{EU}(G) \geq \sqrt{3}m$ and equality holds if and only if G is a complete graph. Now, we state the following result for bipartite graphs.

Lemma 1. *Let $G = G(X, Y)$ be a connected bipartite graph of size m such that $|X|, |Y| \geq 2$. Then $\mathcal{E}_{EU}(G) \geq 2\sqrt{3}m$. Also, the equality holds if and only if G is a complete bipartite graph.*

Proof. First note that $e(x) \geq 2$ for any $x \in V(G)$. Suppose uv is an arbitrary edge in $E(G)$. Hence

$$\sqrt{e^2(u) + e^2(v) + e(u)e(v)} \geq 2\sqrt{3}.$$

It concludes that $\mathcal{E}_{EU}(G) \geq 2\sqrt{3}m$. Now, let $G = G(X, Y)$ such that $x \approx y$ for some $x \in X$ and $y \in Y$. So, $e(x), e(y) \geq 3$. Thus

$$\sqrt{e^2(x) + e^2(y) + e(x)e(y)} \geq 3\sqrt{3}$$

and consequently $\mathcal{E}_{EU}(G) > 2\sqrt{3}m$. \square

Now, we compute the eccentric Euler Sombor index of some well-known graphs with $e(u) \in \{a, b\}$ for any vertex u and some suitable distinct positive integer numbers a and b .

Example 2. (i) If $n \geq 3$, then $\mathcal{E}_{EU}(S_n) = \sqrt{7}(n-1)$.

(ii) If $G = W_n$ with $n \geq 4$, then the eccentricity of the center vertex in G is equal to 1 and for other vertices is equal to 2. So, we have

$$\mathcal{E}_{EU}(W_n) = (\sqrt{7} + 2\sqrt{3})n.$$

(iii) $\mathcal{E}_{EU}(F_{m,n}) = \begin{cases} \sqrt{7}n + 2\sqrt{3}(n-1) & m = 1, n \geq 3 \\ 2\sqrt{7}m + \sqrt{3} & m \geq 2, n = 2 \end{cases}$

(iv) If $T = DS_{p,q}$, then the eccentricity of all leaves of T is equal to 3 and for the other vertices is equal to 2. So, we have

$$\mathcal{E}_{EU}(DS_{p,q}) = \sqrt{19}(p-1) + \sqrt{19}(q-1) + 2\sqrt{3} = \sqrt{19}(n-2) + 2\sqrt{3}.$$

Example 3. The eccentricity of vertices of P_n is shown in Figures 1 and 2.

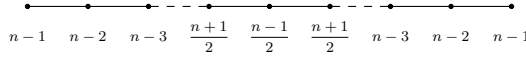


Figure 1. The eccentricity of the vertices of P_n (n is odd).

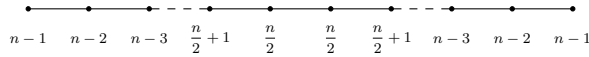


Figure 2. The eccentricity of the vertices of P_n (n is even).

So, if n is an odd number, then we have:

$$\begin{aligned} \mathcal{E}_{EU}(P_n) &= 2 \sum_{i=1}^{\frac{n-1}{2}} \sqrt{(n-i)^2 + (n-i-1)^2 + (n-i)(n-i-1)} \\ &= 2\sqrt{3} \sum_{i=1}^{\frac{n-1}{2}} \sqrt{n^2 - (2i+1)n + i^2 + i + 1/3}; \end{aligned}$$

and for any even number n , we get:

$$\begin{aligned} \mathcal{E}_{EU}(P_n) &= 2 \sum_{i=1}^{\frac{n}{2}-1} \sqrt{(n-i)^2 + (n-i-1)^2 + (n-i)(n-i-1)} + \frac{\sqrt{3}}{2}n \\ &= 2\sqrt{3} \sum_{i=1}^{\frac{n}{2}-1} \sqrt{n^2 - (2i+1)n + i^2 + i + 1/3} + \frac{\sqrt{3}}{2}n. \end{aligned}$$

Recall that a cut edge in a graph G is an edge whose removal increases the number of connected components of G .

