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Research Article

Mond-Weir duality and optimality for nonsmooth vector bilevel programs in terms of approximations

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Abstract: This paper addresses a nonsmooth vector bilevel optimization problem. By reformulating the hierarchical model into a single-level problem using the optimal value approach, we establish sufficient optimality conditions for the nonsmooth extremum problem. These conditions rely on the assumption that the functions involved exhibit generalized convexity, characterized through their approximations. Additionally, we introduce Mond-Weir-type dual models for these problems and prove several duality theorems within the generalized convexity framework. The applicability of these optimality conditions is demonstrated through illustrative examples of nonsmooth vector bilevel optimization problems.

Keywords: bilevel optimization, approximations, sufficient optimality conditions, duality.

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1. Introduction

Stackelberg games, often referred to as bilevel programming problems, feature a hierarchical structure involving two decision-makers: upper-level (also called the leader) and the lower-level (also called the follower). Both aim to minimize their respective objective functions, subject to interrelated constraints. The decision variables are

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divided such that neither player can dominate the other completely. The leader acts first by selecting $z \in \mathbb{R}^{n_1}$, which influences but does not entirely dictate the follower's response. The follower then reacts to the leader's decision by choosing $w \in \mathbb{R}^{n_2}$ to minimize their own cost function. Over time, bilevel programming problems have seen significant advancements, particularly in the development of optimality conditions and duality theories. In modern times, bilevel programming has become a focal point in optimization theory, drawing considerable attention due to its widespread study. Bilevel optimization represents a highly active area of research in optimization theory, owing to its broad applicability across diverse fields such as environmental economics [10, 11], agriculture [6], portfolio management [9], engineering [13, 26], medicine [40].

Simultaneously, problems in this domain pose notable challenges, involving both theoretical and numerical complexities. Numerous researchers have made substantial contributions to this area, as evidenced by the works of [3, 7, 12, 14, 15, 27, 30, 32, 33, 39]. Recent studies have contributed to the advancement of variational analysis, control theory, and nonsmooth optimization from both theoretical and applied perspectives. Treanţă et al. [38] investigated reciprocal solution existence results for classes of vector variational control inequalities and demonstrated their applicability to physical models, thereby strengthening the connection between variational theory and real-world systems. Treanţă and Dragu [34] derived Euler–Lagrange equations for gradient-type Lagrangians and established associated conservation laws, enriching the analytical foundations of variational calculus. In another direction, Guo et al. [20] addressed nonsmooth interval optimization problems by exploiting interval-valued symmetric invexity, providing effective tools for handling uncertainty and nonsmoothness. Furthermore, Treanţă, et al. [35] developed efficiency criteria for multicost variational models driven by generalized functionals, while Treanţă, et al. [36] proposed dual-model-based efficiency criteria for multiple cost control problems. Collectively, these contributions highlight ongoing progress in variational optimization, nonsmooth analysis, and multi-criteria decision-making.

This paper investigates vector bilevel optimization problems using the optimistic approach. We propose that the leader, assuming the follower's cooperative behavior, postulates that the follower will invariably choose the solution within their parametric sub-problem that optimally aligns with the leader's objective function. We examine a vector bilevel optimization problem $(\mathcal{P})$ with the following structure:

$$(\mathcal{P}) : \begin{cases} \min\limits_{z,w} \quad \Theta(z,w) = \big(\Theta_1(z,w), \ldots, \Theta_n(z,w)\big) \\ s.\ t. \\ \qquad\quad \theta_i(z,w) \leq 0, \qquad \forall\ i \in \mathcal{I} = \{1, \ldots, p\}, \\ \qquad\quad w \in \urcorner(z), \end{cases}$$

for any $z \in \mathbb{R}^{n_1}$, $\urcorner(z)$ represents the set of optimal solutions of another optimization problem $(\mathcal{P}_z)$ defined as:

$$(\mathcal{P}_z) : \begin{cases} \min\limits_{w} & \Phi(z,w) \\ s.\ t. & \phi_j(z,w) \leq 0, \quad \forall\, j \in \mathcal{J} = \{1,\ldots,q\}, \end{cases}$$

Here, $\Theta_s,\ \theta_i,\ \Phi,\ \phi_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R},\ i \in \mathcal{I},\ j \in \mathcal{J},\ s \in \mathcal{S} := \{1,\ldots,n\}$ and $n \geq 0,\ p, q, n_1, n_2$ are integers.
Within the context of vector bilevel optimization problem $(\mathcal{P})$, we explore the concept of weak efficiency. Defining the feasible region of $(\mathcal{P})$ as

$$\mathfrak{E} = \{(z,w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : w \in \daleth(z),\ \theta_i(z,w) \leq 0,\ i \in \mathcal{I}\},$$

a solution $(\breve{z}, \breve{w})$ is weakly efficient if,

$$\Theta(z,w) - \Theta(\breve{z},\breve{w}) \notin -int\mathbb{R}^n_+, \qquad \forall\ (z,w) \in \mathfrak{E}.$$

In nonsmooth environments, Theorem 1 presented by Gadhi and Ohda [17] ensures only the necessity of optimality conditions for problem $(\mathcal{P})$. Questions remain about under what assumptions these conditions become sufficient.
Recent research has made significant strides in establishing optimality and duality results for nonsmooth bilevel optimization problems by using generalized subdifferential frameworks. In this context, the contributions of Pandey et al. [30] and Saeed et al. [31] are noteworthy, as they have employed Clarke subdifferentials to develop sufficient conditions for optimality and to explore Wolfe or Mond-Weir type duality results. Their findings are framed mainly within the assumptions of local Lipschitz continuity and generalized convexity. Treanţă et al. [37] examined nondifferentiable bilevel multiobjective optimization models governed by tangential subdifferentials. Their research produced sufficient efficiency criteria and established Mond-Weir type duality via optimal value reformulation. Meanwhile, Gadhi and Zerrifi [19] focused on single objective bilevel optimization problems, developing sufficient optimality conditions by using tangential subdifferentials along with Dini generalized convexity assumptions.
Unlike these approaches, which depend on Clarke or tangential subdifferential frameworks and require specific Lipschitz-type or tangentially convexity conditions, the proposed work employs an approximation tool. Since approximations need not be be closed or bounded and can even apply to non-Lipschitz or discontinuous functions, the developed sufficient efficiency criteria hold under weaker regularity assumptions. This extends the applicability of optimality and duality results to a broader class of nonsmooth bilevel optimization problems.
The application of generalized derivatives in establishing optimality conditions for nonsmooth optimization has been a topic of significant interest and extensive research globally over the past few decades. Various types of generalized derivatives have been proposed, each with distinct requirements for the regularity of the mappings. Some

derivatives are defined for locally Lipschitz mappings, while others are applicable to continuous mappings. Each type is suited to specific classes of problems. In this work, we employ the concept of approximations as a generalized derivative, which was first introduced in [23] and later extended to the second-order in [1].
In nonsmooth analysis, approximations play a crucial role. Notably, for locally Lipschitz functions, both the Clarke subdifferential [8] and approximate Jacobians [22] are approximations. Drawing upon necessary optimality conditions from Gadhi and Odha [17] and the general concept of approximations [1, 4, 16, 23, 25], we establish sufficient optimality conditions for $(\mathcal{P})$ using a generalized convexity introduced by Gadhi et al. [18]. We then formulate the Mond-Weir dual problem and prove duality theorems without any constraint qualification. Illustrative examples are provided to demonstrate our findings. Our methodology leverages the optimal value function of the lower-level problem, denoted by $\nu(z)$ and defined as

$$\nu(z) := \inf_{w} \big\{ \Phi(z,w) : \phi_j(z,w) \leq 0, j \in \mathcal{J} \big\}, \ \forall z \in \mathbb{R}^{n_1}.$$

To our knowledge, no prior work has investigated sufficient optimality conditions and duality results for vector bilevel problems using approximations.
The subsequent sections of the manuscript follow this organization: Section 2 encompasses fundamental definitions and preliminary results. Sections 3 and 4 focus on presenting sufficient optimality conditions and duality theorems.

2. Preliminaries

This paper operates within the framework of n-dimensional Euclidean spaces denoted by $\mathbb{R}^n$, equipped with a standard norm denoted by $\|\cdot\|$. We use familiar notation for the inner product as $\langle \cdot, \cdot \rangle$ and represent the closed line segment as $[z,w]$ for $z, w \in \mathbb{R}^n$. The open line segment $]z,w[$ is defined by

$$]z,w[:= \{\alpha(z-w) + w : 0 < \alpha < 1\}.$$

We consider $\mathcal{K} \neq \emptyset \subseteq \mathbb{R}^n$, cl $\mathcal{K}$, *co* $\mathcal{K}$, *int* $\mathcal{K}$ and pos $\mathcal{K}$ denote the closure, convex hull, interior and convex cone (including the origin) of $\mathcal{K}$ respectively.
Consider $\mathcal{K} \subseteq \mathbb{R}^n$, and $y \in$ cl $(\mathcal{K})$.

- The contingent cone

$$\mathfrak{T}_{\mathcal{K}}(y) := \Big\{ \rho \in \mathbb{R}^n : \exists\ t_k \downarrow 0, \ \exists\ \rho_k \to \rho, \quad y + t_k \rho_k \in \mathcal{K} \Big\}.$$

- The negative polar cone

$$\mathcal{K}^\circ = \{\rho \in \mathbb{R}^n : \langle \rho, y \rangle \leq 0, \ \forall\ y \in \mathcal{K}\}.$$

- The strictly negative polar cone

$$\mathcal{K}^s = \{\rho \in \mathbb{R}^n : \langle \rho, y\rangle < 0, \ \forall\ y \in \mathcal{K}\backslash\{0\}\}.$$

We denote by $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ the collection of all continuous linear map from n-dimensional Euclidean space $\mathbb{R}^n$ to R. Additionally, $\mathcal{B}_{\mathbb{R}^n}$ signifies the closed unit ball centered at the origin in $\mathbb{R}^n$.
Following the established literature, we employ the approximation definition as presented by Allali and Amahroq [1], which is a revised version of the original definition by Jourani and Thibault [23].

