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Research Article

Irregularity-based entropy inequalities determining complete interconnection network complexity

Asfand Fahad[1], Muhammad Imran Qureshi[2,†], Muhammad Anwar Chaudhry[3], Rida Irfan[4], Anas Ashraf[2,‡], Yilun Shang[5,*]

[1]Centre for Advanced Studies in Pure and Applied Mathematics,
Bahauddin Zakariya University, Multan, 60000, Pakistan
asfandfahad1@bzu.edu.pk

[2]Department of Mathematics, COMSATS University Islamabad,
Vehari Campus, Vehari, 61100, Pakistan
[†]imranqureshi@cuivehari.edu.pk
[‡]anskhan739@gmail.com

[3]Department of Mathematics and Statistics, Institute of Southern Punjab, Multan 60800, Pakistan
chaudhry@isp.edu.pk

[4]Department of Mathematics, COMSATS University Islamabad,
Sahiwal Campus, Sahiwal, 57000, Pakistan
ridairfan@cuisahiwal.edu.pk

[5]School of Computer Science, Northumbria University, Newcastle, NE1 8ST, UK
yilun.shang@northumbria.ac.uk

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Abstract: In contemporary scientific research, network analysis theory contributes significantly to diverse systems, including forecasting cetane numbers, examining thermal transport phenomena, addressing sustainability in supply chains, monitoring post-operative health, shaping public policy through social networks, and investigating phase transitions in storage tanks. Various Interconnection networks (ICNs) are discriminated based on attributes such as bandwidth, latency, switch radix, and network topology. Entropy is pivotal in extracting meaningful information from these scientific models, thereby enhancing their efficiency. The butterfly and Benes networks and their derived networks are widely utilized in diverse systems, including IBM, NEC Cenju-3, MIT Transit Project, optical coupler internal structures, permutation routing, chip networks, and multiprocessor systems. To extract meaningful information, a few models to discriminate these networks on the basis of their entropy and complexity have been introduced in recent research. In this paper, we propose an entropy-determining model based on the irregularity of these ICNs, which is applied to obtain mathematical formulae for entropy determination, leading to complexity analysis of these ICNs. The

* *Corresponding Author*

analysis shows that we determine explicit inequalities through numeric data produced by these formulae, thus giving a complete characterization of this family of ICNs and answering the recently posed questions.

Keywords: irregularity-based entropy, complexity, information functional, network analysis, interconnection networks.

AMS Subject Classification: 05C82, 05C92

1. Introduction

In the contemporary realm of scientific inquiry, the theory of network analysis has made significant contributions across a range of areas. These include forecasting cetane numbers [22], examining thermal transport phenomena [20], addressing sustainability in supply chains [37], monitoring postoperative health [38], shaping public policy through social networks [65], and investigating phase transitions in storage tanks [58]. Over the decades, the network complexity determination/analysis contributed significantly in many ways: for example finite automata, decision trees, VLSI chips and threshold circuits [24], robot systems [23], fastest algorithms for linear programming [54], Semantic communication networks to facilitate smart exchanges between different communication agents [64], software library structure [53], evaluation of business operational efficiency [6, 7, 25] and characterizing networks [8, 42]. On the other hand, modeling different scientific systems/objects via graphs has made graph theory a subject of great importance. These models cover computer networks [19, 49, 50, 59], information theory [36], information fusion [15], chemical science [9, 31, 62], medical science [4] and also several classical mathematical notions too [47]. Thus, the range of problems studied via graphs and graph theory is extremely broad and modern. Consequently, it is natural to investigate complexity quantification/analysis via graph-theoretic techniques. A technique utilized for quantifying the graph-based system's complexity is the use of entropy measures [10–12, 40, 41]. Entropy plays a crucial role in extracting meaningful information from scientific models [27, 35, 39, 44]. Consequently, it holds significance in the determination and calculation of entropy measures [32, 48, 61].As far as the entropy determination of graphs is concerned, there have been different applications, including graph characterizations [17], obtaining significant vertices for email database [51], complexity study in derived networks [67], and studying metal-insulator lattice and organic frameworks [14, 68]. Various generalizations of entropy have been proposed in physics and information theory, and a brief comparison with the present approach is relevant. Barrow entropy, as studied by Luciano and Saridakis [35], modifies the Bekenstein–Hawking framework and has been applied to baryon asymmetry and observational constraints. Tsallis entropy [46] provides a non-extensive generalization suited to complex systems, while Kaniadakis entropy [13] offers a one-parameter deformation linking standard and relativistic statistics; Shannon entropy [5] remains the foundational measure in information theory and underlies many structural entropy formulations. In this paper, the

entropy measures we propose are based on graph irregularity and yield explicit formulae for the Benes family of interconnection networks, thereby complementing these approaches and contributing to the discrimination of such networks on the basis of entropy and complexity.
Now, we move towards the formulation of the problem to be studied in this manuscript. Several Interconnection networks, abbreviated as ICNs, are discriminated on the basis of their bandwidth, latency, switch radix, and importantly, network topology [52]. Networks with diverse utilizations, including ICNs, are investigated through their topology, for example, see [19, 30, 43, 50]. For the significance of ICNs in High-Performance Computer Systems, we refer [34] to the readers. The butterfly networks/graphs, used for multiple input linkage to the outputs, are the graphs representing fast Fourier transform networks (FFT networks). The butterfly network uses a series of switch stages and connection patterns to link many inputs to related outputs. Benes network [3], constructed via the back-to-back pairing of two butterfly networks are employed in IBM, NEC Cenju-3, MIT Transit Project, optical coupler internal structures, and permutation routing [33], chip networks [21, 29], and multiprocessor systems [26]. The structural feature-based study of Benes networks was initiated in [66]. Motivated by the structure and applications of these networks, new classes of ICNs were introduced via identifications in Benes networks [18] and some structural parameters of these families were investigated in [55, 57]. For $r \geq 2$, the constructions of a Butterfly network $BF(r)$, Benes network $B(r)$, vertical (and horizontal) cylindrical Benes networks $V(r)$ $(B(r))$ and toroidal Benes network $(T(r))$ are well expressed in [18, 55, 57], see Figures 3, 4, 5 and Figure 6. Moreover, it is also known and easy to validate the order and size of these networks.

