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Research Article

Robust Toland-Fenchel-Lagrange duality for DC problem under uncertainty

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Abstract: This paper concerns the robust duality theory for a DC problem under uncertainty. We use a "robust" qualification condition to establish robust Toland-Fenchel-Lagrange duality property. We deduce robust strong Lagrange duality to the case when we have an uncertain conical convex problem.

Keywords: DC problem, Toland-Fenchel-Lagrange duality, uncertain, conical convex problem, robust.

AMS Subject Classification: 90C17, 90C25, 90C26, 90C46

1. Introduction

In the field of nonconvex optimization, DC problem plays an interesting and important part because of its theoretical aspects as well as its wide range of applications in economics, operations research, optimal control, mechanics and others ([21]). Mathematical problems dealing with DC functions are called DC Problems.
A function $k : X \to \mathbb{R} \cup \{+\infty\}$ is called a DC function, if it can be decomposed as a difference of two convex functions. Most functions encountered in practice are DC function [33]. For example convex and concave functions are particular case of DC functions. Functions which can be represented as differences of convex functions were considered by Alexandrov (1949, 1950) [1, 2] and Landis (1951) [27], and some time

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later by Hartman (1959) who proved a number of important properties. Many real world problems possess this mathematical model [3–6, 21]. There are different types of algorithms which can be applied to problems dealing with DC functions such as the Branch-and-Bound algorithm and the Cutting Plane algorithm ([21]).

The Slater types conditions [17, 26, 29] are constraint qualifications for convex and DC optimization useful for duality theory. Due to the fact that these conditions are often not satisfied for many problems in applications, a Farkas-Minkovski condition [15, 16, 18, 22, 23] has been developed to extends such a type of constraint qualifications. A more general closedness condition called (CC) [15] has been used to derive strong duality and optimality conditions for the DC problem in [19]. It notices that real-world problems of constrained optimization often involve input data that are uncertain due to modelling or measurement errors [9, 12, 13]. Various approaches have been developed for treating uncertainty in optimization like deterministic and stochastic approaches [8, 10, 11, 24, 25, 30, 31]. In this paper, we consider a robust optimization framework [9, 10] for studying duality theory for DC problem with data uncertainty. The duality theory is very important in the area of constrained optimization. We use a closedness condition to establish robust Toland-Fenchel-Lagrange duality.

Let X, Y be a real locally convex Hausdorff topological vector spaces, X^*, Y^*, their respective topological dual, endowed with the weak*-topologies; $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous (l.s.c.) convex functions, C be a closed convex cone of Y, $\mathcal{U}$ be a set of uncertain parameters and $h_u : X \longrightarrow Y$ be a $C-$level-closed convex function for all $u \in \mathcal{U}$.

We consider the following uncertain DC problem :

$$(P): \qquad \inf(f(x) - g(x)) \quad \text{s.t} \quad x \in X, \;\; h_u(x) \in -C.$$

The robust counterpart of the problem (P) is the problem

$$(RP): \qquad \inf(f(x) - g(x)) \quad \text{s.t} \quad x \in X, \;\; h_u(x) \in -C \qquad \forall u \in \mathcal{U}.$$

The outline of the paper is as follows. Section 2 presents notations and preliminaries of convex analysis that will be used later in the paper. In section 3 we recall Lagrange, Fenchel, Fenchel-Lagrange, Toland and Toland-Fenchel-Lagrange duality theories. Section 4 establishes robust Toland-Fenchel-Lagrange duality. In section 5 we apply the robust Toland-Fenchel-Lagrange duality theory for an uncertain conical convex problem.

2. Notations and preliminaries

Let X be a locally convex Hausdorff topological vector space with topological dual X^* and $(.,.)$ be the standard bilinear coupling function between X and X^*.