Definition 1. **([1]):** Let $\breve{y} \in \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}$. The set $\mathcal{A}_\psi(\breve{y}) \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is called approximation of ψ at $\breve{y}$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\psi(y) - \psi(\breve{y}) \in \mathcal{A}_\psi(\breve{y})(y - \breve{y}) + \epsilon\ \|y - \breve{y}\|\ \mathcal{B}_{\mathbb{R}^n}$$

for all $y \in \breve{y} + \delta\mathcal{B}_{\mathbb{R}^n}$.

Remark 1. It is important to note that approximations are, in general, not unique. In particular, for any $\mathcal{A} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, the singleton set $\{\mathcal{A}\}$ constitutes an approximation of ψ at $\breve{y}$ if and only if $\mathcal{A}$ coincides with the Fréchet derivative of ψ at $\breve{y}$. Moreover, as shown in [1, Proposition 2.1.2], if ψ is locally Lipschitz, then its Clarke subdifferential [8] at $\breve{y}$ provides an approximation of ψ.

Proposition 1. *([2]): If $\mathfrak{g},\ \mathfrak{h} : \mathbb{R}^n \to \mathbb{R}$ have approximations $\mathcal{A}_\mathfrak{g}(\breve{y})$ and $\mathcal{A}_\mathfrak{h}(\breve{y})$ at $\breve{y}$, then $\mathcal{A}_\mathfrak{g}(\breve{y}) + \mathcal{A}_\mathfrak{h}(\breve{y})$ is an approximation of $\mathfrak{g} + \mathfrak{h}$ at $\breve{y}$.*

Proposition 2. *([2]): Consider the function $\mathfrak{g}_i : \mathbb{R}^n \to \mathbb{R},\ i \in K := \{1, \ldots, p\}$ have approximations $\mathcal{A}_{\mathfrak{g}_i}(\breve{y})$ at $\breve{y}$. Let*

$$\mathfrak{f}(y) := \max\{\mathfrak{g}_i(y) : i \in K\},$$

and

$$K(\breve{y}) := \{i \in K : \mathfrak{g}_i(\breve{y}) = \mathfrak{f}(\breve{y})\}.$$

Then, $co\{\mathcal{A}_{\mathfrak{g}_i}(\breve{y}) : i \in K(\breve{y})\}$ is an approximation of $\mathfrak{f}$ at $\breve{y}$.

Proposition 3. *([17]): Let $\mathfrak{h} : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, and let $a, b \in \mathbb{R}^n$. Assume that for each $c \in [a, b]$, the function $\mathfrak{h}$ admits a nonempty compact approximation $\mathcal{A}_\mathfrak{h}(c)$ at c. Then, there exists $c \in [a, b]$ and $c^* \in co\,\mathcal{A}_\mathfrak{h}(c)$ such that*

$$\mathfrak{h}(b) - \mathfrak{h}(a) = \langle c^*, b - a\rangle.$$

Remark 2. For functions that are locally Lipschitz continuous, it is sometimes possible to construct subdifferential approximations that are strictly smaller than the Clarke subdifferential, as illustrated in Example 1. Moreover, such approximations can, in some cases, consist of only a finite set of elements.

Example 1. Consider $\mathfrak{h}:\mathbb{R}^2\to\mathbb{R}$ given by:

$$\mathfrak{h}(z_1,z_2)=\begin{cases}2z_1^3\cos\left(\frac{1}{z_1}\right)+\frac{1}{2}|z_2| & \text{if } z_1\neq 0,\\ \frac{1}{2}|z_2| & \text{if } z_1=0.\end{cases}$$

On one hand, $\mathfrak{h}$ is locally Lipschitz at $(\breve{z}_1,\,\breve{z}_2)=(0,0)$ and $\partial_c\mathfrak{h}(\breve{z}_1,\,\breve{z}_2)=[-2,2]\times[-\frac{1}{2},\frac{1}{2}]$. On the other hand $\mathcal{A}_{\mathfrak{h}}(\breve{z}_1,\,\breve{z}_2):=\left\{\left(0,-\frac{1}{2}\right),\left(0,\frac{1}{2}\right)\right\}$ is a compact approximation of $\mathfrak{h}$ at $(\breve{z}_1,\,\breve{z}_2)$. Note that $co\,\mathcal{A}_{\mathfrak{h}}(\breve{z}_1,\,\breve{z}_2)=\{0\}\times\left[-\frac{1}{2},\,\frac{1}{2}\right]$. It is clear that $co\,\mathcal{A}_{\mathfrak{h}}(\breve{z}_1,\,\breve{z}_2)\subsetneq\partial_c\mathfrak{h}(\breve{z}_1,\,\breve{z}_2)$.

Remark 3. Example 1 clearly demonstrates that certain optimality conditions, when formulated using appropriate approximations, can yield precise and meaningful results, even in the case of locally Lipschitz functions.

Now, we re-call the definitions of $\mathcal{A}$-convexity, $\mathcal{A}$-pseudoconvexity and $\mathcal{A}$-quasiconvexity of a function formulated in terms of its approximation. The aforesaid definitions have been derived by Gadhi et al. [18].

Definition 2. **([18]):** Consider a function $\psi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ and $(\breve{y}_1,\,\breve{y}_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Suppose ψ has an approximation $\mathcal{A}_\psi(\breve{y}_1,\,\breve{y}_2)\subset\mathcal{L}(\mathbb{R}^n,\mathbb{R})$ at $(\breve{y}_1,\,\breve{y}_2)$. Then:

- ψ is $\mathcal{A}$-convex at $(\breve{y}_1,\,\breve{y}_2)$ iff the relation

$$\psi(y_1,\,y_2)-\psi(\breve{y}_1,\,\breve{y}_2)\geq\Big\langle\zeta,\;(y_1,\,y_2)-(\breve{y}_1,\,\breve{y}_2)\Big\rangle,\quad\forall\,\zeta\in\mathcal{A}_\psi(\breve{y}_1,\,\breve{y}_2)$$

  holds for all $(y_1,\,y_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

- ψ is $\mathcal{A}$-quasiconvex at $(\breve{y}_1,\,\breve{y}_2)$ iff the relation

$$\psi(y_1,\,y_2)\leq\psi(\breve{y}_1,\,\breve{y}_2)\Longrightarrow\Big\langle\zeta,\;(y_1,\,y_2)-(\breve{y}_1,\,\breve{y}_2)\Big\rangle\leq 0,\quad\forall\,\zeta\in\mathcal{A}_\psi(\breve{y}_1,\,\breve{y}_2)$$

  holds for all $(y_1,\,y_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

- ψ is $\mathcal{A}$-pseudoconvex at $(\breve{y}_1,\,\breve{y}_2)$ iff the relation

$$\psi(y_1,\,y_2)<\psi(\breve{y}_1,\,\breve{y}_2)\Longrightarrow\Big\langle\zeta,\;(y_1,\,y_2)-(\breve{y}_1,\,\breve{y}_2)\Big\rangle<0,\quad\forall\,\zeta\in\mathcal{A}_\psi(\breve{y}_1,\,\breve{y}_2)$$

  holds for all $(y_1,\,y_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

- ψ is strictly $\mathcal{A}$-pseudoconvex at $(\breve{y}_1,\,\breve{y}_2)$ iff the relation

$$\psi(y_1,\,y_2)\leq\psi(\breve{y}_1,\,\breve{y}_2)\Longrightarrow\Big\langle\zeta,\;(y_1,\,y_2)-(\breve{y}_1,\,\breve{y}_2)\Big\rangle<0,\quad\forall\,\zeta\in\mathcal{A}_\psi(\breve{y}_1,\,\breve{y}_2)$$

  holds for all $(y_1,\,y_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},(y_1,\,y_2)\neq(\breve{y}_1,\,\breve{y}_2)$.

3. Sufficient optimality conditions

Let $\breve{n} := 1 + p + q$ and $I := \{1, \dots, \breve{n}\}$. For each $i \in I$, define a function $\Upsilon_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$

$$\Upsilon_i(z, w) := \begin{cases} \theta_i(z, w) & \text{if } \ i \in \{1, \dots, p\} \\ \phi_{i-p}(z, w) & \text{if } \ i \in \{p+1, \dots, p+q\} \\ \Phi(z, w) - \nu(z) & \text{if } \ i = 1 + p + q. \end{cases}$$

Based on Outrata [29], we can rewrite the original problem $(\mathcal{P})$ as an equivalent problem

$$(\mathcal{P}^*) : \begin{cases} \min\limits_{z,w} \ \Theta(z, w) = \big(\Theta_1(z, w), \dots, \Theta_n(z, w)\big) \\ \ s.\ t. \\ \qquad\qquad \Upsilon_i(z, w) \le 0, \quad i \in I, \\ \qquad\qquad (z, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{cases}$$

Remark 4. **([17]):** If conditions $(\mathcal{B}_1)$ and $(\mathcal{B}_2)$ hold, the optimization problem $(\mathcal{P})$ possesses at least one weak efficient solution.
$(\mathcal{B}_1)$: Functions $\Theta_s,\ s \in \mathcal{S},\ \Phi,\ \theta_i,\ i \in \mathcal{I}$ and $\phi_j,\ j \in \mathcal{J}$ exhibit lower semicontinuity on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, while ν displays upper semicontinuity on $\mathbb{R}^{n_1}$.
$(\mathcal{B}_2)$: Problem $(\mathcal{P}^*)$ admits at least one feasible solution, and its feasible set,

$$\Xi = \Big\{(z, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \theta_i(z, w) \le 0,\ i \in \mathcal{I},\ \phi_j(z, w) \le 0,\ j \in \mathcal{J},\ \Phi(z, w) - \nu(z) \le 0\Big\}$$

is bounded.
Under these conditions, the feasible set Ξ becomes a nonempty compact set, and Θ_s is lower semicontinuous for all $s \in \mathcal{S}$.

