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Research Article

Some results on local distance antimagic chromatic number of graphs

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Abstract: Let $G = (V, E)$ be a simple graph of order n without isolated vertices. A bijection $f : V \to \{1, 2, \dots, n\}$ is called a local distance antimagic labeling, if $w(u) \neq w(v)$ for every edge uv of G, where $w(u) = \sum_{x \in N(u)} f(x)$. The local distance antimagic chromatic number of a graph $\chi_{ld}(G)$ is defined as the minimum number of colors taken over all the colorings of G induced by local distance antimagic labelings of G. In this paper, we study the local distance antimagic chromatic number for the join of graphs and the lexicographic product of graphs with the complement of the complete graph.

Keywords: local distance antimagic labeling, local distance antimagic chromatic number, lexicographic product.

AMS Subject Classification: 05C78

1. Introduction

By a graph $G = (V, E)$, we mean a finite, simple, undirected graph having neither multiple edges nor loops. For graph theoretic notations, we refer to Chartrand and Lesniak [3].

The notion of antimagic labeling was introduced by Hartsfield and Ringel [10] in 1990. A graph G is antimagic if the edges of G can be labeled by the numbers $\{1, 2, \dots, |E|\}$ such that the sums of the labels of the edges incident to each vertex (called the weight of a vertex) are all distinct. They conjectured that *every connected*

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graph with at least three vertices admits an antimagic labeling. They also made a weaker conjecture that *every tree with at least three vertices admits an antimagic labeling.* These two conjectures were partly shown to be correct by several authors, but they are still unsolved.

Arumugam and Kamatchi [12] introduced a vertex version of antimagic labeling of a graph as follows: a bijection $f : V \to \{1, 2, \ldots, |V|\}$ is said to be distance antimagic labeling of G if all the vertices have distinct vertex weights, where the weight of a vertex is defined as $w(v) = \sum_{x \in N(v)} f(x)$, where $N(v)$ is the open neighborhood of the vertex v, which is the set of vertices of the graph G adjacent to v. A graph G is called a distance antimagic graph if it admits a distance antimagic labeling f. For details (see [6], [7], [12]).

Arumugam et al. [1] introduced a local version of antimagic labeling of a graph G as follows: a bijection $f : E \to \{1, 2, \ldots, |E|\}$ is called local antimagic labeling if for any two adjacent vertices u and v, $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. Thus, any local antimagic labeling induces a proper vertex coloring of G where the vertex v is assigned the color $w(v)$. The local antimagic chromatic number $\chi_{la}(G)$ is the minimum number of colors taken over all colorings induced by local antimagic labelings of G.

Arumugam et al. [1] conjectured that *a connected graph with at least three vertices admits a local antimagic labeling.* Bensmail et al. [2] solved this conjecture partially. Finally, Haslegrave [11] proved this conjecture using probabilistic tools. Recently, several authors investigated the local antimagic chromatic number for several families of graphs. For further study, (see [1], [13], [14], [15]).

Motivated by local antimagic labeling, Handa et al. [7] introduced the notion of local distance antimagic labeling of a graph G as a bijection $f : V \to \{1, 2, \ldots, |V|\}$ having the property $w(u) \neq w(v)$ for every pair of adjacent vertices $u, v \in V$, where the weight of a vertex v is defined as $w(v) = \sum_{x \in N(v)} f(x)$. A graph that admits such a labeling is called a local distance antimagic graph. A local distance antimagic labeling induces a proper vertex coloring of the graph, with the vertex v assigned the color $w(v)$. The local distance antimagic chromatic number $\chi_{ld}(G)$ is the minimum number of colors taken over all colorings induced by local distance antimagic labelings of G. Clearly $\chi_{ld}(G) \geq \chi(G)$. The similar study is also done independently by Divya et al. [4].

2. Known results

Several authors have studied and found local distance antimagic chromatic numbers for different classes of graphs. For further study, (see [4], [7], [16], [17], [18], [19]). Handa et al. [7] proved the following result, which is useful to get a lower bound for the local distance antimagic chromatic number of a graph.