2.1. Notations

Given a function $k : X \to \mathbb{R} \cup \{+\infty\}$, we note by :

- $\mathrm{dom}k = \{x \in X \mid k(x) < +\infty\}$ the effective domain of k. One says that k is proper if $\mathrm{dom}k$ is non-empty.
- $\mathrm{epi}k = \{(x,t) \in X \times \mathbb{R} \mid k(x) \leq t\}$, the epigraph of the function k. Recall that k is convex if and only if $\mathrm{epi}k$ is convex, k is lower semi-continuous if and only if $\mathrm{epi}k$ is closed.

For $x^* \in X^*$ and $x \in X$, we denote $x^*(x) = (x^*, x)$. The Legendre-Fenchel conjugate of $k : X \to \mathbb{R} \cup \{+\infty\}$ is the function

$$k^* : X^* \to \mathbb{R} \cup \{+\infty\}, \quad k^*(x^*) = \sup_{x \in X}\{(x, x^*) - k(x)\},$$

which is convex and weak* lower semi-continuous.

Given a subset A of X, we denote by i_A the indicator function of A defined on X by $i_A(x) = 0$ if $x \in A$ and $i_A(x) = +\infty$ otherwise, $co(A)$ its convex hull, $\overline{A}$ its closure, $\overline{co}(A)$ its closed convex hull.

On the dual space X^* we only consider the weak* topology, and for any subset B of X^* we simply denote by $\overline{B}$ the weak* closure of B.
Given $E \subset \mathbb{R} \cup \{+\infty\}$, we write $\min E$ (respectively $\max E$) instead of $\inf E$ (respectively $\sup E$) when the infimum (respectively supremum) of E is attained.

Let Y be another locally convex Hausdorff topological vector space and $C \subset Y$ a nonempty closed convex cone. The C-epigraph of a mapping $h : X \to Y$, is the set

$$\mathrm{epi}_C h = \{(x,y) \in X \times Y : y - h(x) \in C\},$$

and the C-level set of h at level $y \in Y$ is defined as $\{x \in X : h(x) \in y - C\}$.
h is convex if $\mathrm{epi}_C h$ is convex or

$$\forall x_1, x_2 \in X, \forall t \in [0,1], \qquad h_u(tx_1 + (1-t)x_2) - th_u(x_1) - (1-t)h_u(x_2) \in -C.$$

We shall say that h is C-epi-closed convex if $\mathrm{epi}_C h$ is closed and convex, and that h is C-level-closed convex if its C-level set at level y is closed and convex for each $y \in Y$.
We denote by

$$C^+ = \{\lambda \in Y^* \ : \ (y\ ,\ \lambda) \geq 0,\ \forall y \in C\},$$

the positive polar cone of C.
Given $\lambda \in C^+$, we denote by λh the function defined on X by

$$\lambda h(x) = (\lambda, h(x)), \qquad \text{for all } x \in X.$$

2.2. Preliminaries

Let us consider the characteristic cone associated to the system of inequalities $\{x \in X \ : \ \lambda h(x) \leq 0, \ \forall \lambda \in C^{+}\}$,

$$K_h = \bigcup_{\lambda \in C^{+}} \mathrm{epi}(\lambda h)^{*}.$$

It is known that, without any convexity assumption on h, K_h is always a convex cone. If h is a $C-$convex mapping and $h^{-1}(-C) \neq \emptyset$. Then ([7])

$$\mathrm{epi} i^{*}_{h^{-1}(-C)} = cl(K_h). \tag{2.1}$$

Lemma 1 ([7]). *Let f, g two proper, l.s.c. convex functions. One has*

$$epi(f+g)^{*} = cl(epif^{*} + epig^{*}).$$

Moreover, $\mathrm{epi} f^{*} + \mathrm{epi} g^{*}$ is weak$^{*}-$closed, if at least one of both functions f or g is continuous at some point of $\mathrm{dom} f \cap \mathrm{dom} g$ (see [15]).