Network/graph	Order	Size
$BF(r)$	$2^r(r+1)$	$r(2^{r+1})$
$B(r)$	$(2r+1)2^r$	$r2^{r+2}$
$H(r)$	$(2r+1)(2^r-1)$	$2r(2^{r+1}-1)$
$V(r)$	$r2^{r+1}$	$r2^{r+2}$
$T(r)$	$2r(2^r-1)$	$2r(2^{r+1}-1)$

The problem of complexity analysis and characterization of Benes networks on the basis of complexity was first proposed in [60]. Their model showed some encouraging trends, but their applicability was positive only on restricted domains, instead of all values of parameters. In [63], a similar problem was investigated, and improved results were shown, but the completeness could not be demonstrated. The study continued in [56] and the answer to characterizations between $B(r)$ and $V(r)$ was given for the whole domain. However, the complete characterizations between $B(r)$ and $H(r)$ could not be given. The main objective of this study is to propose a model that gives complete complexity comparisons of $BF(r)$, $B(r)$, $H(r)$, $V(r)$, and $T(r)$. There are two major components of this model, one is related to entropy measures, and the other is the use of topological indices (TIs). The significance, determination and development, and its connections with complexity have already been described.

For the developments of TIs, their significance, and (more importantly) limitations, we refer [45, 55] to the readers.
Throughout this manuscript, a graph is denoted by $G = (V, E)$, where V is the vertex set and E is the edge set. The *order* of G is the cardinality $|V(G)|$ and the *size* of G is the cardinality $|E(G)|$. For a vertex $u \in V$, the *degree* $d(u)$ is the number of edges incident with u; for an edge $uv \in E$, we write d_u and d_v for the degrees of its end vertices u and v, so that $d_u = d(u)$ and $d_v = d(v)$. A graph is *regular* if every vertex has the same degree. Interconnection networks are abbreviated as ICNs; $BF(r)$, $B(r)$, $H(r)$, $V(r)$, and $T(r)$ denote the butterfly, Benes, horizontal cylindrical Benes, vertical cylindrical Benes, and toroidal Benes networks of dimension r, respectively. An *information functional* is a function ϕ on E with positive values, and ENT_ϕ denotes the corresponding edge-based entropy from Equation 1.1 below. Topological indices are abbreviated as TIs.
Proposed Model To achieve our objective, we propose a measurement scheme by employing multiple well-established notions related to graphs/networks. The first computational tool is an edge-dependent entropy-measuring formula introduced by Dehmer. The work of Dehmer on structural entropy measurements of networks is among the most distinguished ones in this direction, see [10–12]. For $G = (V, E)$ and ϕ, where ϕ is a function on E with positive values, we have:

$$ENT_\phi = log(\sum_{uv \in E} \phi(uv)) - \sum_{uv \in E} \frac{\phi(uv)}{\sum_{uv \in E} \phi(uv)} log(\phi(uv)). \tag{1.1}$$

Note that ϕ provides great flexibility to introduce entropy measures. In the work studied in [56, 60, 63], multiple information functionals ϕ have been used to determine the complexity comparisons between $B(r)$, $H(r)$, $V(r)$ and $T(r)$. Here, we notice that the network structures own regularity on major proportion, and where the structures of these networks are different, the irregularity is obtained in that part. So, in order to reflect this structural feature in measurement, we use irregularity TIs. The Irregularity indices are a noteworthy class of TIs, depending upon the difference of degrees of adjacent vertices. Several mathematicians studied these TIs for the different networks/graphs [1, 2, 16, 28]. We use the TIs from Table 1 to define information functionals ϕ in Equation (1.1) and produce new irregularity entropy measures, see [2, 28].

Irregularity TIs	Formulae
$AL(G)$	$\sum_{uv \in E} \|d_u - d_v\|$
$IRLU(G)$	$\sum_{uv \in E} \frac{\|d_u - d_v\|}{\min(d_u, d_v)}$
$IRRt(G)$	$\frac{1}{2} \sum_{uv \in E} \|d_u - d_v\|$
$IRF(G)$	$\sum_{uv \in E} (d_u - d_v)^2$
$IRA(G)$	$\sum_{uv \in E} (d_u^{-\frac{1}{2}} - d_v^{-\frac{1}{2}})^2$

Table 1. **Irregularity TIs**

In Table 2, we propose entropy-measuring formulae based on the irregularity of the structure of networks/graphs. The formulae given in Table 2 are obtained by defining different functions ϕ in Equation 1.1.

$\phi(u,v)$	Corresponding irregularity Entropy
$\lvert d_u - d_v \rvert$	$ENT_{AL}(G) = log(AL(G)) - \frac{1}{AL(G)} \sum_{uv\in E} \lvert d_u - d_v \rvert log(\lvert d_u - d_v \rvert)$
$\frac{\lvert d_u - d_v \rvert}{\min(d_u,d_v)}$	$ENT_{IRLU}(G) = log(IRLU(G)) - \frac{1}{IRLU(G)} \sum_{uv\in E} \phi(uv) log(\phi(uv))$
$\frac{1}{2}\lvert d_u - d_v \rvert$	$ENT_{IRRt}(G) = log(IRRt(G)) - \frac{1}{IRRt(G)} \sum_{uv\in E} \phi(uv) log(\phi(uv))$
$(d_u - d_v)^2$	$ENT_{IRF}(G) = log(IRF(G)) - \frac{1}{IRF(G)} \sum_{uv\in E} \phi(uv) log(\phi(uv))$
$(d_u^{-\frac{1}{2}} - d_v^{-\frac{1}{2}})^2$	$ENT_{IRA}(G) = log(IRA(G)) - \frac{1}{IRA(G)} \sum_{uv\in E} \phi(uv) log(\phi(uv))$

Table 2. **Irregularity entropy measures corresponding to information functional ϕ**

Figure 1 summarizes the construction of irregularity-based entropy: starting from an interconnection network as a graph $G = (V, E)$, edges are grouped by the degrees (d_u, d_v) of their end vertices (as in the partition tables used in the proofs), each edge is assigned a positive weight $\phi(uv)$ derived from irregularity (Tables 1 and 2), and Dehmer's formula (1.1) yields the corresponding entropy ENT_ϕ.