Remark 5. **([17]):** Examine the set-valued map $\mathcal{T}(z) : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$ defined as

$$\mathcal{T}(z) := \Big\{w \in \mathbb{R}^{n_2} : \phi_j(z, w) \le 0,\ j \in \mathcal{J}\Big\}.$$

Berge [5, Theorem 1] and Aubin [24, Theorem 2] establish that the upper semicontinuity of the optimal value function ν is guaranteed when Φ is upper semicontinuous and $\mathcal{T}$ is lower semicontinuous [5]. Specifically, $\mathcal{T}(z) = \emptyset$, for all $z \in \mathbb{R}^{n_1}$.

We adopt the following assumptions.

◇ Assumption 1:

 – Functions $\Phi,\ \theta_i,\ i \in \mathcal{I}$ and $\phi_j,\ j \in \mathcal{J}$ have nonempty compact approximations $\mathcal{A}_\Phi(\breve{z}, \breve{w}),\ \mathcal{A}_{\theta_i}(\breve{z}, \breve{w}),\ i \in \mathcal{I}(\breve{z}, \breve{w})$ and $\mathcal{A}_{\phi_j}(\breve{z}, \breve{w}),\ j \in \mathcal{J}(\breve{z}, \breve{w})$ at $(\breve{z}, \breve{w})$.

– $\mathcal{T}(\breve{z})$ exhibits separability, and $w \to \Phi(\breve{z}, w)$ is continuous.

◇ Assumption 2: A neighborhood $\mathcal{V}$ of $(\breve{z}, \breve{w})$ exists such that for all (z, w) in $\mathcal{V}$:

– functions $\Theta_s,\ s \in \mathcal{S}$ have nonempty compact approximations $\mathcal{A}_{\Theta_s}(z, w)$ at (z, w).

– the set-valued maps $\mathcal{A}_{\Theta_s},\ s \in \mathcal{S}$ are upper semicontinuous in $\mathcal{V}$.

To derive the necessary optimality conditions for the bilevel optimization problem $(\mathcal{P})$, Gadhi and Ohda [17] proposed the following constraint qualification.

Definition 3. **([17]):** Let $(\breve{z}, \breve{w}) \in \Xi$. Assume that $\Phi,\ \theta_i,\ i \in \mathcal{I}(\breve{z}, \breve{w})$ and $\phi_j,\ j \in \mathcal{J}(\breve{z}, \breve{w})$ have nonempty approximations $\mathcal{A}_\Phi(\breve{z}, \breve{w}),\ \mathcal{A}_{\theta_i}(\breve{z}, \breve{w}),\ i \in \mathcal{I}(\breve{z}, \breve{w})$ and $\mathcal{A}_{\phi_j}(\breve{z}, \breve{w}),\ j \in \mathcal{J}(\breve{z}, \breve{w})$ at $(\breve{z}, \breve{w})$ and that ν has a nonempty approximation $\mathcal{A}_\nu(\breve{z})$ at $\breve{z}$. Let

$$\Omega(\breve{z}, \breve{w}) := \Big(\bigcup_{i \in \mathcal{I}(\breve{z}, \breve{w})} co\ \mathcal{A}_{\theta_i}(\breve{z}, \breve{w}) \Big) \cup \Big(\bigcup_{j \in \mathcal{J}(\breve{z}, \breve{w})} co\ \mathcal{A}_{\phi_j}(\breve{z}, \breve{w}) \Big) \cup co\Big(\mathcal{A}_\Phi(\breve{z}, \breve{w}) - \mathcal{A}_\nu(\breve{z}) \times \{0\} \Big).$$

We say that the nonsmooth Abadie constraint qualification (NACQ) is satisfied at $(\breve{z}, \breve{w})$ if

$$\Omega(\breve{z}, \breve{w})^\circ \subseteq cl\, co\, \mathfrak{T}_\Xi(\breve{z}, \breve{w}), \tag{3.1}$$

where $\mathcal{I}(\breve{z}, \breve{w}) := \{i \in \mathcal{I} \mid \theta_i(\breve{z}, \breve{w}) = 0\}$ and $\mathcal{J}(\breve{z}, \breve{w}) := \{j \in \mathcal{J} \mid \phi_j(\breve{z}, \breve{w}) = 0\}$.

According to [21, Proposition 4.2.6], the expression (3.1) is equivalent to

$$\mathfrak{T}_\Xi(\breve{z}, \breve{w})^\circ \subseteq cl\ pos\ \Omega(\breve{z}, \breve{w}). \tag{3.2}$$

Gadhi and Ohda [17] have established the following theorem, providing necessary optimality conditions of Karush-Kuhn-Tucker type for the considered bilevel optimization problem $(\mathcal{P})$.

Theorem 1. ***([17]):*** *Let $(\breve{z}, \breve{w}) \in \Xi$ be a local weakly efficient solution of $(\mathcal{P})$ and the nonsmooth constraint qualification (NACQ) be satisfied at $(\breve{z}, \breve{w})$ for $(\mathcal{P})$. Assume that $pos\, \Omega(\breve{z}, \breve{w})$ is closed set and both Assumptions 1-2 are satisfied at $(\breve{z}, \breve{w})$. Then there exist $\mathfrak{t}_s \geq 0,\ s \in \mathcal{S},\ \mathfrak{a}_i \geq 0, i \in \mathcal{I}(\breve{z}, \breve{w}),\ \mathfrak{b}_j \geq 0, j \in \mathcal{J}(\breve{z}, \breve{w}),\ \mathfrak{c} \geq 0$ with $\sum_{s \in \mathcal{S}} \mathfrak{t}_s = 1$ such that*

$$\begin{aligned} (0,0) \in \sum_{s=1}^{n} \mathfrak{t}_s\, co\mathcal{A}_{\Theta_s}(\breve{z}, \breve{w}) &+ \sum_{i \in \mathcal{I}} \mathfrak{a}_i\, co\mathcal{A}_{\theta_i}(\breve{z}, \breve{w}) + \sum_{j \in \mathcal{J}} \mathfrak{b}_j\, co\mathcal{A}_{\phi_j}(\breve{z}, \breve{w}) \\ &+ \mathfrak{c}\ co\big(\mathcal{A}_\Phi(\breve{z}, \breve{w}) - \mathcal{A}_\nu(\breve{z}) \times \{0\} \big), \\ \mathfrak{a}_i\, \theta_i(\breve{z}, \breve{w}) = 0, \quad & \mathfrak{b}_j\, \phi_j(\breve{z}, \breve{w}) = 0. \end{aligned} \tag{3.3}$$

Now, we derive and prove the sufficient conditions for a feasible solution to be weak efficient in the considered bilevel optimization problem $(\mathcal{P})$ under generalized convexity assumptions.

Theorem 2. *Let $(\breve{z},\breve{w}) \in \Xi$ and there exist $F^* = (\mathfrak{t}_s,\ \mathfrak{a}_i,\ \mathfrak{b}_j,\ \mathfrak{c}) \in \mathbb{R}_+^{\breve{n}}$, with $\sum_{s=1}^{n} \mathfrak{t}_s = 1$ such that the necessary optimality condition (3.3) be satisfied. Further, assume that $\Theta_s, s \in \mathcal{S}$ is $\mathcal{A}$-pseudoconvex at $(\breve{z},\breve{w})$, on Ξ, θ_i, $i \in \mathcal{I}(\breve{z},\breve{w})$, ϕ_j, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $(\Phi - \nu)$ are $\mathcal{A}$-quasiconvex at $(\breve{z},\breve{w})$ on Ξ. Then $(\breve{z},\breve{w})$ is a weak efficient solution of $(\mathcal{P})$.*

Proof. Contrary to the result, suppose that $(\breve{z},\breve{w}) \in \Xi$ is not a weakly efficient solution of $(\mathcal{P})$. Hence, by Definition, there exists $(z_0, w_0) \in \Xi$ such that

$$\Theta_s(z_0, w_0) < \Theta_s(\breve{z},\breve{w}), \quad \forall\ s \in \mathcal{S}.$$