3. Lagrange, Fenchel, Fenchel-Lagrange dual

In this section, we use perturbational approach [14, 20] to recall some types of dual problems for a given primal problem. Consider in this section the problem

$$(P) \qquad \inf f(x) \quad \text{s.t} \quad g(x) \in -C.$$

3.1. The Lagrange dual

The perturbation function associated to (P) is the function $\phi_L : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\phi_L(x,y) = \begin{cases} f(x) & \text{if } g(x) \in y - C \\ +\infty & \text{otherwise} \end{cases}$$

The dual problem to (P) is

$$(D_L) \qquad \sup_{\lambda \in Y^{*}} \{-\phi^{*}_L(0,\lambda)\}$$

equivalent to,

$$(D_L) \qquad \sup_{\lambda \in Y^{*}} \inf_{x \in X} \{f(x) + \lambda g(x)\}$$

(D_L) is the Lagrange dual problem of (P).

3.2. The Fenchel dual

The perturbation function is $\phi_F : X \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$, defined by

$$\phi_F(x,y) = \begin{cases} f(x+y) & \text{if } \ g(x) \in -C, \\ +\infty & \text{otherwise.} \end{cases}$$

the Fenchel dual problem to (P) is

$$\begin{aligned} (D_F) \quad & \sup_{p \in X^*} \{-\phi_F^*(0,p)\} && \Leftrightarrow \\ (D_F) \quad & \sup_{p \in X^*} \{-f^*(p) + \inf_{x \in A} \langle p, x\rangle\} && \Leftrightarrow \\ (D_F) \quad & \sup_{p \in X^*} \{-f^*(p) - i_A^*(-p)\}, && \end{aligned}$$

where $A = \{x \in X : \ g(x) \in -C\}$.

3.3. Fenchel-Lagrange dual

We consider the perturbation function $\phi_{FL} : X \times X \times Y \to \mathbb{R} \cup \{+\infty\}$,

$$\phi_{FL}(x,y,z) = \begin{cases} f(x+y) & \text{if } g(x) \in z - C, \\ +\infty & \text{otherwise.} \end{cases}$$

The following dual problem

$$\begin{aligned} (D_{FL}) \quad & \sup_{p \in X^*, \lambda \in Y^*} \{-\phi_{FL}^*(0,p,\lambda)\} && \Leftrightarrow \\ (D_{FL}) \quad & \sup_{p \in X^*, \lambda \in C^*} \{-f^*(p) + \inf_{x \in X} [\langle p, x\rangle + \lambda g(x)]\} && \Leftrightarrow \\ (D_{FL}) \quad & \sup_{p \in X^*, \lambda \in C^*} \{-f^*(p) - (\lambda g)^*(-p)\} && \end{aligned}$$

is the Fenchel-Lagrange dual which is a "combination" of Lagrange and Fenchel duals. The Fenchel-Lagrange dual has been introduced in [34] for the case of finite-dimensional optimization problems.

3.4. Toland dual and Toland-Fenchel-Lagrange dual problems

Let us consider the following DC problem :

$$(q) \qquad \inf(f(x) - g(x)) \quad \text{s.t} \quad x \in X, \quad h(x) \in -C.$$

The Toland dual problem associated to (q) is defined by

$$(q_T^*) \qquad \inf_{x^*\in X^*}\{g^*(x^*) - (f + i_{-C}\circ h)^*(x^*)\}.$$

The Toland-Fenchel-Lagrange dual problem is defined by

$$(q_{TFL}^*) \qquad \inf_{x^*\in X^*}\max_{\lambda\in C^+}\{g^*(x^*) - (f + \lambda h)^*(x^*)\}.$$

Let us recall the result on duality for DC problems established by Toland in [32].

Lemma 2 (Toland dual Theorem). *Let X be a locally convex Hausdorff topological vector space. Let $f, g : X \to \mathbb{R}\cup\{+\infty\}$. Assume that g is proper convex, lower-semicontinuous function and f is an arbitrary function. Then*

$$\inf_{x\in X}\{f(x) - g(x)\} = \inf_{x^*\in X^*}\{g^*(x^*) - f^*(x^*)\}$$

The following Lemma present the Toland-Fenchel-Lagrange dual problem for DC problem (q) where there are not data uncertainty [19, 26].

