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Research Article

The hamiltonicity and pancyclicity of split graphs

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Dedicated to our colleague Dr. Odile Favaron on the occasion of her 88th birthday, with admiration and gratitude.

Abstract: A split graph is a graph whose vertex set can be partitioned into two disjoint subsets (either of which may be empty) such that one subset induces a clique and the other induces an independent set. Regarding the hamiltonicity of such graphs, Dai et al. [Discrete Math. 345 (2022), 112826] conjectured that every r-connected $K_{1,r+1}$-free split graph is hamiltonian. In this paper, we provide a partial verification of this conjecture for the case $r = 4$. Precisely, we show that every 4-connected $\{K_{1,5}, K_{1,5} + e\}$-free split graph is hamiltonian.

Furthermore, we address Bondy's meta-conjecture proposed in 1971, which asserts that almost any nontrivial condition guaranteeing a graph to be hamiltonian also implies the graph to be pancyclic, except for a small number of well-characterized exceptional graphs. We prove that this meta-conjecture holds for split graphs.

Keywords: hamiltonian, pancyclic, split graph, $\{K_{1,5}, K_{1,5} + e\}$-free.

AMS Subject Classification: 05C45, 05C38

1. Introduction

All graphs considered in this paper are finite, undirected and simple. For a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. We use $|X|$

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to denote the cardinality of a set X. For a subgraph H of G and a vertex $v \in V(G)$, let $d_H(v)$ and $N_H(v)$ represent the degree and the neighborhood of v in H. When $H = G$, we simplify these to $d(x)$ and $N(x)$. For two subsets $X, Y \subseteq V(G)$, we write $E(X, Y)$ for the set of edges with one endvertex in X and the other in Y; when one of the sets is a singleton $\{x\}$ or $\{y\}$, we write $E(x, Y)$ or $E(X, y)$. For notation and terminology not defined here, we refer the reader to [7].

A graph G is *k-connected* if $|V(G)| > k$ and the removal of any subset $X \subseteq V(G)$ with $|X| < k$ leaves G connected. Let $\mathcal{F}$ be a family of graphs. A graph G is said to be $\mathcal{F}$-free if G contains no graph of $\mathcal{F}$ as an induced subgraph; a graph of $\mathcal{F}$ is called a *forbidden subgraph*. If $\mathcal{F} = \{F\}$, we say that G is *F-free*. For an integer $r \geq 1$, we denote by $K_{1,r} + e$ the graph obtained from $K_{1,r}$ by adding an edge between two leaves of the star (see Figure 1 as an illustration). We always call a $K_{1,3}$-free graph a *claw-free* graph. A family $\mathcal{F}$ of forbidden subgraphs is called a *forbidden pair* if $|\mathcal{F}| = 2$.

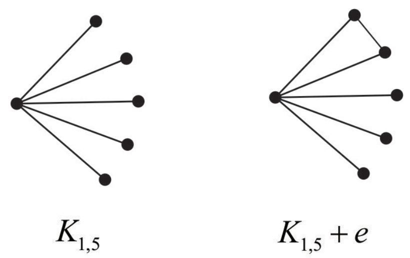

Figure 1. **Illustration of $K_{1,5}$ and $K_{1,5} + e$.**

Let $C = u_0 u_1 \ldots u_m u_0$ be a cycle. Assigning an orientation yields the *positive orientation* $\overrightarrow{C}$; its reverse is $\overleftarrow{C}$. For vertices $u_i, u_j \in V(C)$, the segments from u_i to u_j along $\overrightarrow{C}$ and $\overleftarrow{C}$ are denoted by $u_i \overrightarrow{C} u_j$ and $u_i \overleftarrow{C} u_j$, respectively. For a vertex $v \in V(C)$, its predecessor and successor on $\overrightarrow{C}$ are v^- and v^+, respectively. A cycle on k vertices is a *k-cycle*. Two vertices x, y are consecutive vertices of C if $xy \in E(C)$. A *hamiltonian cycle* is a cycle containing all vertices of G, and a graph containing such a cycle is *hamiltonian*. A *2-factor* of a graph is a collection of pairwise vertex-disjoint cycles whose union covers all vertices. Note that a connected 2-factor is precisely a hamiltonian cycle.