Theorem 1. *Let G be a graph and $e \in E(G)$ such that e is not a cut edge. Then*

$$\mathcal{E}_{EU}(G) \leq \mathcal{E}_{EU}(G - \{e\}) + \sqrt{3n^2 - 3(d_u + d_v)n + d_u^2 + d_v^2 + d_u d_v}.$$

Proof. Let $e = uv$ and $G' = G - \{e\}$. We have

$$\begin{aligned} \mathcal{E}_{EU}(G) &\leq \mathcal{E}_{EU}(G') + \sqrt{e^2(u) + e^2(v) + e(u)e(v)} \\ &\leq \mathcal{E}_{EU}(G') + \sqrt{(n - d_u)^2 + (n - d_v)^2 + (n - d_u)(n - d_v)} \\ &= \mathcal{E}_{EU}(G') + \sqrt{3n^2 - 3(d_u + d_v)n + d_u^2 + d_v^2 + d_u d_v}, \end{aligned}$$

and the result follows. \square

Corollary 1. *Let G be a graph and $e \in E(G)$ such that e is not a cut edge. Then, by Theorem 1 we have:*

$$\mathcal{E}_{EU}(G) \leq \mathcal{E}_{EU}(G - \{e\}) + \sqrt{3}(n - \delta(G)).$$

In the following, we state upper and lower bounds for eccentricity of any vertex of a graph G in terms of radius, diameter, order, minimum and maximum degrees of G .

Lemma 2. *For any vertex u in a connected graph G of order $n \geq 2$, we have:*

- (a) $\text{rad}(G) \leq e(u) \leq \text{diam}(G)$.
- (b) $n - \Delta(G) \leq e(u) \leq n - \delta(G)$.

Proof. It is self-explanatory from the concept of vertex eccentricity. \square

Now, we have the following result.

Proposition 1. *Let G be a connected graph of order n and size m . Then*

- (i) $\sqrt{3}m\text{rad}(G) \leq \mathcal{E}_{EU}(G) \leq \sqrt{3}m\text{diam}(G)$.
- (ii) $\sqrt{3}m(n - \Delta(G)) \leq \mathcal{E}_{EU}(G) \leq \sqrt{3}m(n - \delta(G))$.

Proof. (i) Let u and v be arbitrary vertices of G . By Lemma 2 we have

$$\text{rad}(G) \leq e(u), e(v) \leq \text{diam}(G)$$

So, we conclude that

$$3(\text{rad}(G))^2 \leq e^2(u) + e^2(v) + e(u)e(v) \leq 3(\text{diam}(G))^2.$$

This implies that

$$\sqrt{3}m\text{rad}(G) \leq \mathcal{E}_{EU}(G) \leq \sqrt{3}m\text{diam}(G)$$

and we are done.

(ii) The proof is done with a similar method applied in the proof of (i). \square

Theorem 2. For any tree T of order n ,

$$\mathcal{E}_{EU}(S_n) \leq \mathcal{E}_{EU}(T) \leq \mathcal{E}_{EU}(P_n).$$

The equalities hold if and only if $T = S_n$ and $T = P_n$, respectively.

Proof. Clearly, the only tree of order 2 is $T = P_2 \cong S_2$ and so the proof is straightforward. Now, let $n \geq 3$ and suppose u and v are two arbitrary vertices of $T \neq S_n$. One can easily check that at least one of the $e(u)$ or $e(v)$ is greater than 1. Hence