Lemma 3. *If the closedness condition*

$$(CC) \qquad epif^* + \bigcup_{\lambda\in C^+} epi(\lambda h)^* \qquad is \;\; weak^* - closed$$

holds, then

$$\inf(q) = \inf_{x^*\in X^*}\max_{\lambda\in C^+}\{g^*(x^*) - (f + \lambda h + i_C)^*(x^*)\}.$$

4. Robust Toland-Fenchel-Lagrange duality

Let us define the function $p : X \to \mathbb{R}\cup\{+\infty\}$ by

$$p = \sup_{u\in\mathcal{U}}(f + i_{F_u}) = f + \sup_{u\in\mathcal{U}} i_{F_u} = f + i_F,$$

where

$$F_u = \{x\in X, h_u(x)\in -C\} \;\text{ and }\; F = \bigcap_{u\in\mathcal{U}} F_u = \{x\in X, h_u(x)\in -C, \forall u\in\mathcal{U}\}.$$

We have $\text{dom}p = F\cap\text{dom}f$. Let us recall the following Lemma useful for the sequel.

Lemma 4 ([7]). *If $F \cap \mathrm{dom} f \neq 0$, then*

$$\mathrm{epi} p^* = \overline{co}\left(\bigcup_{u \in \mathcal{U}} \mathrm{epi}(f + i_{F_u})^*\right).$$

Let us consider the "robust" closedness condition

$$(RCC) \qquad \mathrm{epi} f^* + \bigcup_{\lambda \in C^+, u \in \mathcal{U}} \mathrm{epi}(\lambda h_u)^* \ \ \textit{is weak}^* - \textit{closed convex}.$$

Remark 1. Note that a similar condition has been used in [28] to establish the robust strong duality for convex conical optimization problem under uncertainty.

Let us establish the following result which is useful for the sequel.

Theorem 1. *If $F \cap \mathrm{dom} f \neq \emptyset$ and (RCC) holds, then for all $x^* \in X^*$,*

$$p^*(x^*) = \min_{u \in \mathcal{U}} \min_{\lambda \in C^+} \min_{y \in X^*} \{f^*(y) + (\lambda h_u)^*(x^* - y)\}.$$

Proof. Let $x^* \in X^*$. For all $y \in X^*$, $u \in \mathcal{U}$, $\lambda \in C^+$ and $x \in F$. We have

$$f^*(y) \geq (y, x) - f(x)$$

and

$$(\lambda h_u)^*(x^* - y) \geq (x^* - y, x) - (\lambda h_u)(x).$$

Then,

$$\begin{aligned} f^*(y) + (\lambda h_u)^*(x^* - y) &\geq (x^*, x) - f(x) - (\lambda h_u)(x) \\ &\geq (x^*, x) - f(x) - i_F(x) \end{aligned}$$

This implies that $f^*(y) + (\lambda h_u)^*(x^* - y) \geq (f + i_F)^*(x^*)$.
Thus

$$\inf_{u \in \mathcal{U}} \inf_{\lambda \in C^+} \inf_{y \in X^*} \{f^*(y) + (\lambda h_u)^*(x^* - y)\} \geq p^*(x^*).$$

Let us prove the converse inequality.
If $x^* \notin \mathrm{dom} p^*$, we have $p^*(x^*) = +\infty$. Then the equality holds in this case.

Let $x^* \in \text{dom}p^*$.

$$
\begin{aligned}
\text{epi}p^* =& \overline{\text{co}}\left(\bigcup_{u\in\mathcal{U}} \text{epi}(f+i_{F_u})^*\right) \qquad \text{(Lemma 4)}\\
=& \overline{\text{co}}\left(\bigcup_{u\in\mathcal{U}} cl(\text{epi}f^* + \text{epi}i^*_{F_u})\right) \qquad \text{(Lemma 1)}\\
=& \overline{\text{co}}\left(\bigcup_{u\in\mathcal{U}} cl(\text{epi}f^* + cl(K_{h_u}))\right) \qquad \text{(equation (2.1))}\\
=& \overline{\text{co}}\left(\bigcup_{u\in\mathcal{U}} cl(\text{epi}f^* + K_{h_u})\right)\\
=& \overline{\text{co}}\left(\bigcup_{u\in\mathcal{U}} \text{epi}f^* + K_{h_u}\right)\\
=& \text{epi}f^* + \bigcup_{u\in\mathcal{U},\lambda\in C^+} \text{epi}(\lambda h_u)^* \qquad (RCC).
\end{aligned}
$$