Since Θ_s, $s \in \mathcal{S}$ is a $\mathcal{A}$-pseudoconvex at $(\breve{z},\breve{w})$ on Ξ, by Definition 2, we get that the inequality

$$\Big\langle \eta_s,\ (z_0, w_0) - (\breve{z},\breve{w}) \Big\rangle < 0$$

is satisfied for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w})$. Since $\mathfrak{t}_s \in \mathbb{R}_+^n$ with $\sum_{s=1}^{n} \mathfrak{t}_s = 1$, we obtain that the inequality

$$\Bigg\langle \sum_{s=1}^{n} \mathfrak{t}_s \eta_s,\ (z_0, w_0) - (\breve{z},\breve{w}) \Bigg\rangle < 0 \tag{3.4}$$

holds for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w})$.
Using $(z_0, w_0) \in \Xi$ and $(\breve{z},\breve{w}) \in \Xi$ together with the definitions of $\mathcal{I}(\breve{z},\breve{w})$ and $\mathcal{J}(\breve{z},\breve{w})$, we get

$$\theta_i(z_0, w_0) \le \theta_i(\breve{z},\breve{w}) = 0, \quad \forall\ i \in \mathcal{I}(\breve{z},\breve{w}),$$

$$\phi_j(z_0, w_0) \le \phi_j(\breve{z},\breve{w}) = 0, \quad \forall\ j \in \mathcal{J}(\breve{z},\breve{w})$$

$$\Big(\Phi(z_0, w_0) - \nu(z_0)\Big) \le \Big(\Phi(\breve{z},\breve{w}) - \nu(\breve{z})\Big) = 0.$$

Since θ_i, $i \in \mathcal{I}(\breve{z},\breve{w})$, ϕ_j, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $(\Phi - \vartheta)$ are $\mathcal{A}$-quasiconvex at $(\breve{z},\breve{w})$ on Ξ, by Definition 2, the inequalities above yield, respectively, for any $\gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z},\breve{w})$, $i \in \mathcal{I}(\breve{z},\breve{w})$, $\rho_j \in co\mathcal{A}_{\phi_j}(\breve{z},\breve{w})$, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $\xi \in co(\mathcal{A}_{\Phi}(\breve{z},\breve{w}) - \mathcal{A}_{\nu}(\breve{z}) \times \{0\})$,

$$\Big\langle \gamma_i,\ (z_0, w_0) - (\breve{z},\breve{w}) \Big\rangle \le 0, \quad \forall\ i \in \mathcal{I}(\breve{z},\breve{w}),$$

$$\Big\langle \rho_j,\ (z_0, w_0) - (\breve{z},\breve{w}) \Big\rangle \le 0, \quad \forall\ j \in \mathcal{J}(\breve{z},\breve{w}),$$

$$\Big\langle \xi,\ (z_0, w_0) - (\breve{z},\breve{w}) \Big\rangle \le 0.$$

Since $\mathfrak{a}_i \ge 0$, $i \in \mathcal{I}(\breve{z},\breve{w})$, $\mathfrak{b}_j \ge 0$, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $\mathfrak{c} \ge 0$, therefore, the following inequality

$$\Bigg\langle \sum_{i \in \mathcal{I}(\breve{z},\breve{w})} \mathfrak{a}_i \gamma_i + \sum_{j \in \mathcal{J}(\breve{z},\breve{w})} \mathfrak{b}_j \rho_j + \mathfrak{c}\,\xi,\ (z_0, w_0) - (\breve{z},\breve{w}) \Bigg\rangle \le 0$$

holds for any $\gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z},\breve{w}),\ i \in \mathcal{I}(\breve{z},\breve{w}),\ \rho_j \in co\mathcal{A}_{\phi_j}(\breve{z},\breve{w}),\ j \in \mathcal{J}(\breve{z},\breve{w})$ and $\xi \in co(\mathcal{A}_{\Phi}(\breve{z},\breve{w}) - \mathcal{A}_{\nu}(\breve{z}) \times \{0\})$. Taking $\mathfrak{a}_i = 0,\ i \notin \mathcal{I}(\breve{z},\breve{w})$ and $\mathfrak{b}_j = 0,\ j \notin \mathcal{J}(\breve{z},\breve{w})$, we gets

$$\left\langle \sum_{i\in\mathcal{I}} \mathfrak{a}_i\gamma_i + \sum_{j\in\mathcal{J}} \mathfrak{b}_j\,\rho_j + \mathfrak{c}\,\xi,\ (z_0,w_0) - (\breve{z},\breve{w}) \right\rangle \leq 0. \tag{3.5}$$

Adding both sides of (3.4) and (3.5), we get that the inequality

$$\left\langle \sum_{s=1}^{n} \mathfrak{t}_s\eta_s + \sum_{i\in\mathcal{I}} \mathfrak{a}_i\gamma_i + \sum_{j\in\mathcal{J}} \mathfrak{b}_j\,\rho_j + \mathfrak{c}\,\xi,\ (z_0,w_0) - (\breve{z},\breve{w}) \right\rangle < 0$$

holds for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w}),\ \gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z},\breve{w}),\ i \in \mathcal{I},\ \rho_j \in co\mathcal{A}_{\phi_j}(\breve{z},\breve{w}),\ j \in \mathcal{J}$ and $\xi \in co(\mathcal{A}_{\Phi}(\breve{z},\breve{w}) - \mathcal{A}_{\nu}(\breve{z}) \times \{0\})$, contradicting the necessary optimality condition (3.3). This completes the proof of this theorem. □

Remark 6. A similar result can be obtained for efficient solutions by assuming the strict $\mathcal{A}$-pseudoconvexity of Θ_s for $s \in \mathcal{S}$, instead of $\mathcal{A}$-pseudoconvexity.

We now establish the sufficient optimality conditions for $(\mathcal{P})$ under generalized convexity assumptions.

Theorem 3. *Let $(\breve{z},\breve{w}) \in \Xi$ and there exist $F^* = (\mathfrak{t}_s,\ \mathfrak{a}_i,\ \mathfrak{b}_j,\ \mathfrak{c}) \in \mathbb{R}^{\breve{n}}_{+}$, with $\sum_{s=1}^{n} \mathfrak{t}_s = 1$ such that the necessary optimality condition (3.3) be satisfied. Further, assume that Θ_s, $s \in \mathcal{S}$, θ_i, $i \in \mathcal{I}(\breve{z},\breve{w})$, ϕ_j, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $(\Phi - \nu)$ are $\mathcal{A}$-convex at $(\breve{z},\breve{w})$ on Ξ. Then $(\breve{z},\breve{w})$ is a weak efficient solution of $(\mathcal{P})$.*

Proof. Contrary to the result, suppose that $(\breve{z},\breve{w}) \in \Xi$ is not a weakly efficient solution of $(\mathcal{P})$. Hence, by Definition, there exists $(z_0,w_0) \in \Xi$ such that

$$\Theta_s(z_0,w_0) < \Theta_s(\breve{z},\breve{w}), \qquad \forall\ s \in \mathcal{S}.$$

Since $\Theta_s,\ s \in \mathcal{S}$ is a $\mathcal{A}$-convex at $(\breve{z},\breve{w})$ on Ξ, by Definition 2, we get that the inequality

$$\left\langle \eta_s,\ (z_0,w_0) - (\breve{z},\breve{w}) \right\rangle < 0$$

is satisfied for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w})$. Since $\mathfrak{t}_s \in \mathbb{R}^n_+$ with $\sum_{s=1}^{n} \mathfrak{t}_s = 1$, we obtain that the inequality

$$\left\langle \sum_{s=1}^{n} \mathfrak{t}_s\eta_s,\ (z_0,w_0) - (\breve{z},\breve{w}) \right\rangle < 0 \tag{3.6}$$

holds for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w})$.
By (3.3), for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w})$, $\gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z},\breve{w})$, $i \in \mathcal{I}$, $\rho_j \in co\mathcal{A}_{\phi_j}(\breve{z},\breve{w})$, $j \in \mathcal{J}$ and $\xi \in co(\mathcal{A}_\Phi(\breve{z},\breve{w}) - \mathcal{A}_\nu(\breve{z}) \times \{0\})$, one gets,

$$\begin{gathered}\sum_{s=1}^{n} \mathfrak{t}_s\eta_s + \sum_{i\in\mathcal{I}} \mathfrak{a}_i\gamma_i + \sum_{j\in\mathcal{J}} \mathfrak{b}_j\,\rho_j + \mathfrak{c}\,\xi = 0, \\ \mathfrak{a}_i\,\theta_i(\breve{z},\breve{w}) = 0, \qquad \mathfrak{b}_j\,\phi_j(\breve{z},\breve{w}) = 0.\end{gathered} \tag{3.7}$$

Using $(z_0, w_0) \in \Xi$ and $(\breve{z},\breve{w}) \in \Xi$ together with the definitions of $\mathcal{I}(\breve{z},\breve{w})$ and $\mathcal{J}(\breve{z},\breve{w})$, we get

$$\begin{aligned}\theta_i(z_0,w_0) &\le \theta_i(\breve{z},\breve{w}) = 0, \qquad \forall\, i \in \mathcal{I}(\breve{z},\breve{w}),\\ \phi_j(z_0,w_0) &\le \phi_j(\breve{z},\breve{w}) = 0, \qquad \forall\, j \in \mathcal{J}(\breve{z},\breve{w})\\ \Big(\Phi(z_0,w_0) - \nu(z_0)\Big) &\le \Big(\Phi(\breve{z},\breve{w}) - \nu(\breve{z})\Big) = 0.\end{aligned}$$

Since θ_i, $i \in \mathcal{I}(\breve{z},\breve{w})$, ϕ_j, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $(\Phi - \vartheta)$ are $\mathcal{A}$-convex at $(\breve{z},\breve{w})$ on Ξ, by Definition 2, the inequalities above yield, respectively, for any $\gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z},\breve{w})$, $i \in \mathcal{I}(\breve{z},\breve{w})$, $\rho_j \in co\mathcal{A}_{\phi_j}(\breve{z},\breve{w})$, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $\xi \in co(\mathcal{A}_\Phi(\breve{z},\breve{w}) - \mathcal{A}_\nu(\breve{z}) \times \{0\})$,

$$\begin{aligned}&\Big\langle \gamma_i,\ (z_0,w_0) - (\breve{z},\breve{w}) \Big\rangle \le 0, \qquad \forall\, i \in \mathcal{I}(\breve{z},\breve{w}),\\ &\Big\langle \rho_j,\ (z_0,w_0) - (\breve{z},\breve{w}) \Big\rangle \le 0, \qquad \forall\, j \in \mathcal{J}(\breve{z},\breve{w}),\\ &\Big\langle \xi,\ (z_0,w_0) - (\breve{z},\breve{w}) \Big\rangle \le 0.\end{aligned}$$