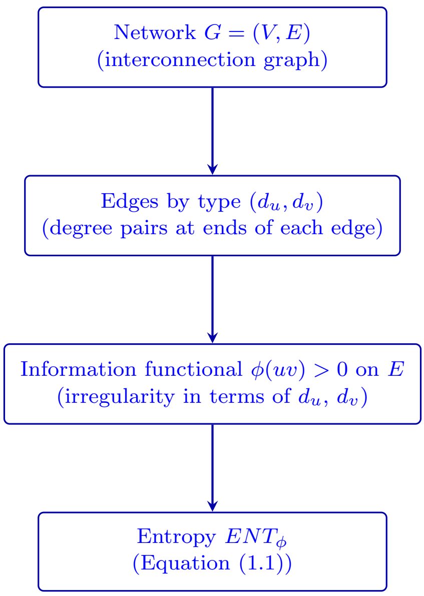

Figure 1. Schematic pipeline from an interconnection network to irregularity entropy ENT_ϕ: network $\rightarrow$ edge classification by (d_u, d_v) $\rightarrow$ choice of ϕ $\rightarrow$ entropy via (1.1).

The paper is organized as follows: we begin by proving explicit formulae of the defined entropies for ICNs via mathematical proofs. Then we utilize these formulae to generate numerical values for irregularity entropies of ICNs to investigate patterns and behaviors of complexity of ICNs. The presentation of results is complemented by graphical representations.

2. Irregularity entropy measures for interconnection networks

In this section, we establish mathematical formulae of the defined irregularity entropy measures for the families of interconnection networks, including $BF(r)$, $B(r)$, $H(r)$, $V(r)$, and $T(r)$. These formulae are obtained by using rigorous mathematical proofs and thus provide authenticity to our work. The proofs are based on the suitable partitions of the edge sets of the specific networks. Thus, for $BF(r)$ and $B(r)$, we give the complete proofs, and for the remaining networks, we provide the partition tables, and the entropy formulae can be obtained from the partitions in a similar way as in the case of $BF(r)$ and $B(r)$.

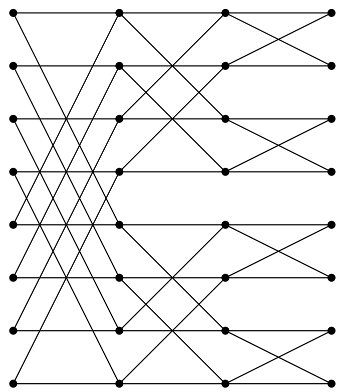

Figure 2. **Graphical representation of** $BF(3)$

The butterfly network for $r = 3$ is shown in Figure 2. For $BF(r)$, the edge set of $BF(r)$ can be partitioned in terms of degrees of its end vertices as shown in Table 3.

Edge Type in terms of the degree of end vertices	Frequency
(2,4)	$4(2^r)$
(4,4)	$(2r-4)2^r$

Table 3. **Edge partition in terms of degree of end vertices of** $BF(r)$

To prove irregularity entropy measures of $BF(r)$, we first compute the irregularity TIs given in Table 1 for $BF(r)$.

Theorem 1. *Let $BF(r)$ be a butterfly network of dimension r, where $r \geq 3$, we have*

1. $AL(BF(r)) = 8 \cdot 2^r$

2. $IRLU(BF(r)) = 4 \cdot 2^r$
3. $IRRt(BF(r)) = 4 \cdot 2^r$
4. $IRF(BF(r)) = 16 \cdot 2^r$
5. $IRA(BF(r)) = 0.17157 \cdot 2^r$

Proof. Follows directly from the formulae in Table 1 and the partition from Table 3. □

In the next theorem, we prove the irregularity entropy measures for $BF(r)$

Theorem 2. *Let $BF(r)$ is an r-dimensional butterfly network with $r \geq 3$, then for all $\phi(uv)$ in Table 2*

$$ENT_{\phi(uv)}(BF(R)) = log(4 \cdot 2^r)$$

Proof. From Table 1, Table 2, and Table 3. After solving, we can see that all the entropy results are the same. So, we can write

$$\begin{aligned} ENT_{AL}(BF(r)) &= log(8 \cdot 2^r) - \frac{1}{8 \cdot 2^r}[4 \cdot 2^r |2-4| \times log|2-4|] \\ &= log(8 \cdot 2^r) - \frac{1}{8 \cdot 2^r}[8 \cdot 2^r \times log(2)] \\ &= log(4 \cdot 2^r). \end{aligned}$$

$$\begin{aligned} ENT_{IRLU}(BF(r)) &= log(4 \cdot 2^r) - \frac{1}{4 \cdot 2^r}[4 * 2^r \frac{|2-4|}{2} \times log\frac{|2-4|}{2}] \\ &= log(4 \cdot 2^r) - \frac{1}{4 \cdot 2^r}[(4 * 2^r) \times log(1)] \\ &= log(4 \cdot 2^r). \end{aligned}$$