As $(x^*, p^*(x^*)) \in \text{epi}p^*$ then, there exist $\overline{\lambda} \in C^+, \overline{u} \in \mathcal{U}, (y,r) \in \text{epi}f^*$, $(v,s) \in \text{epi}(\overline{\lambda}h_{\overline{u}})$ such that :

$$(x^*, p^*(x^*)) = (y,r) + (v,s).$$

then $y + v = x^*$ and $p^*(x^*) = r + s$. As $f^*(y) \le r$, $(\overline{\lambda}h_{\overline{u}})^*(v) \le s$ then $p^*(x^*) \ge f^*(y) + (\overline{\lambda}h_{\overline{u}})^*(v)$, i.e $p^*(x^*) \ge f^*(y) + (\overline{\lambda}h_{\overline{u}})^*(x^* - y) \ge \inf_{u\in\mathcal{U}} \inf_{\lambda\in C^+} \inf_{y\in X^*}\{f^*(y) + (\lambda h_u)^*(x^* - y)\}$. So

$$p^*(x^*) = \min_{u\in\mathcal{U}} \min_{\lambda\in C^+} \min_{y\in X^*}\{f^*(y) + (\lambda h_u)^*(x^* - y)\}.$$

$\square$

Corollary 1. *If $F \cap \text{dom}f \neq \emptyset$ and (RCC) holds, then for all $x^* \in X^*$,*

$$p^*(x^*) = \min_{u\in\mathcal{U}} \min_{\lambda\in C^+}\{(f + \lambda h_u)^*(x^*)\}$$

Proof. Let $x^* \in X^*$. From Theorem 1, there exist $\overline{\lambda} \in C^+$, $\overline{u} \in \mathcal{U}$ and $\overline{y} \in X^*$ such that :

$$p^*(x^*) = f^*(\overline{y}) + (\overline{\lambda}h_{\overline{u}})^*(x^* - \overline{y}).$$

It follows that for all $x \in X$,

$$p^*(x^*) \ge \langle \overline{y}, x\rangle - f(x) + \langle x^* - \overline{y}, x\rangle - \overline{\lambda}h_{\overline{u}}(x) = \langle x^*, x\rangle - (f + \overline{\lambda}h_{\overline{u}})(x).$$

Thus

$$p^*(x^*) \geq (f + \overline{\lambda} h_{\overline{u}})^*(x^*) \geq \inf_{u\in\mathcal{U}} \inf_{\lambda\in C^+} \{(f + \lambda h_u)^*(x^*)\}.$$

Let $\lambda \in C^+,\ u \in \mathcal{U}$, one gets

$$\begin{aligned}(f + \lambda h_u)^*(x^*) &= \sup_{x\in X} \{\langle x^*, x\rangle - (f + \lambda h_u)(x)\} \\ &\geq \sup_{x\in F} \{\langle x^*, x\rangle - f(x)\} \qquad (\text{because } h_u(x) \in -C) \\ &= p^*(x^*).\end{aligned}$$

Thus,

$$\inf_{u\in\mathcal{U}} \inf_{\lambda\in C^+} \{(f + \lambda h_u)^*(x^*)\} \geq p^*(x^*).$$

Hence,

$$\min_{u\in\mathcal{U}} \min_{\lambda\in C^+} \{(f + \lambda h_u)^*(x^*)\} = p^*(x^*).$$

□

Let us give now the robust duality result.