Since $\mathfrak{a}_i \ge 0$, $i \in \mathcal{I}(\breve{z},\breve{w})$, $\mathfrak{b}_j \ge 0$, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $\mathfrak{c} \ge 0$, therefore, the following inequality

$$\Bigg\langle \sum_{i\in\mathcal{I}(\breve{z},\breve{w})} \mathfrak{a}_i\gamma_i + \sum_{j\in\mathcal{J}(\breve{z},\breve{w})} \mathfrak{b}_j\,\rho_j + \mathfrak{c}\,\xi,\ (z_0,w_0) - (\breve{z},\breve{w}) \Bigg\rangle \le 0$$

holds for any $\gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z},\breve{w})$, $i \in \mathcal{I}(\breve{z},\breve{w})$, $\rho_j \in co\mathcal{A}_{\phi_j}(\breve{z},\breve{w})$, $j \in \mathcal{J}(\breve{z},\breve{w})$ and $\xi \in co(\mathcal{A}_\Phi(\breve{z},\breve{w}) - \mathcal{A}_\nu(\breve{z}) \times \{0\})$. Taking $\mathfrak{a}_i = 0$, $i \notin \mathcal{I}(\breve{z},\breve{w})$ and $\mathfrak{b}_j = 0$, $j \notin \mathcal{J}(\breve{z},\breve{w})$, we get

$$\Bigg\langle \sum_{i\in\mathcal{I}} \mathfrak{a}_i\gamma_i + \sum_{j\in\mathcal{J}} \mathfrak{b}_j\,\rho_j + \mathfrak{c}\,\xi,\ (z_0,w_0) - (\breve{z},\breve{w}) \Bigg\rangle \le 0. \tag{3.8}$$

Adding both sides of (3.6) and (3.8), we get the inequality

$$\Bigg\langle \sum_{s=1}^{n} \mathfrak{t}_s\eta_s + \sum_{i\in\mathcal{I}} \mathfrak{a}_i\gamma_i + \sum_{j\in\mathcal{J}} \mathfrak{b}_j\,\rho_j + \mathfrak{c}\,\xi,\ (z_0,w_0) - (\breve{z},\breve{w}) \Bigg\rangle < 0$$

holds for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w})$, $\gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z},\breve{w})$, $i \in \mathcal{I}$, $\rho_j \in co\mathcal{A}_{\phi_j}(\breve{z},\breve{w})$, $j \in \mathcal{J}$ and $\xi \in co(\mathcal{A}_\Phi(\breve{z},\breve{w}) - \mathcal{A}_\nu(\breve{z}) \times \{0\})$. This contradicts (3.7), thus concluding the proof of this theorem. □

Algorithm 1 An algorithm for finding weakly efficient solution for bilevel optimization problem $(\mathcal{P})$

Step 1. Initialization
Input the following data related to the given bilevel optimization problem $(\mathcal{P})$:

- Input Θ_s for all $s \in \mathcal{S}$, θ_i for all $i \in \mathcal{I}$, Φ, ϕ_j for all $j \in \mathcal{J}$.

Step 2. Identify the Feasible Set

- Construct the feasible region as follows:

$$\Xi = \Big\{(z, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \theta_i(z, w) \leq 0,\ i \in \mathcal{I},\ \phi_j(z, w) \leq 0,\ j \in \mathcal{J},\ \Phi(z, w) - \nu(z) \leq 0\Big\}.$$

Step 3. Select a Feasible Point

- If the feasible set Ξ is empty, terminate the algorithm.
- Otherwise, choose any point $(\breve{z}, \breve{w}) \in \Xi$.
- Update the feasible set by excluding $(\breve{z}, \breve{w})$: $\Xi := \Xi \setminus \{(\breve{z}, \breve{w})\}$.

Step 4. Computation of the approximation

- Compute the approximation of each functions Θ_s for all $s \in \mathcal{S}$, θ_i for all $i \in \mathcal{I}$, ϕ_j for all $j \in \mathcal{J}$ and $(\Phi - \nu)$ at $(\breve{z}, \breve{w})$.

Step 5. Choose arbitrary subdifferentials
Choose arbitrary elements from the subdifferentials:

- For each $s \in \mathcal{S}$: $\eta_s \in co\mathcal{A}_{\Theta_s}(\breve{z}, \breve{w})$.
- For each $i \in \mathcal{I}$, $\gamma_i \in co\mathcal{A}_{\theta_i}(\breve{z}, \breve{w})$, $j \in \mathcal{J}$, $\rho_j \in co\mathcal{A}_{\phi_j}(\breve{z}, \breve{w})$ and $\xi \in co(\mathcal{A}_{\Phi}(\breve{z}, \breve{w}) - \mathcal{A}_{\nu}(\breve{z}) \times \{0\})$.

Step 6. Check the nonsmooth Abadie constraint qualification (NACQ)

- If the nonsmooth Abadie constraint qualification holds at $(\breve{z}, \breve{w})$, proceed to the next step.
- If not, return to Step 3.

Step 7. Verification of approximation-based KKT Conditions
Attempt to find multipliers $\mathfrak{t}_s \geq 0,\ s \in \mathcal{S},\ \mathfrak{a}_i \geq 0, i \in \mathcal{I}(\breve{z}, \breve{w}),\ \mathfrak{b}_j \geq 0, j \in \mathcal{J}(\breve{z}, \breve{w}),\ \mathfrak{c} \geq 0$ with $\sum\limits_{s \in \mathcal{S}} \mathfrak{t}_s = 1$ satisfying the Karush-Kuhn-Tucker conditions (3.3):

- If such multipliers can be found, then $(\breve{z}, \breve{w})$ is a local weakly efficient solution of the bilevel optimization problem.
- If not, return to Step 3.

Step 8. Verify Generalized Convexity Conditions
At the point $\bar{\mathfrak{z}}$, confirm the following:

- For each $s \in \mathcal{S}$: Check whether Θ_s is $\mathcal{A}$-pseudoconvex at $(\breve{z}, \breve{w})$ over Ξ.
- For Φ, θ_i for all $i \in \mathcal{I}$, ϕ_j for all $j \in \mathcal{J}$ are $\mathcal{A}$-quasiconvex at $(\breve{z}, \breve{w})$ over Ξ.

If these conditions are not met, the problem cannot be solved using the current framework-return to Step 3.

Step 9. Output the Solution
The point $(\breve{z}, \breve{w})$ obtained through this process is a weakly efficient solution for bilevel optimization problem.

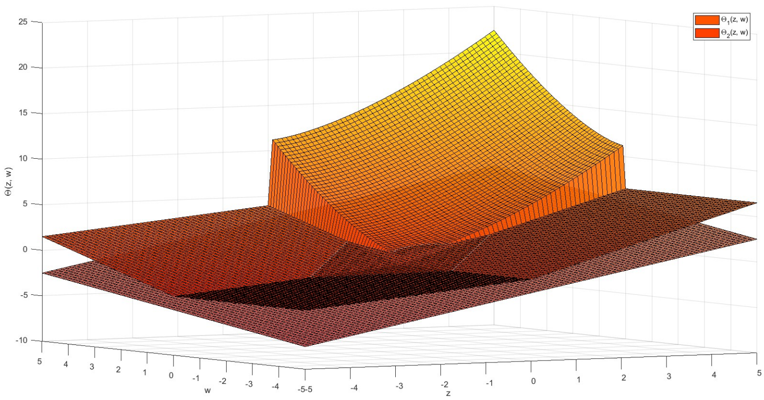

Figure 1. The graph of the objective functions $\Theta_1(z, w)$ and $\Theta_2(z, w)$ of $(\mathcal{P}_1)$ considered in Example 2.

Now, we present the example of an vector bilevel optimization problem to illustrate the optimality conditions established in this section.