Proposition 1. *[7] Let G be a local distance antimagic graph. If u and v are vertices such that $|N(u) \triangle N(v)| = 1$ or 2, where $\triangle$ denotes the symmetric difference of sets, then $w(u) \neq w(v)$.*

Theorem 1. *[7, 19] The cycle C_n, $n \geq 3$, is local distance antimagic with*

$$\chi_{ld}(C_n) = \begin{cases} 2 & n = 4, \\ 3 & n \in \{3, 12\}, \\ 4 & n \in \{6, 8, 10, 14\}, \\ 5 & n \in \{5, 7, 9\} \end{cases} \quad \text{and} \quad \begin{cases} 4 \leq \chi_{ld}(C_n) \leq 5, & n \in \{11, 13\}, \\ 4 \leq \chi_{ld}(C_n) \leq 6, & n \geq 15. \end{cases}$$

Theorem 2. *[7, 19] The path P_n, $n \geq 2$, is local distance antimagic with*

$$\chi_{ld}(P_n) = \begin{cases} 2 & n \in \{2, 3\}, \\ 3 & n \in \{5, 11\}, \\ 4 & n \in \{4, 6, 7, 8, 9, 10\} \end{cases} \quad \text{and} \quad 4 \leq \chi_{ld}(P_n) \leq \begin{cases} 5 & n \geq 12, \ n \text{ even}, \\ 6 & n \geq 13, \ n \text{ odd}. \end{cases}$$

Theorem 3. *[7] The complete multipartite graph $G = K_{n_1, n_2, \ldots, n_r}$ is local distance antimagic with $\chi_{ld}(G) = r$.*

Theorem 4. *[7] The wheel W_n, $n \geq 3$, is local distance antimagic with $3 \leq \chi_{ld}(W_n) \leq 7$.*

Theorem 5. *[7] The complete graph K_n, $n \geq 2$, is local distance antimagic with $\chi_{ld}(K_n) = n$.*

Theorem 6. *[4, 7] The friendship graph F_n, $n \geq 2$, is local distance antimagic with $\chi_{ld}(F_n) = 2n + 1$.*

Theorem 7. *[4] The bistar graph $B_{m,n}$, $m, n \geq 2$, is local distance antimagic with $\chi_{ld}(B_{m,n}) = 4$.*

In our investigation, we also require the concept of the magic rectangle.

A magic rectangle $MR(a, b) = (m_{i,j})$ is an $a \times b$ array with entries $1, 2, \ldots, ab$ each appearing once, with all its row sums equal to a constant ρ and all its column sums equal to a constant σ. The sum of the entries in $MR(a, b)$ is $\frac{ab(ab+1)}{2}$ and the magic constants are

$$\sigma = \sum_{i=1}^{a} m_{ij} = \frac{a(ab + 1)}{2}, \text{ for any } j \in \{1, 2, \ldots, b\} \text{ and}$$

$$\rho = \sum_{j=1}^{b} m_{ij} = \frac{b(ab + 1)}{2}, \text{ for any } i \in \{1, 2, \ldots, a\}.$$

Harmuth [8, 9] proved that such arrays exist whenever a and b are of the same parity, except when exactly one of a and b is 1, or $a = b = 2$.

Theorem 8. *[8, 9] A magic rectangle $MR(a, b)$ exists if and only if $a, b > 1$, $ab > 4$ and $a \equiv b \pmod 2$.*

3. The main results

In this paper, we study the local distance antimagic chromatic number for the join of graphs and the lexicographic product of some classes of graphs with the complement of the complete graph, and check if the results of the chromatic number of a graph hold good for the local distance antimagic chromatic number also. We assume that all graphs G considered in the paper admit a local distance antimagic labeling, with local distance antimagic chromatic number as $\chi_{ld}(G)$.

Handa, in his thesis [6], studied the local distance antimagic labeling of graphs. He posted the following problem.

Problem 1. Given a fixed integer $p \in \{2, 3, \ldots, n\}$, does there exist a graph G of order n, with $\chi_{ld}(G) = p$?

We provide two solutions to the problem.