Theorem 2 (robust Toland-Fenchel-Lagrange duality). *If $F \cap \mathrm{dom} f \neq \emptyset$ and (RCC) holds, then ,*

$$\inf(RP) = \inf_{x^*\in X^*} \max_{u\in\mathcal{U},\lambda\in C^+} \{g^*(x^*) - (f + \lambda h_u)^*(x^*)\}. \tag{4.1}$$

Proof. We have $\inf(RP) = \inf(p - g)$.
According to the Toland dual Theorem (Lemma 2), one gets :

$$\inf(RP) = \inf_{x\in X} \{(p - g)(x)\} = \inf_{x^*\in X^*} \{g^*(x^*) - p^*(x^*)\}.$$

Using corollary 1, for all $x^* \in X^*$,

$$\begin{aligned}\inf(RP) &= \inf_{x^*\in X^*} \{g^*(x^*) - \min_{u\in\mathcal{U}} \min_{\lambda\in C^+} \{(f + \lambda h_u)^*(x^*)\}\} \\ &= \inf_{x^*\in X^*} \max_{u\in\mathcal{U},\lambda\in C^+} \{g^*(x^*) - (f + \lambda h_u)^*(x^*)\}.\end{aligned}$$

□

Remark 2. So, we have established the equality between the robust value and the robust Toland-Fenchel-Lagrange dual value. It is therefore necessary to find a condition achieving this equality, because it doesn't always fullfields. Thus this condition allows us to address this uncertain problem because the dual problem is in general easier to solve.

Remark 3. Using Theorem 1, (4.1) becomes

$$\inf(RP) = \inf_{x^*\in X^*} \max_{u\in\mathcal{U},\lambda\in C^+} \max_{y\in X^*}\{g^*(x^*) - f^*(y) - (\lambda h_u)^*(x^* - y)\}.$$

Example 1. Consider the following uncertain optimization problem :

$$(P): \qquad \inf\{x_1 - x_1^2 - x_2^2\} \quad \text{s.t} \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (1+u)x_2 \le 2.$$

Where u is uncertain and it belongs to $\mathcal{U} = [-\frac{1}{2}, \frac{1}{2}]$. Let $X = \mathbb{R}^2, C = C^+ = \mathbb{R}^+, Y = \mathbb{R}$, $f(x) = x_1, g(x) = x_1^2 + x_2^2$ and for each $u \in \mathcal{U}, h_u(x) = (1+u)x_2 - 2$.
f, g are convex, continuous functions on $\mathbb{R}^2$. For each $u \in \mathcal{U}, h_u$ is C-level-closed convex and continuous. Furthermore $(0,0) \in F = \{(x_1,x_2) \in \mathbb{R}^2 \mid h_u(x_1,x_2) \le 0 \quad \forall u \in \mathcal{U}\}$ and $\mathrm{dom} f = \mathbb{R}$. Then $F \cap \mathrm{dom} f \neq \emptyset$.
For all $y = (y_1, y_2) \in \mathbb{R}^2$,

$$f^*(y) = \sup_{(x_1,x_2)\in\mathbb{R}^2}\{y_1x_1 + y_2x_2 - x_1\} = \begin{cases} 0 & \text{if } y_1 = 1,\ y_2 = 0 \\ +\infty & \text{else} \end{cases}.$$

For each $u \in \mathcal{U}$, for all $(y_1, y_2) \in \mathbb{R}^2$ and for all $\lambda \in C^+ = \mathbb{R}^+$, we have

$$(\lambda h_u)^*(y_1,y_2) = \sup_{(x_1,x_2)\in\mathbb{R}^2}\{y_1x_1 + y_2x_2 - \lambda(1+u)x_2 + 2\lambda\} = \begin{cases} 2\lambda & \text{if } y_1 = 0,\ y_2 = \lambda(1+u) \\ +\infty & \text{else} \end{cases}.$$

So,

$$\mathrm{epi} f^* + \bigcup_{\lambda\in\mathbb{R}^+,u\in\mathcal{U}} \mathrm{epi}(\lambda h_u)^* = \{(1, \lambda(1+u), 2\lambda + r) : \quad \lambda, r \in \mathbb{R}^+, u \in \mathcal{U}\}.$$