$$\sqrt{e^2(u) + e^2(v) + e(u)e(v)} \geq \sqrt{7}.$$

Also, there exists $e = xy \in E(T)$ such that $e(x), e(y) \geq 2$, since $T \neq S_n$. It concludes that $\mathcal{E}_{EU}(T) > \sqrt{7}(n-1)$ and consequently $\mathcal{E}_{EU}(S_n) < \mathcal{E}_{EU}(T)$, by Example 2. Next, we prove the other inequality. Let $T \neq P_n$ be a tree of order $n \geq 3$. Thus, $\text{diam}(T) \leq n-2$. If $\text{diam}(T) \leq n-3$, then $\mathcal{E}_{EU}(T) \leq \sqrt{3}(n-1)(n-3)$, by Proposition 1(i). On the other hand, According to Proposition 1(ii), we have $\mathcal{E}_{EU}(P_n) \geq \sqrt{3}(n-1)(n-2)$. These imply that $\mathcal{E}_{EU}(T) < \mathcal{E}_{EU}(P_n)$. If $\text{diam}(T) = n-2$, then T is a tree in which an isolated vertex is adjacent to one of the vertices of the path P_{n-1} by an edge as a leaf. Among the mentioned trees, the tree of Figure 3 has the maximum index. Clearly,

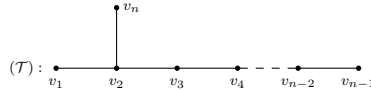


Figure 3. The tree with the maximum eccentric Euler Sombor index among trees of $\text{diam}(T) = n-2$.

$$\mathcal{E}_{EU}(T) = \mathcal{E}_{EU}(P_{n-1}) + \sqrt{(n-2)^2 + (n-3)^2 + (n-2)(n-3)}.$$

By Example 3, one can see that $\mathcal{E}_{EU}(P_n) > \mathcal{E}_{EU}(T)$ and the proof is complete. \square

Note that if G is a connected graph of order $n = 3$, then either $G = P_3 = S_3$ or $G = K_3$. On the other hand, $\mathcal{E}_{EU}(S_3) = 2\sqrt{7}$ and $\mathcal{E}_{EU}(K_3) = 3\sqrt{3}$. Thus $\mathcal{E}_{EU}(S_3) > \mathcal{E}_{EU}(K_3)$. For $n \geq 4$, we have the following:

Theorem 3. For any connected graph G of order $n \geq 4$,

$$\mathcal{E}_{EU}(S_n) \leq \mathcal{E}_{EU}(G).$$

The equality holds if and only if $G = S_n$.

Proof. Let G be a graph of size m . According to Theorem 2, it is enough to prove theorem in the case $m > n - 1$. First, note that $\mathcal{E}_{EU}(S_n) = \sqrt{7}(n - 1) < \frac{\sqrt{3}}{2}n(n - 1) = \mathcal{E}_{EU}(K_n)$, since $n \geq 4$. So, let $G \neq K_n$. Thus, there exist two non-adjacent vertices x and y in $V(G)$. Therefore, $e(x), e(y) \geq 2$. If for any arbitrary edge $e = uv$ of G , $e(u) + e(v) \geq 3$, then $\mathcal{E}_{EU}(S_n) < \mathcal{E}_{EU}(G)$, since $m > n - 1$. Else, there exists adjacent vertices u and v in $V(G)$ such that $e(u) = e(v) = 1$. This implies that all other $n - 2$ vertices of G are adjacent to both u and v (see Figure 4).

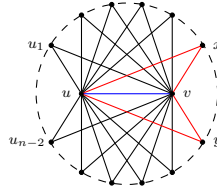


Figure 4. A graph in which two adjacent vertices u and v have eccentricity 1.

Hence G has at least $2(n - 2) + 1 = 2n - 3$ edge and

$$\mathcal{E}_{EU}(G) \geq 4\sqrt{7} + (2n - 7)\sqrt{3} > \sqrt{7}(n - 1).$$

Thus $\mathcal{E}_{EU}(G) > \mathcal{E}_{EU}(S_n)$ and the proof is complete. □

We finish this section by the following conjecture.

Conjecture 4. For any connected graph G of order $n \geq 4$,

$$\mathcal{E}_{EU}(G) \leq \mathcal{E}_{EU}(P_n).$$

Note that Conjecture 4 holds for trees by Theorem 2.