$$\begin{aligned} ENT_{IRRt}(BF(r)) &= log(4 \cdot 2^r) - \frac{1}{4 \cdot 2^r}[\frac{1}{2}|2-4|(4 \cdot 2^r) \times log\frac{1}{2}|2-4|] \\ &= log(4 \cdot 2^r) - \frac{1}{4 \cdot 2^r}[(4 \cdot 2^r) \times log(1)] \\ &= log(4 \cdot 2^r). \end{aligned}$$

$$\begin{aligned} ENT_{IRF}(BF(r)) &= log(16 \cdot 2^r) - \frac{1}{(16 \cdot 2^r)}[4 \cdot 2^r (2-4)^2 \times log(2-4)^2] \\ &= log(16 \cdot 2^r) - \frac{1}{(16 \cdot 2^r)}[(16 \cdot 2^r) \times log(4)] \\ &= log(4 \cdot 2^r). \end{aligned}$$

$$\begin{aligned} ENT_{IRA}(BF(r)) &= log(0.1715)2^r - \frac{1}{(0.1715)2^r}[4 \cdot 2^r(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}})^2 \times log(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}}) \\ &= log(0.1715)2^r - \frac{1}{(0.1715)2^r}[4 \cdot 2^r(\frac{3-2\sqrt{2}}{4}) \times log(\frac{3-2\sqrt{2}}{4})] \\ &= log(0.1715)2^r - \frac{1}{(0.1715)2^r}[(0.1715)2^r \times log(0.0428)] \\ &= log(4 \cdot 2^r). \end{aligned}$$

□

Let G be a graph of the Benes network. So, the cardinality of the vertex set for the Benes network is $2^r(2r+1)$ and for the edge set is $r2^{r+2}$. The Benes network of dimension 3 is shown in Figure 3.

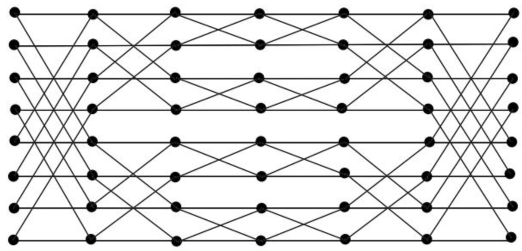

Figure 3. **Graphical representation of $B(3)$**

The edge set of $B(r)$ can be partitioned in terms of degrees of its end vertices as shown in Table 4. As the cardinality of the edges with end degrees $(2,4)$ for the

Edge Type	Number of Edges
(2,4)	$4(2^r)$
(4,4)	$4(r-1)2^r$

Table 4. **Edge partition of Benes network**

Benes network is exactly the same as in the Butterfly network $BF(r)$, that is, $(4 \cdot 2^r)$. So, the results for the Benes network and the entropy of the Benes network are the same as the butterfly network.
Let $G = H(r)$ be a graph of the horizontal cylindrical Benes network. Clearly, the cardinality of vertex set for horizontal cylindrical Benes network is $(2r+1)(2^r-1)$ and for edge set is $2^r(2^{r+1}-1)$. The graph of the horizontal cylindrical Benes network for $r=3$ is shown in Figure 4.
We can see that the edges of $H(r)$ admit the following partition.

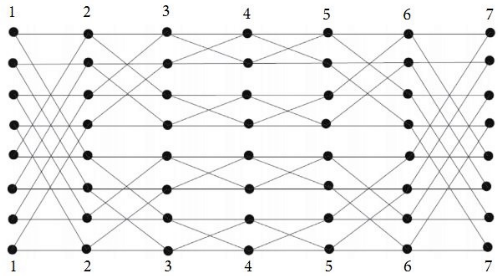

Figure 4. **Graphical representation of $H(3)$**

Edge Type	Number of Edges
(2,6)	4
(3,4)	4
(3,6)	2
(4,2)	$2^{r+2} - 12$
(4,4)	$4(r-1)(2^r - 3)$
(4,6)	$8(r-1)$
(6,6)	$2(r-1)$

Table 5. **Edge partition of $H(r)$**

Theorem 3. *Let $G = H(r)$ is an r-dimensional horizontal cylindrical Benes network with $r \geq 3$, then*

1. $AL(H(r)) = 2 \cdot 2^{r+2} + 16r - 14$
2. $IRLU(H(r)) = 2^{r+2} + 4r - 4.6666$
3. $IRRt(H(r)) = 2^{r+2} + 8r - 7$
4. $IRF(H(r)) = 4 \cdot 2^{r+2} + 4.8r + 6$
5. $IRA(H(r)) = 2^{r+2}(1.4571) + 8r(0.4166) + 1.5883$

Proof. Using Table 1 and Table 5

$$\begin{aligned} AL(H(r)) &= |2-6|4 + |3-4|4 + |3-6|2 + |4-2|(2^{r+2} - 12) + |4-6|(8r-8) \\ &= 2 \cdot 2^{r+2} + 16r - 14. \end{aligned}$$

$$\begin{aligned} IRLU(H(r)) &= \frac{|2-6|}{2} \cdot 4 + \frac{|3-4|}{3} \cdot 4 + \frac{|3-6|}{3} \cdot 2 + \frac{|4-2|}{2}(2^{r+2} - 12) \\ &+ \frac{|4-6|}{4}(8r-8) \\ &= 2^{r+2} + 4r - 4.6666. \end{aligned}$$

$$
\begin{aligned}
IRRt(H(r)) &= \frac{1}{2}|2-6|\cdot 4+\frac{1}{2}|3-4|\cdot 4+\frac{1}{2}|3-6|\cdot 2+\frac{1}{2}|4-2|(2^{r+2}-12)\\
&+ \frac{1}{2}|4-6|(8r-8)\\
&= 2^{r+2}+8r-7.
\end{aligned}
$$

$$
\begin{aligned}
IRF(H(r)) &= (2-6)^2\cdot 4+(3-4)^2\cdot 4+(3-6)^2\cdot 2+(4-2)^2(2^{r+2}-12)\\
&+ (4-6)^2(8r-8)\\
&= 4.2^{r+2}+4.8r+6.
\end{aligned}
$$