1.1. Hamiltonicity of split graphs

The hamiltonian problem is NP-complete in general and remains computationally intractable even for various restricted graph classes. Since Dirac's seminal degree condition for hamiltonicity [9], the study of sufficient conditions guaranteeing hamiltonian cycles has been one of the central themes in graph theory (see, e.g., the surveys [12, 14, 16]).

A longstanding open problem in this area is the Matthews–Sumner Conjecture [18], which asserts that every 4-connected claw-free graph is hamiltonian. The systematic study of forbidden pairs for hamiltonicity was initiated by Bedrossian [4], who characterized all pairs of connected graphs $\{F_1, F_2\}$ such that every 2-connected $\{F_1, F_2\}$-free graph is hamiltonian. Faudree and Gould [10] subsequently refined this characterization by providing a complete classification of forbidden pairs for hamiltonicity of 2-connected graphs with order at least 10. Since then, the characterization of forbidden pairs for various graph properties has attracted substantial attention; we refer to [15] for hamiltonian properties, and to [2, 3, 13] for 2-factors.

A graph $G = (V(G), E(G))$ is a *split graph* if its vertex set $V(G)$ can be partitioned into two disjoint subsets D and I (either of which may be empty) such that D induces a clique and I induces an independent set. We call (D, I) the split partition of G. Foldes and Hammer [11] gave an equivalent characterization of split graphs: a connected graph is a split graph if and only if it is $\{C_4, C_5, 2K_2\}$-free.

The hamiltonian problem for split graphs has been investigated from both algorithmic and structural perspectives. Renjith and Sadagopan [21] developed polynomial-time algorithms for determining the hamiltonicity of $K_{1,r+1}$-free split graphs when $r = 2, 3$, and proved that the problem becomes NP-complete when $r \geq 4$. In the same paper, they established a necessary and sufficient condition for the hamiltonicity of $K_{1,3}$-free split graphs.

Theorem 1. (Renjith and Sadagopan [21]) *Let G be a $K_{1,3}$-free split graph. Then G is hamiltonian if and only if G is 2-connected.*

Subsequently, Dai et al. [8] extended this line of research by providing a sufficient condition for the hamiltonicity of $K_{1,4}$-free split graphs.

Theorem 2. (Dai et al. [8]) *Let G be a $K_{1,4}$-free split graph. If G is 3-connected, then G is hamiltonian.*

Motivated by Theorem 1 and Theorem 2, Dai et al. [8] proposed the following natural conjecture, which interpolates between these two results.

Conjecture 3. (Dai et al. [8]) Let G be a $K_{1,r+1}$-free split graph. If G is r-connected, then G is hamiltonian.

In this paper, we provide a partial verification of Conjecture 3 for the case $r = 4$. Specifically, we prove the following result.

Theorem 4. *Let G be a $K_{1,5}$-free split graph. If G is 4-connected and $(K_{1,5} + e)$-free, then G is hamiltonian.*

The additional condition of being $(K_{1,5}+e)$-free is necessary in our proof technique, as the presence of this induced subgraph obstructs the extension arguments required for constructing a hamiltonian cycle.
Our approach to prove Theorem 4 relies on two fundamental tools. The first concerns the existence of 2-factors in highly connected graphs with forbidden stars.

Theorem 5. (Aldred et al.[1]) *If G is an r-connected $K_{1,r+1}$-free graph, then G has a 2-factor.*

The second tool addresses hamiltonicity in balanced bipartite graphs through a sharp degree-sum condition. For a split graph G with split partition (D, I) and $|D| = |I|$, the graph $G - E(D)$ obtained by deleting all edges within the clique D forms a balanced bipartite graph. The following classical result of Moon and Moser [19] provides a sufficient condition for such bipartite graphs to be hamiltonian.

Theorem 6. (Moon and Moser [19]) *Let $G = (X, Y)$ be a balanced bipartite graph of order $2n \geq 4$. If $d_G(x) + d_G(y) > n$ for every pair of nonadjacent vertices $x \in X$ and $y \in Y$, then G is hamiltonian.*

1.2. Pancyclicity of split graphs

A graph G is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$. Clearly, a pancyclic graph is hamiltonian, but the converse is not true. In 1971, Bondy [6] proposed the following meta-conjecture, which has guided much of the subsequent research in hamiltonian graph theory.