Then $\mathrm{epi} f^* + \bigcup_{\lambda\in\mathbb{R}^+,u\in\mathcal{U}} \mathrm{epi}(\lambda h_u)^*$ is a closed convex subset.
Also, for all $y = (y_1, y_2) \in \mathbb{R}^2$, we have

$$g^*(y) = \sup_{(x_1,x_2)\in\mathbb{R}^2}\{y_1x_1 + y_2x_2 - x_1^2 - x_2^2\} = \frac{y_1^2}{4} + \frac{y_2^2}{4}$$

and

$$(f+\lambda h_u)^*(y) = \sup_{(x_1,x_2)\in\mathbb{R}^2}\{y_1x_1 + y_2x_2 - x_1 - \lambda(1+u)x_2 + 2\lambda\} = \begin{cases} 2\lambda & \text{if } y_1 = 1,\ y_2 = \lambda(1+u) \\ +\infty & \text{else.} \end{cases}$$

Hence,

$$\inf_{x^*\in X^*} \max_{\lambda\in C^+,u\in\mathcal{U}}\{g^*(x^*) - (f+\lambda h_u)^*(x^*)\} = \inf_{x_2\in\mathbb{R}} \max_{u\in\mathcal{U}}\left\{\frac{1}{4} + \frac{x_2^2}{4} - \frac{2x_2}{1+u}\right\}.$$

Note that,

$$\max_{u\in\mathcal{U}}\left\{\frac{1}{4} + \frac{x_2^2}{4} - \frac{2x_2}{1+u}\right\} = \begin{cases} \frac{1}{4} + \frac{x_2^2}{4} - \frac{4x_2}{3} & \text{if } x_2 \ge 0 \\ \frac{1}{4} + \frac{x_2^2}{4} - 4x_2 & \text{if } x_2 \le 0. \end{cases}$$

So,

$$\inf_{x^*\in X^*} \max_{\lambda\in C^+,u\in\mathcal{U}}\{g^*(x^*) - (f+\lambda h_u)^*(x^*)\} = -\frac{103}{36}.$$

From the robust Toland-Fenchel-Lagrange duality theorem (Theorem 2), it follows that the robust value of the problem (P) is $-\frac{103}{36}$.

5. Robust Toland-Fenchel-Lagrange duality for an uncertain conical convex problem

Consider the problem (P) and suppose that $g \equiv 0$. One gets the following uncertain conical convex problem :

$$(P'): \qquad \inf f(x) \quad \text{s.t} \quad x \in X, \;\; h_u(x) \in -C,$$

with his robust counterpart,

$$(RP'): \qquad \inf f(x) \quad \text{s.t} \quad x \in X, \;\; h_u(x) \in -C, \; \forall u \in \mathcal{U}.$$

In this case the robust Toland-Fenchel-Lagrange dual of (P') is

$$(D'_{TFL}) \qquad \max_{u \in \mathcal{U}, \lambda \in C^+} \{-(f + \lambda h_u)^*(0)\}$$

Because

$$g^*(x^*) = \begin{cases} 0 & \text{if } x^* = 0 \\ +\infty & \text{else} \end{cases}.$$

So, we can derive the robust strong Lagrange duality result.

Corollary 2. *If $F \cap \text{dom} f \neq \emptyset$ and (RCC) holds, then ,*

$$\inf(RP') = \max_{u \in \mathcal{U}, \lambda \in C^+} \inf_{x \in X} \{f(x) + \lambda h_u(x)\}. \tag{5.1}$$

Proof. According to the robust Toland-Fenchel-Lagrange duality Theorem, one has

$$\inf(RP') = \max_{u \in \mathcal{U}, \lambda \in C^+} \{-(f + \lambda h_u)^*(0)\}.$$

As $\max_{u \in \mathcal{U}, \lambda \in C^+} \{-(f+\lambda h_u)^*(0)\} = \max_{u \in \mathcal{U}, \lambda \in C^+} \inf_{x \in X} \{f(x)+\lambda h_u(x)\}$ we have the result. □

Remark 4. So, Theorem 2 allows us to obtain the robust duality result in [[28], Corollary 3.1]. Thus Theorem 2 generalizes the robust strong Lagrange duality.