Example 2. Consider the following nondifferentiable vector bilevel optimization problem defined by

$$(\mathcal{P}_1): \begin{cases} \min\limits_{z,w} \ \Theta(z,w) = \Big(\Theta_1(z,w), \Theta_2(z,w)\Big) \\ \ s.t. \\ \qquad \theta_1(z,w) = -z \leq 0, \\ \qquad \theta_2(z,w) = z^2 - 1 \leq 0, \\ \qquad w \in \urcorner(z), \end{cases}$$

where,

$$\Theta_1(z,w) = \begin{cases} |z|^{\frac{3}{2}} + |w|^{\frac{3}{2}} - 1, & \text{if } z > 0, w > 0, \\ \frac{1}{2}|z| + \frac{1}{2}|w| + z + \frac{1}{2}w - 1, & \text{if } z \leq 0 \text{ or } w \leq 0, \end{cases}$$

and

$$\Theta_2(z,w) = z + \frac{1}{2}w.$$

For any $z \in \mathbb{R}$, $\urcorner(z)$ represents the solution set of the the lower level problem defined as follows

$$(\mathcal{P}_1)_z: \begin{cases} \min\limits_{w} \ \ \Phi(z,w) = w - \frac{2}{3}z + z^4 \\ s.t. \qquad \phi_1(z,w) = w^2 - w \leq 0. \end{cases}$$

The graphical representations of the objective functions are shown in Figure 1. From the calculation of $(\mathcal{P}_1)_z$, we obtain that $\urcorner(z) = \{0\}$, and $\nu(x) = -\frac{2}{3}z + z^4$, therefore $\Phi(z,w) - \nu(z) = w$. Then, $\Xi = \big\{(z,w) : z \in [0,1],\ w \in \{0\}\big\}$, and its geometric representation is shown in Figure 2. For $(\breve{z}, \breve{w}) = (0,0) \in \Xi$, we have

$$\mathfrak{T}_{\Xi}(\breve{z}, \breve{w}) = \{(z,0) : z \geq 0\}, \quad \mathcal{I}(\breve{z}, \breve{w}) = \{1\}, \quad \mathcal{J}(\breve{z}, \breve{w}) = \{1\}.$$

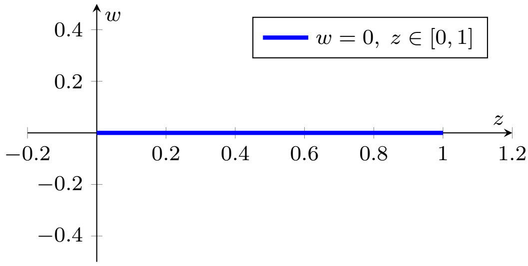

Figure 2. **The feasible region of $(\mathcal{P}_1)$ considered in Example 2.**

The sets $\mathcal{A}_{\Theta_1}(\breve{z},\breve{w}) = \{(\frac{1}{2},1),(\frac{3}{2},0),(\frac{1}{2},0),(0,0)\}$, $\mathcal{A}_{\Theta_2}(\breve{z},\breve{w}) = \{(1,\frac{1}{2})\}$ are compact approximations of Θ_1 and Θ_2. It can be shown, by Definition 2, that the objective functions Θ_1 and Θ_2 are $\mathcal{A}$-pseudoconvex at $(\breve{z},\breve{w})$. Note that, by Definition 1, the sets $\mathcal{A}_{\theta_1}(\breve{z},\breve{w}) = \{(-1,0)\}$, $\mathcal{A}_{\phi_1}(\breve{z},\breve{w}) = \{(0,-1)\}$, $(\mathcal{A}_{\Phi}(\breve{z},\breve{w}) - \mathcal{A}_{\nu}(\breve{z}) \times \{0\}) = \{(0,1)\}$ are compact approximations of θ_1, ϕ_1 and $(\Phi - \nu)$, which are $\mathcal{A}$-quasiconvex at $(\breve{z},\breve{w})$. Further, it can be shown, by Definition 2, that θ_1, ϕ_1 and $(\Phi - \nu)$ are $\mathcal{A}$-quasiconvex at $(\breve{z},\breve{w})$. It can be shown that the nonsmooth constraint qualification (NACQ) holds at $(\breve{z},\breve{w})$. Indeed, by Definition 3, one has

$$\Omega(\breve{z},\breve{w}) = \{(-1,0),\ (0,-1),\ (0,1)\}.$$

Thus,

$$\Omega(\breve{z},\breve{w})^{\circ} = \{(x,0) : x \in \mathbb{R}^{+}\}, \quad \text{and} \quad pos\ \Omega(\breve{z},\breve{w}) = \{(z,w) \in \mathbb{R}^2 : z \leq 0\}.$$

Notice that $pos\ \Omega(\breve{z},\breve{w})$ is closed and

$$\Omega(\breve{z},\breve{w})^{\circ} \subseteq cl\, co\, \mathfrak{T}_{\Xi}(\breve{z},\breve{w}),$$

by Definition 3, that the nonsmooth constraint qualification (NACQ) holds at $(\breve{z},\breve{w})$ for $(\mathcal{P}_1)$. For $\mathfrak{t}_1 = \frac{1}{3}$, $\mathfrak{t}_2 = \frac{2}{3}$, $\mathfrak{a}_1 = \frac{5}{6}$, $\mathfrak{b}_1 = 2$ and $\mathfrak{c} = \frac{4}{3}$, the following relation

$$\tfrac{1}{3}\left(\tfrac{1}{2},\ 1\right) + \tfrac{2}{3}\left(1,\ \tfrac{1}{2}\right) + \tfrac{5}{6}(-1,0) + 2(0,-1) + \tfrac{4}{3}(0,\ 1) = (0,\ 0)$$

holds, which means that the necessary optimality condition (3.3) is satisfied at $(\breve{z},\breve{w})$ with these Lagrange multipliers. Hence, by Theorem 2, it follows that $(\breve{z},\breve{w})$ is a weak efficient solution of $(\mathcal{P}_1)$.

4. Dual formulation and characterization outcomes for $(\mathcal{P})$

In this section, for the considered vector bilevel optimization problem $(\mathcal{P})$, we define its vector Mond-Weir type dual problem [28]. Then we prove several duality results between problems $(\mathcal{P})$ and (MDP) under generalized convexity assumption imposed on the functions constituting them.

For $(z,w) \in \mathfrak{E}$, we define the following vector dual problem in the sense of Mond-Weir related to the considered vector bilevel optimization problem $(\mathcal{P})$ as follows:

$$
(\text{MDP}) \begin{cases} \max \Theta(\kappa,\upsilon) = \Big(\Theta_1(\kappa,\upsilon),\ldots,\Theta_n(\kappa,\upsilon)\Big) \\ s.\ t. \\ \quad (0,0) \in \sum_{s=1}^{n} \mathfrak{t}_s^* co\mathcal{A}_{\Theta_s}(\kappa,\upsilon) + \sum_{i\in\mathcal{I}} \mathfrak{a}_i^* co\mathcal{A}_{\theta_i}(\kappa,\upsilon) \\ \qquad\quad + \sum_{j\in\mathcal{J}} \mathfrak{b}_j^* co\mathcal{A}_{\phi_j}(\kappa,\upsilon) + \mathfrak{c}^* co\Big(\mathcal{A}_{\Phi}(\kappa,\upsilon) - \mathcal{A}_{\nu}(s)\times\{0\}\Big), \\ \quad \mathfrak{a}_i^*\theta_i(\kappa,\upsilon) \geq 0, \quad \mathfrak{b}_j^*\phi_j(\kappa,\upsilon) \geq 0,\ i\in\mathcal{I},\quad j\in\mathcal{J}, \\ \qquad\quad \mathfrak{c}^*\big(\Phi(\kappa,\upsilon) - \nu(\kappa)\big) \geq 0, \\ \quad (\mathfrak{t}_1^*,\ldots,\mathfrak{t}_n^*,\ \mathfrak{a}_1^*,\ldots,\mathfrak{a}_p^*,\ \mathfrak{b}_1^*,\ldots,\mathfrak{b}_q^*,\ \mathfrak{c}^*) \geq 0,\ \sum_{s=1}^{n}\mathfrak{t}_s^* = 1. \end{cases}
$$

Theorem 4 illustrates that the objective function value at any feasible solution of $(\mathcal{P})$ is always greater than or equal to the objective function value at a feasible solution of (MDP). This finding is supported by an example for better understanding.

Theorem 4. ***(Weak Duality):*** *Let (z,w) and $\big((\kappa,\upsilon),\mathfrak{t}^*,\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\big)$ be any feasible solutions for $(\mathcal{P})$ and (MDP), respectively. Suppose that each function $\Theta_s,\ s\in\mathcal{S}=\{1,\ldots,n\}$ is $\mathcal{A}$-pseudoconvex at (κ,υ), $\theta_i,\ i=\{1,\ldots,p\}$, $\phi_j,\ j=\{1,\ldots,q\}$ and $(\Phi-\nu)$ are $\mathcal{A}$-quasiconvex at (κ,υ). Then there exists $s_0\in\mathcal{S}$ such that*

$$\Theta_{s_0}(z,w) \geq \Theta_{s_0}(\kappa,\upsilon).$$

Proof. Contrary to the result, suppose that there exist a feasible point (z,w) of $(\mathcal{P})$ and a feasible point $\big((\kappa,\upsilon),\mathfrak{t}^*,\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\big)$ of (MDP) such that

$$\Theta_s(z,w) < \Theta_s(\kappa,\upsilon), \qquad \forall\, s\in\mathcal{S}. \tag{4.1}$$

Since $\mathfrak{t}_s^* \geq 0$, (4.1) gives

$$\sum_{s=1}^{n} \mathfrak{t}_s^*\,\Theta_s(z,w) < \sum_{s=1}^{n} \mathfrak{t}_s^*\,\Theta_s(\kappa,\upsilon). \tag{4.2}$$

Since $\Theta_s,\ s\in\mathcal{S}=\{1,\ldots,n\}$ is $\mathcal{A}$-pseudoconvex at (κ,υ), by Definition 2, we get that the inequality

$$\Big\langle \sum_{s=1}^{n} \mathfrak{t}_s^*\,\eta_s,\ (z,w)-(\kappa,\upsilon)\Big\rangle < 0 \tag{4.3}$$

is satisfied for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\kappa,\upsilon)$, $s \in \mathcal{S} = \{1,\dots,n\}$. Since $\big((\kappa,\upsilon),\mathfrak{t}^*,\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\big)$ is a feasible point of (MDP), therefore, for any $\eta_s \in co\mathcal{A}_{\Theta_s}(\kappa,\upsilon)$, $\gamma_i \in co\mathcal{A}_{\theta_i}(\kappa,\upsilon)$, $i = 1,\dots,p$, $\rho_j \in co\mathcal{A}_{\phi_j}(\kappa,\upsilon)$, $j = 1,\dots q$ and $\xi \in co(\mathcal{A}_\Phi(\kappa,\upsilon) - \mathcal{A}_\nu(\kappa) \times \{0\})$, one gets,

$$\sum_{s=1}^{n} \mathfrak{t}_s^* \eta_s = -\sum_{i=1}^{p} \mathfrak{a}_i^* \gamma_i - \sum_{j=1}^{q} \mathfrak{b}_j^* \rho_j - \mathfrak{c}^* \xi, \tag{4.4}$$

and

$$\mathfrak{a}_i^* \theta_i(\kappa,\upsilon) \geq 0, \quad \mathfrak{b}_j^* \phi_j(\kappa,\upsilon) \geq 0, \quad \mathfrak{c}^* \big(\Phi(\kappa,\upsilon) - \nu(s)\big) \geq 0, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}.$$