> **Bondy's meta-Conjecture**: Almost any nontrivial condition that implies a graph is hamiltonian also implies that the graph is pancyclic, with only a small number of well-characterized exceptional graphs.

Bondy [6] supported this meta-conjecture with the following theorem, which strengthens Ore's condition [20] for hamiltonicity.

Theorem 7. (Bondy [6]) *Let G be a graph of order $n \geq 3$. If $d(x)+d(y) \geq n$ for any two nonadjacent vertices x and y of G, then G is either pancyclic, or isomorphic to $K_{n/2,n/2}$.*

Subsequent research has produced further evidence in favor of Bondy's meta-conjecture, such as [5, 17, 22]. In particular, Schmeichel and Hakimi [23] demonstrated that degree conditions on consecutive vertices of a hamiltonian cycle often guarantee its pancyclicity.

Theorem 8. (Schmeichel and Hakimi [23]) *Let G be a graph containing a hamiltonian cycle $C = x_0x_1x_2 \dots x_{n-1}x_0$ with $n \geq 3$. If $d(x_0) + d(x_{n-1}) \geq n$, then G is either* (i) *pancyclic, or* (ii) *bipartite, or* (iii) *missing only an $(n-1)$-cycle.*

For $K_{1,3}$-free split graphs, Dai et al. [8] established that 2-connectedness is both necessary and sufficient for pancyclicity.

Theorem 9. (Dai et al. [8]) Let G be a $K_{1,3}$-free split graph. Then G is pancyclic if and only if G is 2-connected.

The main contribution of this paper regarding pancyclicity is the following theorem, which confirms that Bondy's meta-conjecture holds in its strongest possible form for the class of split graphs: hamiltonicity and pancyclicity are equivalent.

Theorem 10. *Let G be a split graph of order $n \geq 3$. Then G is pancyclic if and only if G is hamiltonian.*

Theorem 9 follows immediately from Theorem 1 and Theorem 10. Moreover, combining Theorem 2, Theorem 4 and Theorem 10, we obtain the following corollaries concerning pancyclicity.

Corollary 1. *Let G be a $K_{1,4}$-free split graph. If G is 3-connected, then G is pancyclic.*

Corollary 2. *Let G be a $K_{1,5}$-free split graph. If G is 4-connected and $(K_{1,5} + e)$-free, then G is pancyclic.*

2. Proof of Theorem 4

Let $G = (V(G), E(G))$ be a 4-connected $\{K_{1,5}, K_{1,5} + e\}$-free split graph with split partition (D, I). We shall prove that G contains a hamiltonian cycle. Throughout this section, we write $N_I(v) = N_G(v) \cap I$ for each vertex $v \in D$, and denote by G' the bipartite graph $G - E(D)$ with bipartition (D, I).

Observation 11. **(i)** For every vertex $v \in D$, $|N_I(v)| \leq 4$.

(ii) If $x \in D$ is a vertex with $|N_I(x)| = 4$, then for every vertex $y \in D \setminus \{x\}$, $|N_I(x) \cap N_I(y)| \geq 2$.

Proof. (i) If $|N_I(v)| \geq 5$ for some $v \in D$, then since I is independent, the subgraph induced by $\{v\} \cup N_I(v)$ in G contains an induced $K_{1,5}$, contradicting the hypothesis. (ii) If there exists a vertex $y \in D \setminus \{x\}$ such that $|N_I(x) \cap N_I(y)| \leq 1$, then since $xy \in E(G)$ (as D is a clique) and I is independent, the set $\{x, y\} \cup N_I(x)$ induces either a $K_{1,5}$ or a $K_{1,5} + e$, a contradiction. □

Since G is 4-connected and $K_{1,5}$-free, we can immediately get the following observation from Theorem 5.

Observation 12. The graph G contains a 2-factor.

By Observation 12, let $\mathcal{F} = \{C_1, C_2, \ldots, C_k\}$ be a 2-factor of G with the minimum possible number k of cycles. We assume $k \geq 2$, for otherwise G is hamiltonian and we have done. For each cycle $C_i \in \mathcal{F}$, we define $\text{diff}(C_i) = |V(C_i) \cap D| - |V(C_i) \cap I|$.