$$
\begin{aligned}
IRA(H(r)) &= (\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}})^2\cdot 4+(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}})^2\cdot 4\\
&+ (\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}})^2\cdot 2+(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{2}})^2\cdot 2^{r+2}\\
&- 12+(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{6}})^2(8r-8)\\
&= (\frac{1}{2}+\frac{1}{6}-\frac{2}{\sqrt{12}})\cdot 4+(\frac{1}{3}+\frac{1}{4}-\frac{2}{\sqrt{12}})\\
&\cdot\ 4+(\frac{1}{2}+\frac{1}{6}-\frac{2}{\sqrt{12}})\cdot 4\\
&+ (\frac{1}{3}+\frac{1}{6}-\frac{2}{\sqrt{18}})+(\frac{1}{4}+\frac{1}{2}-\frac{2}{\sqrt{8}})\\
&\cdot\ 2^{r+2}-12+(\frac{1}{4}+\frac{1}{6}-\frac{2}{\sqrt{24}})(8r-8)\\
&= 2^{r+2}(1.4571)+8r(0.4166)+1.5883.
\end{aligned}
$$

□

Theorem 4. *Let $H(r)$ is an r-dimensional entropic horizontal cylindrical Benes network with $r\geq 3$, then*

1. $ENT_{AL}(H(r)) = log(2\cdot 2^{r+2}+16r-14)-\frac{1}{(2\cdot 2^{r+2}+16r-14)}$ $[2\cdot 2^{r+2}(0.3010)+16r(0.3010)+0.4556]$
2. $ENT_{IRLU}(H(r)) = log(2^{r+2}+4r-4.6666)-\frac{1}{(2^{r+2}+4r-4.6666)}$ $[-4r(0.3010)+5.3844]$
3. $ENT_{IRRt}(H(r)) = log(2^{r+2}+8r-7)-\frac{2.3344}{2^{r+2}+8r-7}$
4. $ENT_{IRF}(H(r)) = log(4\cdot 2^{r+2}+32r+6)-\frac{1}{(4\cdot 2^{r+2}+4.8r+6)}[4\dot{2}^r(0.6020)+32r(0.6020)+$ $55.7082]$
5. $ENT_{IRA}(H(r)) = log(2^{r+2}(1.4571)+8r(0.4166)+1.5883)$ $-\frac{1}{log(2^{r+2}(1.4571)+8r(0.4166)+1.5883)}[2^{r+2}(0.0428)+8r(0.0084)+0.2056]$

Proof. From Table 1, Table 2 and Table 5

$$\begin{aligned}
ENT_{AL}(H(r)) &= log(2\cdot 2^{r+2}+16r-14)-\frac{1}{(2\cdot 2^{r+2}+16r-14)}[4|2-6|\times log|2-6| \\
&+ |3-4|4\times log|3-4|+|3-6|2\times log|3-6|+|4-2|(2^{r+2}-12) \\
&\times log|4-2|+|4-6|\times 8(r-1)\times log|4-6|] \\
&= log(2\cdot 2^{r+2}+16r-14)-\frac{1}{(2\cdot 2^{r+2}+16r-14)}[16log(4)+4log(1) \\
&+ 6log(3)+(2^{r+2})2log(2)+(8r-8)2log(2)] \\
&= log(2\cdot 2^{r+2}+16r-14)-\frac{1}{(2\cdot 2^{r+2}+16r-14)} \\
&\quad [2*2^{r+2}(0.3010)+16r(0.3010)+0.4556].
\end{aligned}$$

$$\begin{aligned}
ENT_{IRL}(H(r)) &= log(2^r r+2(0.6931)+8r(0.4054)-4.6300) \\
&- \frac{1}{(2^r r+2(0.6931)+8r(0.4054)}\times[4|ln2-ln4]\times log|ln2-ln4|+4|ln3-ln4| \\
&\times log|ln3-ln4|+2|ln3-ln6|\times log|ln3-ln6|+(2^{r+2}-12)|ln4-ln2| \\
&+ (8r-8)||ln4-ln6|\times log|ln4-ln6| \\
&= log(2^{r+2}(0.6931)-8r(0.4054)-4.6300) \\
&- \frac{1}{(2^{r+2}(0.6931)-8r(0.4054)-4.6300)}[2^{r+2}(0.6931)+8r(0.4054)+1.9318].
\end{aligned}$$

$$\begin{aligned}
ENT_{IRLU}(H(r)) &= log(2^{r+2}+4r-4.6666)-\frac{1}{(2^{r+2}+4r-4.6666)}[4\frac{|2-6|}{2}\times log\frac{|2-6|}{2} \\
&+ 4\frac{|3-4|}{3}\times log\frac{|3-4|}{3}+2\frac{|3-6|}{3}\times log\frac{|3-6|}{3}+(2^{r+2}-12)\frac{|4-2|}{2} \\
&\times log\frac{|4-2|}{2}+(8r-8)\frac{|4-6|}{4}\times log\frac{|4-6|}{4} \\
&= log(2^{r+2}+4r-4.6666)-\frac{1}{(2^{r+2}+4r-4.6666)}[-4r(0.3010)+5.3844].
\end{aligned}$$

$$\begin{aligned}
ENT_{IRRt}(H(r)) &= log(2^{r+2}+8r-7)-\frac{1}{(2^{r+2}+8r-7)}[4\times\frac{1}{2}|2-6|\times log\frac{1}{2}|2-6| \\
&+ 4\times\frac{1}{2}|3-4|\times log\frac{1}{2}|3-4|+2\times\frac{1}{2}|3-6|\times log\frac{1}{2}|3-6| \\
&+ (2^{r+2}-12)\frac{1}{2}|4-2|\times log\frac{1}{2}|4-2|+(8r-8)\frac{1}{2}|4-6|\times log\frac{1}{2}|4-6|] \\
&= log(2^{r+2}+8r-7)-\frac{2.3344}{2^{r+2}+8r-7}.
\end{aligned}$$