3. Eccentric Euler Sombor index of thorny graphs

Let G be a connected graph of order n with vertex set $V(G) = \{v_1, \dots, v_n\}$ and let $\mathcal{T} = (t_1, \dots, t_n)$ be an n -tuple of non-negative integers. The *thorny graph* G^{t*} is the graph obtained by attaching t_i pendent vertices to the vertex v_i of G , for $i = 1, \dots, n$. The pendent vertices attached to the vertex v_i are called *thorns* of v_i . We denote the set of t_i thorns of v_i by V_i and the total number of thorns attached to G by t . The concept of thorny graphs was introduced by Gutman [5] and eventually found a variety of chemical applications. In this section, we will calculate the value of the eccentric Euler Sombor index of thorny graphs with $t > 0$.

Theorem 5. Let $n \geq 2$ be an integer number. Then

$$\mathcal{E}_{EU}(K_n^*) = \begin{cases} (n+t-1)\sqrt{7} + \sqrt{3}(n-1)(n-2) & |V_j| = t \text{ for some } j \\ \sqrt{3n(n-1)} + \sqrt{19}t & \text{otherwise.} \end{cases}$$

Proof. If there exists some $1 \leq j \leq n$ with $|V_j| = t$, then the rest of V_i 's are the empty set. So, $e(v_j) = 1$ and $e(x) = 2$ for any $x \in V(K_n^*) \setminus \{v_j\}$. Therefore,

$$\mathcal{E}_{EU}(K_n^*) = (n+t-1)\sqrt{7} + 2\sqrt{3}\left[\frac{1}{2}(n-1)(n-2)\right] = (n+t-1)\sqrt{7} + \sqrt{3}(n-1)(n-2).$$

Else, $e(v_i) = 2$ and $e(x) = 3$ for any $x \in V_i$, $i = 1, \dots, n$. Thus,

$$\mathcal{E}_{EU}(K_n^*) = 2\sqrt{3}\left[\frac{1}{2}n(n-1)\right] + \sqrt{19}t = \sqrt{3n(n-1)} + \sqrt{19}t.$$

and we are done. \square

Theorem 6. Let $p, q \geq 2$ be integer numbers. Let $\mathcal{T}_1 = (t_1, \dots, t_p)$ and $\mathcal{T}_2 = (t_{p+1}, \dots, t_{p+q})$ such that $t_1 \geq \dots \geq t_p$ and $t_{p+1} \geq \dots \geq t_{p+q}$. Then we have

$$\mathcal{E}_{EU}(K_{p,q}^*) = \begin{cases} 2\sqrt{2}q + \sqrt{19}(pq - q + t) & t_1 = t \\ \sqrt{19}pq + \sqrt{37}t & t' = t, t_2 \neq 0 \\ 2\sqrt{2}p + \sqrt{19}(pq - p + t) & t_{p+1} = t \\ \sqrt{19}pq + \sqrt{37}t & t' = 0, t_{p+2} \neq 0 \\ 3\sqrt{3}(p-1)(q-1) + \sqrt{19}(p+q+t-2) + 2\sqrt{2} & 0 \neq t_1 = t' \neq t, t_{p+2} = 0 \\ \sqrt{19}(q+t') + 3\sqrt{3}q(p-1) + \sqrt{37}(t-t') & 0 \neq t_1 = t', t_{p+2} \neq 0 \\ \sqrt{19}(p+t-t') + 3\sqrt{3}p(q-1) + \sqrt{37}t' & 0 \neq t_{p+1} = t-t', t_2 \neq 0 \\ 3\sqrt{3}pq + \sqrt{37}t & t_2, t_{p+2} \neq 0 \end{cases},$$

where $t' = t_1 + \dots + t_p$.

Proof. Let $V(K_{p,q}) = X \cup Y$ such that both $X = \{v_1, \dots, v_p\}$ and $Y = \{v_{p+1}, \dots, v_{p+q}\}$ are independent sets. First suppose $t_1 = 0$.