Claim 1. (i) *For every cycle* $C_i \in \mathcal{F}$*,* $\text{diff}(C_i) \geq 0$*.*
(ii) *There exists at most one cycle* $C_i \in \mathcal{F}$ *with* $\text{diff}(C_i) > 0$*.*

Proof. (i) If there is a cycle $C_i \in \mathcal{F}$ with $\text{diff}(C_i) < 0$, then $|V(C_i) \cap I| > |V(C_i) \cap D|$. This forces that C_i contains two consecutive vertices in I, contradicting the fact that I is an independent set.
(ii) Suppose that there are two distinct cycle C_i and C_j in $\mathcal{F}$ with $\text{diff}(C_i) > 0$ and $\text{diff}(C_j) > 0$. Then each of C_i and C_j contains two consecutive vertices in D. Let x, x^+ be consecutive vertices of C_i in D, and let y, y^+ be consecutive vertices of C_j in D. Since D induces a clique, we have $xy^+, yx^+ \in E(G)$. Then $C^* = x^+ \overrightarrow{C_i} x\, y^+ \overrightarrow{C_j} y\, x^+$ is a cycle of G. Consequently, $\mathcal{F}^* = (\mathcal{F} \backslash \{C_i, C_j\}) \cup \{C^*\}$ is a 2-factor with $|\mathcal{F}^*| < |\mathcal{F}|$, contradicting the minimality of k. □

By Claims 1, we may assume without loss of generality that $\text{diff}(C_1) \geq 0$ and $\text{diff}(C_i) = 0$ for all $i \in \{2, \ldots, k\}$. Consequently, C_i is an even cycle for $i \in \{2, \ldots, k\}$. We now distinguish two cases depending on the value of $\text{diff}(C_1)$ to complete the proof of Theorem 4.

Case 1. $\text{diff}(C_1) = 0$.
In this situation $|D| = |I|$, and the bipartite graph G' is balanced. Moreover, every hamiltonian cycle of G avoids edges of $E(D)$. Thus, G is hamiltonian if and only if G' is hamiltonian.

Claim 2. *The bipartite graph* G' *is a* 2*-connected* 4*-regular graph.*

Proof. We establish the regularity and connectivity in two steps.

Step 1. G' is 4-regular.
From Observation 11 (i), $d_{G'}(v) = |N_I(v)| \leq 4$ for every vertex $v \in D$. Since G is 4-connected and I is an independent set, every vertex $u \in I$ satisfies $d_{G'}(u) = d_G(u) \geq \delta(G) \geq \kappa(G) \geq 4$. Counting edges across the bipartition gives

$$4|I| \leq \sum_{u \in I} d_{G'}(u) = |E(G')| = \sum_{v \in D} d_{G'}(v) \leq 4|D|.$$

Since $|D| = |I|$, all inequalities in the above chain must hold with equality. Consequently:

- $\sum_{u \in I} d_{G'}(u) = 4|I|$, which forces $d_{G'}(u) = 4$ for every $u \in I$;

- $\sum_{v\in D} d_{G'}(v) = 4|D|$, which forces $d_{G'}(v) = 4$ for every $v \in D$.

Thus, G' is 4-regular.

Step 2. G' is 2-connected.
Since G' is 4-regular, by Observation 11 (ii), any two distinct vertices in D have at least 2 common neighbors in I. This implies that G' is connected.
If G' is not 2-connected, then there exists a cut-vertex $z \in V(G')$ whose removal disconnects G'. Since G' is bipartite and 4-regular, each component must contain vertices from both $D \setminus \{z\}$. Let G_1 and G_2 be two components of $G' - z$. Take $u_1 \in V(G_1) \cap D$ and $u_2 \in V(G_2) \cap D$. Then $|N_I(u_1) \cap N_I(u_2)| = \emptyset$. This contradicts Observation 11 (ii). Therefore, G' is 2-connected. □

Choose two vertices $x, y \in D$ such that $|N_{G'}(x) \cap N_{G'}(y)| = t$ is as small as possible. By Observation 11 (ii) and Claim 2, we have $2 \le t \le 4$.