$$\begin{aligned}
ENT_{IRF}(H(r)) &= log(4\cdot 2^{r+2}+32r+6) - \frac{1}{(4\cdot 2^{r+2}+32r+6)}[4(2-6)^2\times log(2-6)^2 \\
&+ 4(3-4)^2\times log(3-4)^2 + 2(3-6)^2\times log(3-6)^2 + (2^{r+2}-12)(4-2)^2 \\
&\times log(4-2)^2 + (8r-8)(4-6)^2\times log(4-6)^2] \\
&= log(4\cdot 2^{r+2}+32r+6) - \frac{1}{(4\cdot 2^{r+2}+32r+6)}[4\cdot 2^r(0.6020) \\
&+ 32r(0.6020)+55.7082].
\end{aligned}$$

$$\begin{aligned}
ENT_{IRA}(H(r)) &= log(2^{r+2}(1.4571)+8r(0.4166)+1.5883) \\
&- \frac{1}{(2^{r+2}(1.4571)+8r(0.4166)+1.5883)} \\
&\cdot[4\cdot(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}})^2\cdot log(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}})^2]+4\times \\
&(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}})^2\times log(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}})^2+2\times(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}})^2 \\
&\times\ log(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}})^2+(2^{r+2}-12)(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{2}})^2\times \\
&\cdot\ log(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{2}})^2+(8r-8)(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{6}})^2\times(\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{6}})^2 \\
&= log(2^{r+2}(1.4571)+8r(0.4166)+1.5883) \\
&- \frac{1}{log(2^{r+2}(1.4571)+8r(0.4166)+1.5883)} \\
&[2^{r+2}(0.0428)+8r(0.0084)+0.2056].
\end{aligned}$$

□

Let $G = V(R)$ be a graph of vertical cylindrical Benes networks. The cardinality of vertex set for $VCB(r)$ network is $r(2^{r+1})$ and for edge set is $r(2^{r+2})$. The graph of the vertical cylindrical Benes network for $r = 3$ is shown in Figure 5. As $V(R)$ is a regular

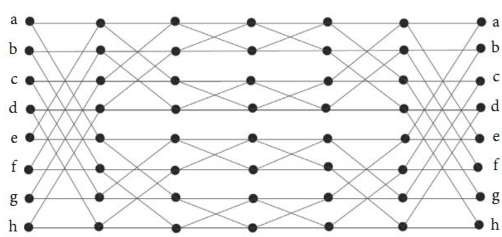

Figure 5. **Graphical representation of** $V(3)$

graph of degree 4, its irregularity and entropy become zero. Hence, in our case, all predefined irregularity indices become zero. Let G be a graph of toroidal cylindrical Benes networks. Clearly, cardinality of vertex set for $T(r)$ network is $2r(2^r - 1)$ and for edge set is $2r(2^{r+1} - 1)$. The toroidal cylindrical Benes network for $r = 3$ is shown in Figure 6. We can see that the edges of $T(r)$ admit the following partition.

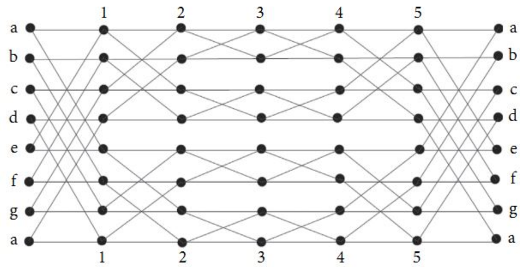

Figure 6. **Graphical representation of $T(3)$**

Edge Type	Number of Edges
$(4, 4)$	$2r.(2^{r+1} - 6)$
$(4, 6)$	$8r$
$(6, 6)$	2^r

Table 6. **Edge partition of $T(r)$**

Theorem 5. *Let $T(r)$ is an r-dimensional toroidal cylindrical Benes network with $r \geq 3$, then*

1. $AL(T(r)) = 16r$
2. $IRLU(T(r)) = 0.5 \cdot 8r$
3. $IRRt(T(r)) = 8r$
4. $IRF(T(r)) = 32r$
5. $IRA(T(r)) = 0.0084 \cdot 8r$

Proof. Using Table 1 and Table 6

$$\begin{aligned} AL(T(r)) &= |4 - 6|8r \\ &= 16r. \end{aligned}$$

$$\begin{aligned} IRLU(T(r)) &= \frac{|4 - 6|}{4} \cdot 8r \\ &= 0.5.8r. \end{aligned}$$

$$\begin{aligned} IRRt(T(r)) &= \frac{1}{2}|4-6| \cdot 8r \\ &= 8r. \end{aligned}$$

$$\begin{aligned} IRF(T(r)) &= (4-6)^2 \cdot 8r \\ &= 32r. \end{aligned}$$

$$\begin{aligned} IRA(T(r)) &= (\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}})^2 \cdot 8r \\ &= 0.0084 \cdot 8r. \end{aligned}$$

□

Theorem 6. *Let $T(r)$ is an r-dimensional toroidal cylindrical Benes network with $r \geq 3$, then for all $\phi(uv)$ in Table 2*

$$ENT_{\phi(uv)}(T(r)) = log(8r).$$

Proof. Using Table 1, Table 2, and Table 6. We obtain the same results for all irregularity entropies.