If $t_{p+1} = t$, then v_{p+1} and all vertices in X have eccentricity 2 and the eccentricity of the other vertices of $K_{p,q}^*$ is equal to 3. Thus,

$$\mathcal{E}_{EU}(K_{p,q}^*) = \sqrt{12}p + \sqrt{19}[p(q-1) + t] = 2\sqrt{2}p + \sqrt{19}(pq - p + t).$$

If $t_{p+1} < t$, then all vertices in X and Y have eccentricity 2 and 3, respectively. Also, the eccentricity of each thorn is equal to 4. Therefore,

$$\mathcal{E}_{EU}(K_{p,q}^*) = \sqrt{19}pq + \sqrt{37}t.$$

If $t_{p+1} = 0$, then one can calculate $\mathcal{E}_{EU}(K_{p,q}^*)$ in a similar way.

Next, let $t_1 \neq 0, t$ and $t_2 = 0$.

If $t_{p+2} = 0$, then the eccentricity of v_1 and v_{p+1} is equal to 2 and the eccentricity of the other vertices of $K_{p,q}^*$ is equal to 3. Thus,

$$\begin{aligned} \mathcal{E}_{EU}(K_{p,q}^*) &= \sqrt{19}[(p-1) + (q-1)] + \sqrt{27}(p-1)(q-1) + \sqrt{12} + \sqrt{19}t \\ &= 3\sqrt{3}(p-1)(q-1) + \sqrt{19}(p+q+t-2) + 2\sqrt{2}. \end{aligned}$$

If $t_{p+2} \neq 0$, then the eccentricity of v_1 is equal to 2, the eccentricity of each thorn of v_1 and the other vertices of $K_{p,q}$ is equal to 3 and the eccentricity of any thorn of vertices in Y is equal to 4. So,

$$\mathcal{E}_{EU}(K_{p,q}^*) = \frac{\sqrt{19}(q+t') + \sqrt{27}q(p-1) + \sqrt{37}(t-t')}{\sqrt{19}(q+t') + 3\sqrt{3}q(p-1) + \sqrt{37}(t-t')} =$$

If $t_{p+1} \neq 0, t$ and $t_{p+2} = 0$, then one can calculate $\mathcal{E}_{EU}(K_{p,q}^*)$ in a similar way.

Finally, if $t_2, t_{p+2} > 0$, then the eccentricity of vertices of $K_{p,q}$ is equal to 3 and the eccentricity of each thorn is equal to 4. Hence,

$$\mathcal{E}_{EU}(K_{p,q}^*) = \sqrt{27}pq + \sqrt{37}t = 3\sqrt{3}pq + \sqrt{37}t,$$

and we are done. □

we finish this section by computing the eccentric Euler Sombor index of the thorny star graph S_n^* . Note that if v_1 is the central vertex of S_n , then one can assume that $t_1 = 0$. This is because that the thorny star graph S_n^* with $\mathcal{T} = \{t_1, \dots, t_n\}$ is isomorphic to the thorny star graph $S_{n+t_1}^*$ with $\mathcal{T}' = \{ \underbrace{0, \dots, 0}_{(t_1+1)\text{-times}}, t_2, \dots, t_n \}$.

Theorem 7. *Let $n \geq 3$ be an integer number. Then we have*

$$\mathcal{E}_{EU}(S_n^*) = \begin{cases} \sqrt{19}(n-2+t) + 2\sqrt{3} & t_2 = t \\ \sqrt{19}(n-1) + \sqrt{37}t & o.w. \end{cases}.$$