Subcase 1.1. $t = 4$.
Since G' is 4-regular, $N_{G'}(x) = N_{G'}(y)$. Let $N_{G'}(x) = N_{G'}(y) = \{v_1, v_2, v_3, v_4\}$. Then $|D| = |I| \ge 4$. For any $z \in D \setminus \{x, y\}$, the minimality of t together with the regularity of G' implies $N_{G'}(x) = N_{G'}(z) = \{v_1, v_2, v_3, v_4\}$. This forces $G' \cong K_{4,4}$, which is hamiltonian. Hence G is hamiltonian, a contradiction.

Subcase 1.2. $t = 3$.
Let $N_{G'}(x) = \{v_1, v_2, v_3, v_4\}$ and $N_{G'}(y) = \{v_1, v_2, v_3, v_5\}$. Then

$$|D| = |I| \ge 5. \tag{2.1}$$

For any $z \in D \setminus \{x, y\}$, the minimality of t yields $|N_{G'}(x) \cap N_{G'}(z)| \ge 3$. Since $d_{G'}(z) = 4$ by Claim 2, the neighborhood $N_{G'}(z)$ must contain at least two vertices from $\{v_1, v_2, v_3\}$. By Claim 2 and a counting argument now gives

$$|D \setminus \{x, y\}| \le \frac{2|\{v_1, v_2, v_3\}| + 3|\{v_4\}|}{3} = 3.$$

Thus,

$$|D| = |I| \le 5. \tag{2.2}$$

By (2.1) and (2.2), we have $|D| = |I| = 5$. Since $d_{G'}(u) + d_{G'}(v) = 8 > 5$ for every pair of nonadjacent vertices $u \in D$ and $v \in I$, G' is hamiltonian by Theorem 6. Therefore G is hamiltonian, a contradiction.

Subcase 1.3. $t = 2$.
Let $N_{G'}(x) = \{v_1, v_2, v_3, v_4\}$ and $N_{G'}(y) = \{v_1, v_2, v_5, v_6\}$. Then

$$|D| = |I| \ge 6. \tag{2.3}$$

Again by the minimality of t, every vertex $z \in D\setminus\{x,y\}$ satisfies $|N_{G'}(x)\cap N_{G'}(z)| \geq 2$ and $|N_{G'}(y) \cap N_{G'}(z)| \geq 2$. Counting incidences yields

$$|D \setminus \{x, y\}| \leq \frac{2|\{v_1, v_2\}| + 3|\{v_3, v_4\}|}{2} = 5.$$

Thus,

$$|D| = |I| \leq 7. \tag{2.4}$$

By (2.3) and (2.4), we have $6 \leq |D| = |I| \leq 7$. Since $d_{G'}(x) + d_{G'}(y) = 8 > 7$ for every pair of nonadjacent vertices $x \in D$ and $y \in I$, G' is hamiltonian by Theorem 6 again. Therefore G is hamiltonian, a contradiction.

Case 2. $\mathrm{diff}(C_1) > 0$.
In this case, there are two consecutive vertices u, u^+ on C_1 such that $u, u^+ \in D$.

Claim 3. *$N_I(u) \cup N_I(u^+) \subseteq V(C_1)$.*

Proof. By symmetry, suppose to the contrary that $N_I(u) \not\subseteq V(C_1)$. Then there exists a vertex $v \in N_I(u) \cap V(C_i)$ for some $C_i \in \mathcal{F} \setminus \{C_1\}$. Since I is independent, $\{v^-, v^+\} \subseteq D$, where v^- and v^+ are the predecessor and successor of v on C_i, respectively. Then $C^* = u\overleftarrow{C_1}u^+v^+\overrightarrow{C_i}vu$ is a cycle of G. Hence $\mathcal{F}' = (\mathcal{F} \setminus \{C_1, C_i\}) \cup \{C^*\}$ is a 2-factor with fewer cycles than $\mathcal{F}$, a contradiction. □

Set $G^* = G' - V(C_1)$. Then G^* is a balanced bipartite graph with bipartition $(D \setminus V(C_1), I \setminus V(C_1))$. Since $\mathcal{F} \setminus \{C_1\}$ is a 2-factor of G^*, the minimum degree $\delta(G^*) \geq 2$.