Solution: Consider the graph $G = K_{p-1} \vee \overline{K_{n-p+1}}$. We claim that $\chi_{ld}(G) = p$. Define a labeling f for G, by labeling the $p-1$ vertices of K_{p-1}, using integers from the set $\{1, 2, 3 \ldots, p-1\}$ in any order and then labeling the remaining $n-p+1$ vertices of $\overline{K_{n-p+1}}$, using integers from the set $\{p, p+1, \ldots, n\}$ in any order. Clearly, the weight of any vertex in $\overline{K_{n-p+1}}$ is $\frac{(p-1)p}{2}$, while the weight of any vertex v in K_{p-1} is $\frac{(p-1)p}{2} - f(v) + \frac{(n-p+1)(n+p)}{2}$. All the $p-1$ vertices of K_{p-1} receive distinct weights under f, while all the vertices in $\overline{K_{n-p+1}}$ receive the same weight, which is distinct from the weights of all the vertices of K_{p-1}, as the maximum value that $f(v)$ can take is $p-1$ for $v \in V(K_{p-1})$. Thus, f is a local distance antimagic labeling and so the total number of distinct weights is p. Thus $\chi_{ld}(G) \leq p$. Now, $\chi(G) = \chi(K_{p-1} \vee \overline{K_{n-p+1}}) = \chi(K_{p-1}) + \chi(\overline{K_{n-p+1}}) = p-1+1 = p$, and since $\chi_{ld}(G) \geq \chi(G) = p$, we get $\chi_{ld}(G) = p$.
We now present the second solution to the problem. For $p = 2, 3, \ldots, n-1$, construct a p partite graph G with the first $p-1$ partite sets having x_i elements, for $i = 1, 2, \ldots, p-1$ where $x_i \geq 1$ and the last partite set having $n - \sum_{i=1}^{p-1} x_i$ elements, provided $n > \sum_{i=1}^{p-1} x_i$, that is, $G = K_{x_1, x_2, \ldots, x_{p-1}, n - \sum_{i=1}^{p-1} x_i}$. Clearly G is graph of order n such that $\chi_{ld}(G) = p$. For $p = n$, we know that K_n is graph of order n, such that $\chi_{ld}(K_n) = n$.

3.1. χ_{ld} of join of graphs

In previous problem, observe that $\chi_{ld}(K_{p-1} \vee \overline{K_{n-p+1}}) = \chi_{ld}(K_{p-1}) + \chi_{ld}(\overline{K_{n-p+1}})$. Motivated by this result, we determine the χ_{ld} of the join of two graphs. If G and H are two graphs then $\chi(G \vee H) = \chi(G) + \chi(H)$ (see [3]). Further, using the definition of χ_{ld}, we have an obvious lower bound for $\chi_{ld}(G \vee H)$, i.e.,

$\chi_{ld}(G \vee H) \geq \chi(G \vee H) = \chi(G) + \chi(H)$.

Our next goal is to provide an upper bound for the local distance antimagic chromatic number of the join of two graphs G and H, one of which is regular.

Theorem 9. *Let G be a graph of order n and H be an r-regular graph of order m such that $n \leq m$. Let δ_G be the minimum degree of a vertex of G. If*

$$r(m+n) - \frac{r(r-1)}{2} - \frac{\delta_G(\delta_G+1)}{2} < nm + \frac{(m-n)(m+n+1)}{2}, \tag{3.1}$$

then $\chi_{ld}(G \vee H) \leq \chi_{ld}(G) + \chi_{ld}(H)$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{x_1, x_2, \ldots, x_m\}$ be the vertex sets G and H, respectively. Let f be the local distance antimagic labeling of G that assigns $\chi_{ld}(G)$ distinct weights to vertices of G and let g be the local distance antimagic labeling of H that assigns $\chi_{ld}(H)$ distinct weights to vertices of H. Using the labelings f and g, we define a new labeling h for vertices of $G \vee H$ by $h(v_i) = f(v_i)$, $1 \leq i \leq n$ and $h(x_j) = g(x_j) + n$, where $1 \leq j \leq m$.
Now the weights of the vertices are as follows:

$$\begin{aligned} w_{G\vee H}(v_i) &= w_G(v_i) + \frac{m(m+1)}{2} + nm && \text{where } 1 \leq i \leq n, \\ w_{G\vee H}(x_j) &= w_H(x_j) + \frac{n(n+1)}{2} + nr && \text{where } 1 \leq j \leq m. \end{aligned}$$