Example 2. Let us consider the following uncertain conical convex problem

$$(P): \qquad \inf x_1^2 \quad \text{s.t} \quad x = (x_1, x_2) \in \mathbb{R}^2, \;\; (1+u)x_2 \leq 2.$$

Where u is uncertain and it belongs to $\mathcal{U} = [-\frac{1}{2}, \frac{1}{2}]$. Let $X = \mathbb{R}^2, C = C^+ = \mathbb{R}^+, Y = \mathbb{R}$, $f(x) = x_1^2$ and for each $u \in \mathcal{U}, h_u(x) = (1+u)x_2 - 2$.

f, g are convex, continuous functions on $\mathbb{R}^2$. For each $u \in \mathcal{U}, h_u$ is C-level-closed convex and continuous. Furthermore $(0,0) \in F = \{(x_1,x_2) \in \mathbb{R}^2 \mid h_u(x_1,x_2) \leq 0 \ \ \forall u \in \mathcal{U}\}$ and $\mathrm{dom} f = \mathbb{R}$. Then $F \cap \mathrm{dom} f \neq \emptyset$.
For all $y = (y_1, y_2) \in \mathbb{R}^2$,

$$f^*(y) = \sup_{(x_1,x_2)\in\mathbb{R}^2} \{y_1x_1 + y_2x_2 - x_1\} = \begin{cases} \frac{y_1^2}{4} & \text{if } y_2 = 0 \\ +\infty & \text{else} \end{cases} .$$

For each $u \in \mathcal{U}$, for all $(y_1, y_2) \in \mathbb{R}^2$ and for all $\lambda \in C^+ = \mathbb{R}^+$, we have

$$(\lambda h_u)^*(y_1,y_2) = \sup_{(x_1,x_2)\in\mathbb{R}^2} \{y_1x_1 + y_2x_2 - \lambda(1+u)x_2 + 2\lambda\} = \begin{cases} 2\lambda & \text{if } y_1 = 0, \ y_2 = \lambda(1+u) \\ +\infty & \text{else} \end{cases} .$$

So,

$$\mathrm{epi} f^* + \bigcup_{\lambda\in\mathbb{R}^+, u\in\mathcal{U}} \mathrm{epi}(\lambda h_u)^* = \left\{ \left(y_1, \lambda(1+u), \frac{y_1^2}{4} + 2\lambda + r \right) : \ \ y_1 \in \mathbb{R}, \lambda, r \in \mathbb{R}^+, u \in \mathcal{U} \right\}.$$

Then $\mathrm{epi} f^* + \bigcup_{\lambda\in\mathbb{R}^+, u\in\mathcal{U}} \mathrm{epi}(\lambda h_u)^*$ is a closed convex subset.
Therefore let us calculate $\max_{u\in\mathcal{U},\lambda\in C^+} \inf_{x\in X} \{f(x) + \lambda h_u(x)\}$. We have

$$\inf_{x\in X} \{f(x) + \lambda h_u(x)\} = \inf_{x\in X} \{x_1^2 + \lambda(1+u)x_2 - 2\lambda\} = \begin{cases} 0 & \text{if } \lambda = 0 \\ -\infty & \text{else.} \end{cases}$$

Then

$$\max_{u\in\mathcal{U},\lambda\in C^+} \inf_{x\in X} \{f(x) + \lambda h_u(x)\} = 0$$

Let us recall that

$$(RP): \qquad \inf x_1^2 \quad \text{s.t} \quad x = (x_1,x_2) \in \mathbb{R}^2, \ \ (1+u)x_2 \leq 2 \ \ \forall u \in \left[-\frac{1}{2}, \frac{1}{2} \right].$$

So, it follows from Corollary 2 that $\inf(RP) = 0$.

Conclusion

Considering an uncertain DC problem we have established robust Toland-Fenchel-Lagrange duality which allowed us to deduce the robust Lagrange duality theory for an uncertain conical convex problem.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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