Claim 4. *The bipartite graph G^* is 2-regular.*

Proof. Suppose that there is a vertex $x \in V(G^*) \cap D$ with $|N_{G^*}(x)| \geq 3$. Let $\{y_1, y_2, y_3\} \subseteq N_{G^*}(x)$. The construction of G^* implies $\{y_1, y_2, y_3\} \subseteq I \setminus V(C_1)$. By Claim 3, $\{y_1, y_2, y_3\} \cap (N_I(u) \cup N_I(u^+)) = \emptyset$. Since D induces a clique, the subgraph induced by $\{x, u, u^+, y_1, y_2, y_3\}$ in G is isomorphic to $K_{1,5} + e$, a contradiction. Thus $|N_{G^*}(x)| \leq 2$ for every $x \in V(G^*) \cap D$. Since $\delta(G^*) \geq 2$ and G^* is a balanced bipartite graph, it follows that G^* is 2-regular. □

By Claim 4, G^* is the union of cycles $C_2, C_3, \ldots, C_k$. Write $V(G^*) \cap I = \{x_1, x_2, \ldots, x_r\}$ with $r \geq 2$.
Since G is 4-connected, $d_G(x_i) \geq 4$ for each $i \in \{1, 2, \ldots, r\}$. Moreover, since G^* is 2-regular, we have $|N_{C_1}(x_i)| \geq 2$ for each i. Let M denote the set of vertices on C_1 that are adjacent to at least one vertex of $\{x_1, x_2, \ldots, x_r\}$ in G^*. By Claim 3, $M \subseteq (D \setminus \{u, u^+\}) \cap V(C_1)$. Let M^- and M^+ denote the sets of predecessors and successors of vertices in M along C_1, respectively. By Claim 3 again, $M^- \cup M^+ \subseteq I$.

Claim 5. *For every vertex $y \in M$, $uy^- \notin E(G)$ and $u^+y^+ \notin E(G)$.*

Proof. Suppose that there is a vertex $y \in M$ such that $uy^- \in E(G)$. Let x_i be the vertex in $V(G^*) \cap I$ with $yx_i \in E(G)$ and let C_2 be the cycle containing x_i. Since $x_i^-, u^+ \in D$, $C^* = u^+\overrightarrow{C_1}y^-u\overleftarrow{C_1}yx_i\overrightarrow{C_2}x_i^-u^+$ is a cycle of G (see Figure 2). Hence, $\mathcal{F}' = (\mathcal{F} \setminus \{C_1, C_2\}) \cup \{C^*\}$ is a 2-factor of G with $|\mathcal{F}'| < |\mathcal{F}|$, contradicting the choice of $\mathcal{F}$. Therefore $uy^- \notin E(G)$. Similarly, $u^+y^+ \notin E(G)$. □

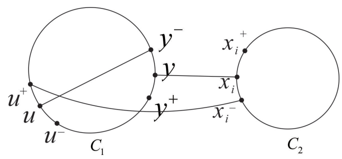

Figure 2. **Illustration of Claim 5.**

By Claim 5, we have $E(u, M^-) = \emptyset$ and $E(u^+, M^+) = \emptyset$.

Claim 6. *For every vertex $y \in M$,*
(i) *y is adjacent to exactly one vertex of $\{x_1, x_2, \ldots, x_r\}$;*
(ii) *either $uy^+ \in E(G)$ or $u^+y^- \in E(G)$.*

Proof. (i) If there is a vertex $y \in M$ and two vertices $x_i, x_j \in \{x_1, x_2, \ldots, x_r\}$ such that $yx_i, yx_j \in E(G)$, then by Claim 3 and Claim 5, the set $\{y, u, y^-, y^+, x_i, x_j\}$ induces either a $K_{1,5}$ of G (if $uy^+ \notin E(G)$) or a $K_{1,5} + e$ (if $uy^+ \in E(G)$) of G, a contradiction. Thus, y is adjacent to at most one vertex of $\{x_1, x_2, \ldots, x_r\}$. Since $y \in M$, by the definition of M, y is adjacent to exactly one vertex of $\{x_1, x_2, \ldots, x_r\}$.
(ii) Suppose that there is a vertex $y \in M$ such that $uy^+ \notin E(G)$ and $u^+y^- \notin E(G)$. Let x_i be the unique neighbor of y in $\{x_1, x_2, \ldots, x_r\}$ (by part (i)). By Claim 3 and Claim 5, the set $\{y, y^-, y^+, u, u^+, x_i\}$ induces a $K_{1,5} + e$ of G, a contradiction. Therefore, either $uy^+ \in E(G)$ or $u^+y^- \in E(G)$. □