$$\begin{aligned} ENT_{AL}(T(r)) &= log(16r) - \frac{1}{16r}[8r|2-4| \times (log|2-4|)] \\ &= log(16r) - \frac{1}{16r}[16r \times log(2)] \\ &= log(8r). \end{aligned}$$

$$\begin{aligned} ENT_{IRLU}(T(r)) &= log(0.5)8r - \frac{1}{(0.5)8r}[8r \cdot \frac{|4-6|}{4} \times log\frac{|4-6|}{4}] \\ &= log(0.5)8r - \frac{1}{(0.5)8r}[8r \cdot \frac{1}{2} \times log\frac{1}{2}] \\ &= log(0.5)8r - \frac{1}{(0.5)8r}[(0.5)8r \times log(0.5)] \\ &= log(8r). \end{aligned}$$

$$\begin{aligned} ENT_{IRRt}(T(r)) &= log(8r) - \frac{1}{8r}[8r \cdot \frac{1}{2} \cdot |4-6| \times log\frac{1}{2}|4-6|] \\ &= log(8r) - \frac{1}{8r}[8r \times log(1)] \\ &= log(8r). \end{aligned}$$

$$\begin{aligned} ENT_{IRF}(T(r)) &= log(32r) - \frac{1}{32r}[8r(4-6)^2 \times log(4-6)^2] \\ &= log(32r) - \frac{1}{32r}[32r \times log(4)] \\ &= log(8r). \end{aligned}$$

$$\begin{aligned} ENT_{IRA}(T(r)) &= log(0.8418)8r - \frac{1}{(0.8418)8r}[8r(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}})^2 \times log(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}})^2] \\ &= log(0.8418)8r - \frac{1}{(0.8418)8r}[8r.(\frac{5-2\sqrt{6}}{12}) \times log(\frac{5-2\sqrt{6}}{12})] \\ &= log(0.8418)8r - \frac{1}{(0.8418)8r}[(0.8412)8r \times log(0.8412)] \\ &= log(8r). \end{aligned}$$

□

3. Complexity comparison of ICNs via numeric data and concluding remarks

The current section explicitly presents the advantages of the proposed model and the generalized entropy formulae proved in the previous section. We induce numeric data from the entropy formulae of the $BF(r)$, $B(r)$, $H(r)$, $V(r)$, and $T(r)$ for $r = 3, 4, \ldots, 9$. Since $V(r)$ is regular and all the proposed entropies for $V(r)$ attain the value 0, the entropy values are minimum among all the entropies of ICNs. Moreover, all the entropies of $B(r)$ are the same as the entropies of $BF(r)$. Interestingly, the entropy values of $BF(r)$, $B(r)$, and $T(r)$ do not depend on ϕ. All these facts conclude that Table 7 is sufficient to represent numeric values of $BF(r)$, $B(r)$, and $T(r)$ for $r = 3, 4, \ldots, 9$. Moreover, Table 8 denotes the numeric data of entropy measures for $H(r)$ for $r = 3, 4, \ldots, 9$. All these facts combine together to produce the following important inequalities which show complete entropy pattern among all the studied ICNs

$$ENT_\phi(V(r)) < ENT_\phi(T(r)) < ENT_\phi(BF(r)) = ENT_\phi(B(r)) < ENT_\phi(H(r)), \tag{3.1}$$

for all the ϕ's defined in this paper and for $r = 3, 4 \ldots, 9$. Furthermore, these trends are also shown graphically in Figures 7, 8, 9. Figure 7 displays the entropy trends for the Benes network $B(r)$ over $r = 3, 4, \ldots, 9$, and Figure 8 the corresponding trends for the horizontal cylindrical Benes network $H(r)$; in both cases the entropy measures increase with r, with $H(r)$ attaining higher values than $B(r)$ for each r, in accordance with the inequality (3.1). Figure 9 illustrates the entropy trends for the toroidal Benes network $T(r)$, which lie between those of $V(r)$ (zero, being regular) and $B(r)$, thus supporting the complete ordering of the ICNs given above.

With inequalities in (3.1), we are able to conclude the problem of entropy and complexity comparisons of the ICNs. Note that the problem formulated in [60], with initial investigations and minimum success, emerged towards significant patterns in [63], having partial success in [56], is completely solved by our proposed model. From these results, it becomes evident that, in determining the entropy of ICNs, irregularity plays a vital role compared to distance. Furthermore, our model aligns with the criteria for endorsing a new formula or model, as outlined in [9, 36]. This success prompts future questions for investigation: (1) What impact does our proposed model have on the discriminability of nano-tubes? (2) Can the model be applied to discriminate chemicals without experimentation, such as distinguishing isomers based on their structure? (3) If the answers to the first two questions are negative, can we design a generalized model applicable to discriminating a broader range of objects? However, if the answers to the initial questions are affirmative, our proposed model may surpass the standards set in [9, 36].

r	$ENT_{\phi(uv)}(B(r))$	$ENT_{\phi(uv)}(T(r))$
3	3.46574	3.17805
4	4.15888	3.46574
5	4.85203	3.68888
6	5.54518	3.87120
7	6.23832	4.02535
8	6.93147	4.15888
9	7.26462	4.27667

Table 7. Numeric data comparison between $B(r)$ and $T(r)$ induced from irregularity based information functional entropy

r	ENT_{IRLU}	ENT_{IRRt}	ENT_{IRA}	ENT_{AL}	ENT_{IRF}
3	3.62701	3.84418	4.05125	4.23631	4.86084
4	4.31437	4.46241	4.67302	4.85455	5.52693
5	4.96961	5.06690	5.31250	5.45904	6.18126
6	5.62466	5.68590	5.96939	6.07802	6.84196
7	6.28857	6.32556	6.63964	6.71771	7.51209
8	6.96185	6.98348	7.31898	7.37563	8.19017
9	7.64240	7.65476	8.00406	8.04691	8.87389

Table 8. Numeric data induced from irregularity based information functional entropy for $H(r)$

Table 7 shows that, for every $r = 3, 4, \ldots, 9$, the irregularity entropy of the Benes network $B(r)$ exceeds that of the toroidal Benes network $T(r)$, in line with the ordering $ENT_\phi(T(r)) < ENT_\phi(B(r))$ in (3.1). Both columns increase strictly with r, so entropy grows monotonically as the parameter r increases for these two families. Turning to Table 8, for each fixed r the five entropy values for $H(r)$ satisfy $ENT_{IRLU}(H(r)) < ENT_{IRRt}(H(r)) < ENT_{IRA}(H(r)) < ENT_{AL}(H(r)) < ENT_{IRF}(H(r))$, i.e. the same ordering of information functionals holds across the whole tabulated range. The largest values are attained for ENT_{IRF}, because the functional $\phi(uv) = (d_u - d_v)^2$ assigns greater weight to edges with large degree imbalance than do $|d_u - d_v|$, $\min(d_u, d_v)^{-1}|d_u - d_v|$, or the inverse-square-root terms, so the induced probability mass on edges is most "spread" in that case. Finally, each of the five entropies increases monotonically with r, reflecting growing structural irregularity in $H(r)$ as the dimension parameter increases.