Proof. Let v_1 be the central vertex of S_n and $\mathcal{T} = \{0, t_2, \dots, t_n\}$. If $t_2 = t$, then $S_n^* \cong DS_{n-2, t+1}$ and consequently by Example 2 we obtain:

$$\mathcal{E}_{EU}(S_n^*) = \sqrt{19}(n-2+t) + 2\sqrt{3}.$$

Next, let $t_2 \neq t$. Then the eccentricity of v_1 is equal to 2, the eccentricity of the other vertices of S_n is equal to 3 and the eccentricity of any thorn of S_n^* is equal to 4. So,

$$\mathcal{E}_{EU}(S_n^*) = \sqrt{19}(n-1) + \sqrt{37}t.$$

and the result follows. □

4. Numerical work

In Figure 5 correlation between Euler Sombor index and eccentric Euler Sombor index in case of trees with $n = 10$ is depicted. As one may see, the correlation is weak and data points are grouped into several clusters. Such finding indicates that both $EU(G)$ and $\mathcal{E}_{EU}(G)$ contain different information on underlying graph, which may be helpful when developing QSAR/QSPR models. The correlation between these two

indices is significantly improved when first Zagreb index $M_1(G)$ is included [11]. This degree-based index is defined as:

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v).$$

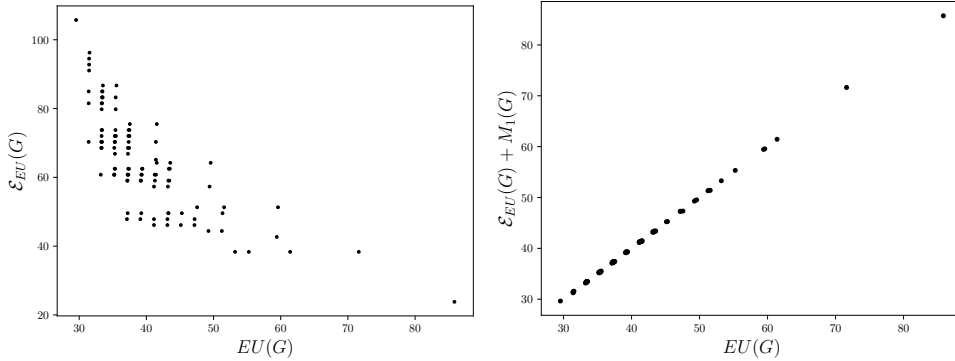


Figure 5. Left: Correlation between $EU(G)$ and $\mathcal{E}_{EU}(G)$. Right: Correlation between $EU(G)$ and $\mathcal{E}_{EU}(G) + M_1$ linear model. Both correlations are in case of trees with $n = 10$.

One of the main characteristics of graph indices is their degeneracy D . The degeneracy is equal to: $D = 1 - \frac{N - N_{TI}}{N}$, where N denotes the number of graphs within a set, and N_{TI} stands for the number of graphs that can not be distinguished by a topological index. In Figure 6 the percentage of degeneracy of the eccentric Euler Sombor index is dep in the case of chemical trees (i.e., $\Delta(G) = 4$).

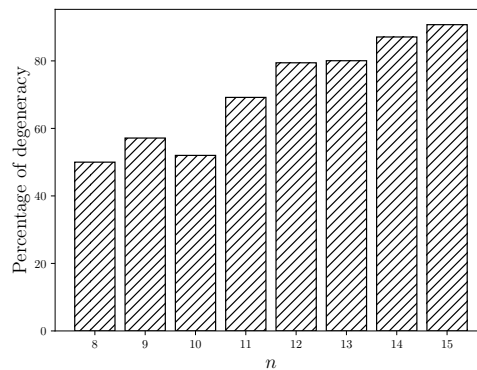


Figure 6. The percentage of degeneracy of $\mathcal{E}_{EU}(G)$ in case of chemical trees with $n = 8-15$.

As one may see, $\mathcal{E}_{EU}(G)$ shows relatively high degeneracy, especially in large data sets. However, such behavior of $\mathcal{E}_{EU}(G)$ is in accordance with other degree-based and distance-based graph indices.

5. Conclusion

In this work, novel eccentricity-based Sombor-type graph index is presented, and we call it eccentric Euler Sombor index of a graph G , $\mathcal{E}_{EU}(G)$. Some bounds in terms of graph structure (size, order, ...) are presented, as well as some properties of $\mathcal{E}_{EU}(G)$, such as its degeneracy. Also, the values of $\mathcal{E}_{EU}(G)$ for some classes of graphs are provided.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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