Claim 7. *For any distinct vertices $y_i, y_j \in M$ that appear in the order u, u^+, y_i, y_j along C_1, we have $uy_i^+ \notin E(G)$ or $u^+y_j^- \notin E(G)$.*

Proof. Assume, for a contradiction, that there exist two such vertices $y_i, y_j \in M$ with $uy_i^+ \in E(G)$ and $u^+y_j^- \in E(G)$.
(i) y_i and y_j are adjacent to the same vertex x_i of $\{x_1, x_2, \ldots, x_r\}$.
Without loss of generality, let C_2 be the cycle containing x_i. Since $x_i^-, y_j \in D$, $C^* = u^+\overrightarrow{C_1}y_ix_i\overrightarrow{C_2}x_i^-y_j\overrightarrow{C_1}uy_i^+\overrightarrow{C_1}y_j^-u^+$ is a cycle of G.
(ii) y_i and y_j are adjacent to different vertices x_i and x_j of $\{x_1, x_2, \ldots, x_r\}$.
Assume without loss of generality that C_2 and C_3 are the cycles containing x_i and x_j (possibly $C_2 = C_3$), respectively. Since $x_i^-, x_j^- \in D$, we have $x_i^-x_j^- \in E(G)$. If

$C_2 = C_3$, assume without loss of generality that x_i, x_j appear in this order along C_2, then $C^* = u^+\overrightarrow{C_1}y_ix_i\overrightarrow{C_2}x_j^-x_i^-\overleftarrow{C_2}x_jy_j\overrightarrow{C_1}uy_i^+\overrightarrow{C_1}y_j^-u^+$ is a cycle of G (see Figure 3 (a)). If $C_2 \neq C_3$, then $C^* = u^+\overrightarrow{C_1}y_ix_i\overrightarrow{C_2}x_i^-x_j^-\overleftarrow{C_3}x_jy_j\overrightarrow{C_1}uy_i^+\overrightarrow{C_1}y_j^-u^+$ is a cycle of G (see Figure 3(b)). In each case, $\mathcal{F}' = (\mathcal{F} \setminus \{C_1, C_2\}) \cup \{C^*\}$ (or, when $C_2 \neq C_3$, $\mathcal{F}' = (\mathcal{F} \setminus \{C_1, C_2, C_3\}) \cup \{C^*\}$) is a 2-factor of G with $|\mathcal{F}'| < |\mathcal{F}|$. This contradicts the choice of $\mathcal{F}$. □

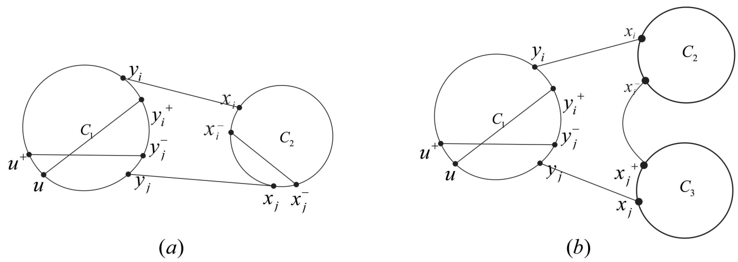

Figure 3. **Illustration of Claim 7.**

Claim 8. $|E(u, M^+)| \leq 3$ *and* $|E(u^+, M^-)| \leq 3$.