From (4.3), we gets

$$\Big\langle -\sum_{i=1}^{p} \mathfrak{a}_i^* \gamma_i - \sum_{j=1}^{q} \mathfrak{b}_j^* \rho_j - \mathfrak{c}^* \xi,\ (z,w) - (\kappa,\upsilon) \Big\rangle < 0. \tag{4.5}$$

Since (z,w) is a feasible point of $(\mathcal{P})$ and $\big((\kappa,\upsilon),\mathfrak{t}^*,\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\big)$ is a feasible point of (MDP), we have, respectively

$$\mathfrak{a}_i^* \theta_i(z,w) \leq 0 \leq \mathfrak{a}_i^* \theta_i(\kappa,\upsilon), \qquad i = \{1,\dots,p\},$$

$$\mathfrak{b}_j^* \phi_j(z,w) \leq 0 \leq \mathfrak{b}_j^* \phi_j(\kappa,\upsilon), \quad j = \{1,\dots q\},$$
$$\mathfrak{c}^* \big(\Phi(z,w) - \nu(z)\big) \leq 0 \leq \mathfrak{c}^* \big(\Phi(\kappa,\upsilon) - \nu(s)\big).$$

Since θ_i, $i = \{1,\dots,p\}$, ϕ_j, $j = \{1,\dots q\}$ and $(\Phi - \vartheta)$ are $\mathcal{A}$-quasiconvex at (κ,υ), by Definition 2, the inequalities above yield, respectively, for any $\gamma_i \in co\mathcal{A}_{\theta_i}(\kappa,\upsilon)$, $i = \{1,\dots p\}$, $\rho_j \in co\mathcal{A}_{\phi_j}(\kappa,\upsilon)$, $j = \{1,\dots q\}$ and $\xi \in co(\mathcal{A}_\Phi(\kappa,\upsilon) - \mathcal{A}_\nu(\kappa) \times \{0\})$,

$$\begin{aligned} &\Big\langle \mathfrak{a}_i^* \gamma_i,\ (z,w) - (\kappa,\upsilon) \Big\rangle \leq 0, \qquad \forall\, i = \{1,\dots,p\}, \\ &\Big\langle \mathfrak{b}_j^* \rho_j,\ (z,w) - (\kappa,\upsilon) \Big\rangle \leq 0, \qquad \forall\, j = \{1,\dots q\}, \\ &\Big\langle \mathfrak{c}^* \xi,\ (z,w) - (\kappa,\upsilon) \Big\rangle \leq 0. \end{aligned} \tag{4.6}$$

By summing these inequalities, we arrive at a contradiction with (4.5), thereby completing the proof. □

The subsequent outcome pertains to the strong duality.

Theorem 5. ***(strong duality):*** *Let $(\breve{z},\breve{w}) \in \Xi$ be a weak efficient solution of $(\mathcal{P})$ and nonsmooth constraint qualification (NACQ) be satisfied at $(\breve{z},\breve{w})$. Then, there exist multipliers $\mathfrak{t}_s \in \mathbb{R}_+^n$, $\mathfrak{a}_i \in \mathbb{R}_+^p$, $\mathfrak{b}_j \in \mathbb{R}_+^q$, $\mathfrak{c} \in \mathbb{R}_+$ with $\sum_{s=1}^n \mathfrak{t}_s = 1$ such that $\big((\breve{z},\breve{w}),\mathfrak{t},\mathfrak{a},\mathfrak{b},\mathfrak{c}\big)$ is feasible in (MDP). If the assumptions of the weak duality theorem (Theorem 4) are also fulfilled, then $\big((\breve{z},\breve{w}),\mathfrak{t},\mathfrak{a},\mathfrak{b},\mathfrak{c}\big)$ represents a weakly efficient solution of (MDP).*

Proof. Since $(\breve{z},\breve{w})$ is a weakly efficient solution of $(\mathcal{P})$, and all the hypotheses of Theorem 4 are satisfied, there exist multipliers $\mathfrak{t}_s \in \mathbb{R}^n_+$, $\mathfrak{a}_i \in \mathbb{R}^p_+$, $\mathfrak{b}_j \in \mathbb{R}^q_+$, $\mathfrak{c} \in \mathbb{R}_+$ with $\sum_{s=1}^n \mathfrak{t}_s = 1$ such that

$$\begin{aligned}(0,0) \in \sum_{s=1}^{n} \mathfrak{t}_s\, co\mathcal{A}_{\Theta_s}(\breve{z},\breve{w}) + \sum_{i\in\mathcal{I}} \mathfrak{a}_i\, co\mathcal{A}_{\theta_i}(\breve{z},\breve{w}) + \sum_{j\in\mathcal{J}} \mathfrak{b}_j\, co\mathcal{A}_{\phi_j}(\breve{z},\breve{w}) \\ + \mathfrak{c}\ co\big(\mathcal{A}_{\Phi}(\breve{z},\breve{w}) - \mathcal{A}_{\nu}(\breve{z}) \times \{0\}\big), \\ \mathfrak{a}_i\,\theta_i(\breve{z},\breve{w}) = 0, \qquad \mathfrak{b}_j\,\phi_j(\breve{z},\breve{w}) = 0.\end{aligned}$$

Since $\Phi(\breve{z},\breve{w}) - \nu(\breve{z}) = 0$, it follows that $\big((\breve{z},\breve{w}),\mathfrak{t},\mathfrak{a},\mathfrak{b},\mathfrak{c}\big)$ is feasible for the dual problem (MDP).
Assume, for contradiction, that $\big((\breve{z},\breve{w}),\mathfrak{t},\mathfrak{a},\mathfrak{b},\mathfrak{c}\big)$ is not a weakly efficient solution of (MDP). Then, there exists $\big((\kappa_0,\upsilon_0),\mathfrak{t}^*,\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\big)$ feasible for the dual problem (MDP), such that

$$\Theta(\kappa_0,\upsilon_0) > \Theta(\breve{z},\breve{w}).$$

This contradicts Theorem 4, as $(\breve{z},\breve{w})$ is feasible for $(\mathcal{P})$ and $\big((\kappa_0,\upsilon_0),\ \mathfrak{t}^*,\mathfrak{a}^*,\mathfrak{b}^*,\mathfrak{c}^*\big)$ is feasible for (MDP). Therefore, $\big((\breve{z},\breve{w}),\mathfrak{t},\mathfrak{a},\mathfrak{b},\mathfrak{c}\big)$ must be a weakly efficient solution of (MDP). □

We conclude the paper by presenting an example that demonstrates the applicability of our duality results.

Example 3. Revisiting problem $(\mathcal{P})$ as discussed in Example 2, one proceeds to explore its Mond-Weir type dual problem

$$(\text{MDP}_1)\quad \begin{cases} max\ \Theta(\kappa,\upsilon) = \Big(\Theta_1(\kappa,\upsilon),\ \Theta_2(\kappa,\upsilon)\Big) \\ s.\ t. \\ \qquad 0 \in \mathfrak{t}_1\ \mathcal{A}_{\Theta_1}(\kappa,\upsilon) + \mathfrak{t}_2\ \mathcal{A}_{\Theta_2}(\kappa,\upsilon) + \mathfrak{a}_1\ \mathcal{A}_{\theta_1}(\kappa,\upsilon) \\ \qquad\qquad + \mathfrak{b}_1\ \mathcal{A}_{\phi_1}(\kappa,\upsilon) + \mathfrak{b}_2\ \mathcal{A}_{\phi_2}(\kappa,\upsilon) \\ \qquad\qquad + \mathfrak{c}\big(\mathcal{A}_{\Phi}(\kappa,\upsilon) - \mathcal{A}_{\nu}(\kappa) \times \{0\}\big), \\ \\ \quad \mathfrak{a}_1\theta_1(\kappa,\upsilon) \geq 0, \quad \mathfrak{b}_1\phi_1(\kappa,\upsilon) \geq 0, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}, \\ \qquad \mathfrak{c}\big(\Phi(\kappa,\upsilon) - \vartheta(\kappa)\big) \geq 0, \\ \qquad \bar{\Lambda}^* = (\mathfrak{t}_1,\mathfrak{t}_2,\ \mathfrak{a}_1,\mathfrak{b}_1,\ \mathfrak{b}_2,\ \mathfrak{c}) \geq 0, \qquad \text{with}\ \ \mathfrak{t}_1+\mathfrak{t}_2 = 1, \end{cases}$$

where,

$$\Theta_1(\kappa,\upsilon) = \begin{cases} |\kappa|^{\frac{3}{2}} + |\upsilon|^{\frac{3}{2}}, & \text{if } \kappa > 0, \upsilon > 0, \\ \frac{1}{2}|\kappa| + \frac{1}{2}|\upsilon| + \kappa + \frac{1}{2}\upsilon - 1, & \text{if } \kappa \leq 0 \text{ or } \upsilon \leq 0, \end{cases}$$

and

$$\Theta_2(\kappa,\upsilon) = z + \frac{1}{2}w.$$

It is noted that $(\kappa, \upsilon) = (-1, 0)$ serves as a feasible point for (MDP_1). Specifically, for $\left(\frac{1}{4}, \frac{3}{4}, \frac{7}{8}, 0, \frac{9}{8}, \frac{1}{2}\right)$, it follows that

$$\frac{1}{4}\Big(\frac{1}{2}, 1\Big) + \frac{3}{4}\Big(1, \frac{1}{2}\Big) + \frac{7}{8}\Big(-1, 0\Big) + \frac{9}{8}\Big(0, -1\Big) + \frac{1}{2}\Big(0, 1\Big) = (0,\ 0).$$