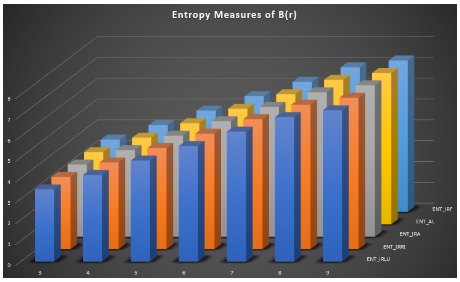

Figure 7. **Entropy trends for $B(r)$**

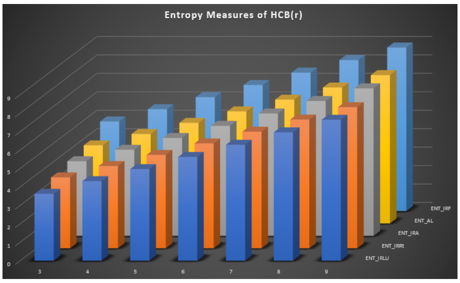

Figure 8. **Entropy trends for $H(r)$**

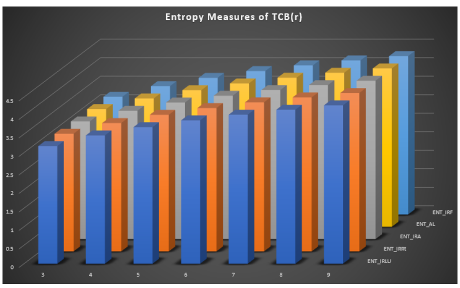

Figure 9. **Entropy trends for $T(r)$**

4. Application to predict robustness measures

To demonstrate applications of newly introduced entropy measures, we develop connections between these entropies and robustness measures, such as average distance (A.D.) of the $BF(r)$, $B(r)$, and $HCB(r)$. More precisely, we use these entropy measures in ML algorithms and choose models and attributes that fit best to predict the robustness measures. The following 3 models produced the best results while predicting the robustness measure average distance of $BF(r)$, $B(r)$, and $HCB(r)$.
Linear Regression. Governing equations: $\hat{y} = \beta_0 + \sum_{i=1}^{p} \beta_i \, x_i$.
For $BF(r)$: Features used: $ENT_{IRRT}(BF(r))$: Fitted model: $\hat{y} = 3.8175 ENT_{IRRT}(BF(r))$
For $B(r)$: Features used: $ENT_{IRLU}(B(r))$: Fitted model: $\hat{y} = 4.0623 ENT_{IRLU}(B(r))$
For $HCB(r)$: Features used: $ENT_{IRA}(HCB(r))$: Fitted model: $\hat{y} = 4.5187 ENT_{IRA}(HCB(r))$.

Table 9. **Performance metrics of ML models via entropy measures across networks**

Model	**Features**	R^2	**MAE**	**MSE**	**RMSE**
Linear Regression($BF(r)$)	$ENT_{IRRT}(BF(r))$	0.9964	0.1556	0.0284	0.1684
Linear Regression($B(r)$)	$ENT_{IRLU}(B(r))$	0.9981	0.1198	0.0168	0.1296
Linear Regression($HCB(r)$)	$ENT_{IRA}(HCB(r))$	0.9992	0.0843	0.0077	0.0876

The model performance evaluation across networks reveals that the best predictive models achieved R^2 values of 0.9964, 0.9981, 0.9992, with mean absolute error (MAE) and root mean square error (RMSE) metrics indicating a high degree of accuracy when using the newly introduced parameters. This exceptional performance suggests that the newly introduced entropies are highly effective in predicting the robustness measure average distance.

5. Conclusion

In this paper, we addressed the problem of entropy-based complexity comparison for the Benes family of interconnection networks, including the butterfly network $BF(r)$, the Benes network $B(r)$, the horizontal cylindrical Benes network $H(r)$, the vertical cylindrical Benes network $V(r)$, and the toroidal Benes network $T(r)$. We proposed an entropy-determining model founded on graph irregularity, using irregularity topological indices to define information functionals in the edge-based entropy formula of Dehmer, and we obtained explicit mathematical formulae for the corresponding entropy measures for each of these networks. The main outcome is a complete characterization: for $r \geq 3$ and for every information functional ϕ considered here, the entropy values satisfy the strict inequality $ENT_\phi(V(r)) < ENT_\phi(T(r)) < ENT_\phi(BF(r)) = ENT_\phi(B(r)) < ENT_\phi(H(r))$, thereby solving the problem posed in earlier work and demonstrating that irregularity plays a vital role in discriminating these ICNs compared with distance-based approaches. Furthermore, we showed that the newly introduced entropy measures perform effectively when used as features to predict robustness measures such as average distance via standard regression, with high R^2 and low error metrics across the networks. The proposed model aligns with the criteria for endorsing a new formula or model as discussed in the literature. Natural directions for future work include investigating the discriminability of nano-tubes and other graph families under this model, exploring applications to chemical structure comparison without experimentation, and developing generalized frameworks applicable to a broader range of interconnection and molecular structures.

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Data availability: No additional data set is required to support the study.

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