Proof. Suppose $|E(u, M^+)| \geq 4$. Choose vertices$y_1, y_2, y_3, y_4 \in M$ such that $\{uy_1^+, uy_2^+, uy_3^+, uy_4^+\} \subseteq E(u, M^+)$. By Claim 3, $\{y_1^+, y_2^+, y_3^+, y_4^+\} \subseteq I \cap V(C_1)$. Since $E(u^+, M^+) = \emptyset$ by Claim 5, the set $\{u, u^+, y_1^+, y_2^+, y_3^+, y_4^+\}$ induces a $K_{1,5}$ of G, a contradiction. So $|E(u, M^+)| \leq 3$. Similarly, $|E(u^+, M^-)| \leq 3$. □

By Claim 6 (i) and the fact $|N_{C_1}(x_i)| = |N_{C_1}(x_i) \cap M| \geq 2$ for every $1 \leq i \leq r$, we have $|M| \geq 2r$. By Claims 6 (ii) and 8, we have

$$\begin{aligned} 6 &\geq |E(u, M^+)| + |E(u^+, M^-)| \\ &= |E(\{u, u^+\}, M^- \cup M^+)| \\ &\geq |M^- \cup M^+| > |M| \geq 2r. \end{aligned} \tag{2.5}$$

So $r \leq 2$. Since $r \geq 2$, we have $r = 2$. Consequently $|\mathcal{F}| = 2$, C_2 is a 4-cycle and $|M| = 4$ or $|M| = 5$.

Write $C_2 = x_1x_1^+x_2x_2^+$. By inequality (2.5), we have $|E(u, M^+)| \geq 3$ or $|E(u^+, M^-)| \geq 3$. By symmetry, assume $|E(u, M^+)| \geq 3$. Then, by Claim 8, $|E(u, M^+)| = 3$. Let $\{y_1, y_2, y_3\} \subseteq M$ be such that $\{uy_1^+, uy_2^+, uy_3^+\} \subseteq E(G)$. By Claim 3 and Claim 5, the set $\{u^+, y_1^+, y_2^+, y_3^+,\} \subseteq I$. Since G is not hamiltonian, the set $\{x_1^+, y_1^+, y_2^+, y_3^+,\}$ is independent. Therefore, $\{u, u^+, x_1^+, y_1^+, y_2^+, y_3^+,\}$ induces a $K_{1,5} + e$, a contradiction.

Now, we have completed the proof of Theorem 4. □

3. Proof of Theorem 10

Let G be a split graph of order $n \geq 3$ with split partition (D, I). The necessity of Theorem 10 is trivial. We now prove the sufficiency. Assume that $C = x_1x_2 \dots x_nx_1$ is a hamiltonian cycle of G. If $I = \emptyset$, then G is a complete graph and hence pancyclic. Next, we assume $I \neq \emptyset$. By Claim 1 (i), $|D| \geq |I|$.

Case 1. $|D| \geq |I| + 1$.
Since $|D| > |I| \geq 1$, there exist two consecutive vertices $x_i, x_{i+1} \in D$ such that $\{x_{i-1}, x_{i+2}\} \cap I \neq \emptyset$. Without loss of generality, assume $\{x_n, x_1\} \subseteq D$ and $x_2 \in I$. Since D induces a clique in G, we have

$$
\begin{aligned}
d_G(x_1) + d_G(x_n) &= (d_D(x_1) + d_I(x_1)) + (d_D(x_n) + d_I(x_n)) \\
&\geq (|D| - 1 + |\{x_2\}|) + (|D| - 1 + 0) \\
&= 2|D| - 1 \\
&\geq |D| + |I| = n.
\end{aligned}
$$

By Theorem 8, G is either (i) pancyclic, or (ii) bipartite, or (iii) missing only an $(n-1)$-cycle. Since $|D| \geq 2$, G is not bipartite. Moreover, since G is a split graph and $x_2 \in I$, we have $x_3 \in D$. Then $C' = x_3x_4 \dots x_nx_1x_3$ is an $(n-1)$-cycle of G. Therefore, G is pancyclic.

Case 2. $|D| = |I|$.
Then $n = 2|D|$ is even. Since I is an independent set and C is a hamiltonian cycle, the vertices of D and I must alternate along C. Thus, we can assume $D = \{x_1, x_3, x_5, \dots, x_{n-1}\}$ and $I = \{x_2, x_4, x_6, \dots, x_n\}$.
For any integer ℓ with $3 \leq \ell \leq n$:

- if ℓ is odd, then $C_\ell = x_1x_2 \dots x_\ell x_1$ is an ℓ-cycle of G;
- if ℓ is even, then $C_\ell = x_1x_3x_4x_5 \dots x_{\ell+1}x_1$ is an ℓ-cycle of G.

Therefore, G is pancyclic.
This completes the proof of Theorem 10. □

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