Which implies

$$\begin{aligned} 0 \in \mathfrak{t}_1\ \mathcal{A}_{\Theta_1}(-1,0) + \mathfrak{t}_2\ \mathcal{A}_{\Theta_2}(-1,0) + \mathfrak{a}_1\ \mathcal{A}_{\theta_1}(-1,0) + \mathfrak{a}_2\ \mathcal{A}_{\theta_2}(-1,0) \\ + \mathfrak{b}_2\ \mathcal{A}_{\phi_2}(-1,0) + \mathfrak{c}\big(\mathcal{A}_{\Phi}(-1,0) - \mathcal{A}_{\nu}(-1) \times \{0\}\big), \end{aligned}$$

and $\mathfrak{t}_1 + \mathfrak{t}_2 = 1$. Moreover,

$$\mathfrak{a}_1\theta_1(-1,0) = \frac{1}{4} \geq 0,\ \ \mathfrak{a}_2\theta_2(-1,0) = 0 \geq 0,\ \ \mathfrak{b}_1\phi_1(-1,0) = 0 \geq 0,$$

and

$$\mathfrak{c}\big(\Phi(-1,0) - \vartheta(-1)\big) = 0 \geq 0.$$

Here, the sets $\mathcal{A}_{\Theta_1}(-1,0) = \left\{\left(\frac{1}{2}, 1\right),\ \left(\frac{1}{2}, 0\right)\right\}$, $\mathcal{A}_{\Theta_2}(-1,0) = \{(1, \frac{1}{2})\}$ are compact approximations of Θ_1 and Θ_2 at $(\kappa, \upsilon) = (-1, 0)$. It can be shown, by Definition 2, that the functions Θ_1 and Θ_2 are $\mathcal{A}$-pseudoconvex at $(\kappa, \upsilon) = (-1, 0)$. Note that, by Definition 1, the sets $\mathcal{A}_{\theta_1}(-1,0) = \{(-1,0)\}$, $\mathcal{A}_{\theta_2}(-1,0) = \{(0,0)\}$, $\mathcal{A}_{\phi_1}(-1,0) = \{(0,-1)\}$, $(\mathcal{A}_{\Phi}(-1,0) - \mathcal{A}_{\nu}(-1) \times \{0\}) = \{(0,1)\}$ are compact approximations of θ_1, θ_2, ϕ_1 and $(\Phi - \nu)$, which are $\mathcal{A}$-quasiconvex at $(\kappa, \upsilon) = (-1, 0)$. Further, it can be shown, by Definition 2, that θ_1, ϕ_1 and $(\Phi - \nu)$ are $\mathcal{A}$-quasiconvex at $(-1, 0)$.

As, we know that $\Xi = \left\{(z, w) : z \in [0,1],\ w = z^3\right\}$ be the feasible region of the primal problem $(\mathcal{P}_1)$, then for any $(z, w) = (0, 0) \in \Xi$, and a feasible solution $\left((-1,0),\ \frac{1}{4}, \frac{3}{4}, \frac{7}{8}, 0, \frac{9}{8}, \frac{1}{2}\right)$ of the dual problem (MDP_1), we have

$$\Theta_2(z, w) - \Theta_2(\kappa, \upsilon) \geq 0.$$

Concludes that Theorem 4 holds true for both the primal problem $(\mathcal{P}_1)$, and its corresponding Dual problem (MDP_1).

5. Conclusion

This paper has examined a vector bilevel programming problem in which the underlying functions are characterized through suitable approximation frameworks. We have derived sufficient optimality conditions for weakly efficient solutions under appropriate generalized convexity assumptions formulated via approximations. Building upon these sufficient conditions, we have proposed algorithmic procedures for identifying weakly efficient solutions to the considered nondifferentiable multiobjective bilevel optimization problem.

In addition, several Mond-Weir type duality results have been established between the nondifferentiable vector bilevel programming problem and its corresponding vector Mond-Weir dual formulation. These duality theorems hold under suitable generalized convexity assumptions and further strengthen the theoretical foundation of the proposed framework. To the best of our knowledge, this work represents one of the first contributions addressing both optimality conditions and duality theory for nondifferentiable vector bilevel optimization problems via approximations.
The current research presents several compelling opportunities for further investigation across diverse mathematical fields. The proposed research questions, along with the identified limitations and potential extensions, are clearly defined below:

- Extending the proposed efficiency criteria to interval optimization settings may provide deeper theoretical insight into hierarchical decision-making models under data uncertainty and imprecision.
- Another promising direction is the development of numerical methods for computing approximations and the associated multiplier conditions. Designing implementable algorithms grounded in the proposed approximation framework would help bridge the gap between abstract optimality theory and practical computational procedures.
- The sufficient optimality conditions are derived under specific generalized convexity assumptions (such as $\mathcal{A}$-convexity, $\mathcal{A}$-pseudoconvexity and $\mathcal{A}$-quasiconvexity). If these assumptions are violated, the obtained criteria may fail to guarantee weak efficiency.

  In such cases, alternative generalized convexity concepts (e.g. invexity-type conditions) or weaker stationarity concepts may be explored.

Appendix A.

A1. It is important to note that the approximations need not be closed. This can be illustrated by considering the function

$$\psi(y) = \begin{cases} \sqrt{y}, & y \geq 0, \\ 0, & y < 0. \end{cases}$$

at the point $\breve{y} = 1$, for which the open set $\left(\frac{1}{4}, \frac{3}{4}\right)$ serves as an approximation.

A2. **Verification of Generalized Convexity in Example 2**

Now, we verify the generalized convexity properties of the functions introduced in Example 2. In particular, we show that the objective functions Θ_1 and Θ_2 are $\mathcal{A}$-pseudoconvex, and that $\theta_1,\ \phi_1$ and $(\Phi - \nu)$ are $\mathcal{A}$-quasiconvex at $(\breve{z}, \breve{w}) = (0,0) \in \Xi$. Let $(z, w) \in \Xi$ be arbitrary.

- $$\Big\langle \mathcal{A}_{\Theta_1}(\breve{z},\breve{w}),\, (z,w)-(\breve{z},\breve{w})\Big\rangle = az+bw \geq 0,$$
$$\forall\,(a,b)\in\big\{\big(\tfrac{1}{2},1\big),\big(\tfrac{3}{2},0\big),\big(\tfrac{1}{2},0\big),\big(0,0\big)\big\}$$

$$\Rightarrow \Theta_1(z,w)=\begin{cases}|z|^{\frac{3}{2}}+|w|^{\frac{3}{2}}-1, & \text{if } z>0, w>0,\\ \frac{1}{2}|z|+\frac{1}{2}|w|+z+\frac{1}{2}w-1, & \text{if } z\leq 0 \text{ or } w\leq 0,\end{cases} \geq \Theta_1(0,0).$$

This establishes that Θ_1 is $\mathcal{A}$-pseudoconvex at $(0,0)$.

- $$\Big\langle \mathcal{A}_{\Theta_2}(\breve{z},\breve{w}),\, (z,w)-(\breve{z},\breve{w})\Big\rangle = z+\tfrac{1}{2}w \geq 0,$$
$$\Rightarrow \Theta_2(z,w)=z+\tfrac{1}{2}w\geq \Theta_1(0,0).$$

This establishes that Θ_2 is $\mathcal{A}$-pseudoconvex at $(0,0)$.

- $\theta_1(z,w)\leq\theta_1(\breve{z},\breve{w})\Rightarrow -z\leq 0$

$$\Longrightarrow \Big\langle \mathcal{A}_{\theta_1}(\breve{z},\breve{w}),\, (z,w)-(\breve{z},\breve{w})\Big\rangle = -z\leq 0.$$

This establishes that θ_1 is $\mathcal{A}$-quasiconvex at $(0,0)$.

- $\phi_1(z,w)\leq\phi_1(\breve{z},\breve{w})\Rightarrow -w\leq 0$

$$\Longrightarrow \Big\langle \mathcal{A}_{\phi_1}(\breve{z},\breve{w}),\, (z,w)-(\breve{z},\breve{w})\Big\rangle = -w\leq 0.$$

This establishes that ϕ_1 is $\mathcal{A}$-quasiconvex at $(0,0)$.

- $\Phi(z,w)-\nu(z,w)\leq\Phi(\breve{z},\breve{w})-\nu(\breve{z},\breve{w})\Rightarrow w\leq 0$

$$\Longrightarrow \Big\langle \mathcal{A}_{\Phi(\breve{z},\breve{w})-\mathcal{A}_{\nu}(\breve{z})\times\{0\}},\, (z,w)-(\breve{z},\breve{w})\Big\rangle = w\leq 0.$$

This establishes that $(\Phi-\nu)$ is $\mathcal{A}$-quasiconvex at $(0,0)$.

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