Note that as adjacent vertices among v_i's have distinct weights in G, they have distinct weights in $G \vee H$ too. Similarly, the adjacent vertices among x_j's have distinct weights. Therefore the labeling h induces $\chi_{ld}(G)$ distinct weights on vertices of G and $\chi_{ld}(H)$ distinct weights on vertices of H in $G \vee H$. Next, we show that $\min\limits_{v_i \in V(G)} w_{G\vee H}(v_i) > \max\limits_{x_j \in V(H)} w_{G\vee H}(x_j)$.
As $n \leq m$, we have,

$$\frac{m(m+1)}{2} + nm > \frac{n(n+1)}{2} + nr.$$

Further, note that the maximum possible value of $w_{G\vee H}(x_j)$ is obtained when the value of $w_H(x_j)$, is the maximum, i.e.,

$$= m + (m-1) + \cdots + (m-r+1) + \frac{n(n+1)}{2} + nr = r(n+m) - \frac{r(r-1)}{2} + \frac{n(n+1)}{2}, \tag{3.2}$$

while the minimum possible value of $w_{G\vee H}(v_i)$ is obtained when the value of $w_G(v_i)$ is the minimum, i.e.,

$$= 1 + 2 + \cdots + \delta_G + \frac{m(m+1)}{2} + nm = \frac{\delta_G(\delta_G+1)}{2} + \frac{m(m+1)}{2} + nm. \tag{3.3}$$

Now subtracting Equation 3.2 from Equation 3.3, we have,

$$\begin{aligned}&= \frac{\delta_G(\delta_G+1)}{2} + \frac{m(m+1)}{2} + nm - r(n+m) + \frac{r(r-1)}{2} - \frac{n(n+1)}{2}\\ &= \frac{\delta_G(\delta_G+1)}{2} + \frac{r(r-1)}{2} - r(m+n) + nm + \frac{(m-n)(m+n+1)}{2}\\ &> 0 \quad \text{using Equation 3.1.}\end{aligned}$$

Therefore, the weight of any vertex of G in $G \vee H$ always exceeds the weight of any vertex of H in $G \vee H$. Hence h is a local distance antimagic labeling for $G \vee H$ and $\chi_{ld}(G \vee H) \leq \chi_{ld}(G) + \chi_{ld}(H)$. □

Corollary 1. *If G is a graph of order n and H is an r-regular graph of order m, such that $n \leq m$ and $r \leq n$, then $\chi_{ld}(G \vee H) \leq \chi_{ld}(G) + \chi_{ld}(H)$.*

Corollary 2. *If $n \leq m$ and G is a graph of order n, then $\chi_{ld}(G \vee O_m) \leq \chi_{ld}(G) + 1$.*

Corollary 3. *If $m < n$ and G is an r-regular graph of order n and if*

$$r(m+n) - \frac{r(r-1)}{2} < nm + \frac{(n-m)(n+m+1)}{2},$$

then $\chi_{ld}(G \vee O_m) \leq \chi_{ld}(G) + 1$.

Theorem 9, gives a sufficient condition of a local distance antimagic labeling for the join of two graphs G and H (one of which is regular) such that $\chi_{ld}(G \vee H) \leq \chi_{ld}(G) + \chi_{ld}(H)$. This bound is sharp. Equality holds for the graphs G and H satisfying $\chi_{ld}(G) = \chi(G)$ and $\chi_{ld}(H) = \chi(H)$, respectively, i.e., $\chi_{ld}(G \vee H) \leq \chi_{ld}(G) + \chi_{ld}(H) \leq \chi(G) + \chi(H) = \chi(G \vee H)$. We know that $\chi_{ld}(G \vee H) \geq \chi(G \vee H)$, hence $\chi_{ld}(G \vee H) = \chi(G \vee H) = \chi(G) + \chi(H) = \chi_{ld}(G) + \chi_{ld}(H)$.

Next we present few more graphs G and H, for which $\chi_{ld}(G \vee H) = \chi_{ld}(G) + \chi_{ld}(H)$.

Theorem 10. *For positive integers n and m, $\chi_{ld}(F_n \vee O_m) = 2n + 2$.*

Proof. Let $V(F_n) = \{c\} \cup \{u_i, v_i :\ 1 \leq i \leq n\}$ be the vertex set of F_n and $V(O_m) = \{x_j :\ 1 \leq j \leq m\}$ be the vertex set of O_m . We define a bijection $f\colon V(F_n \vee O_m) \to \{1, 2, \ldots, 2n + 1 + m\}$ by

$$f(v) = \begin{cases} j & \text{if } v = x_j \text{ where } 1 \leq j \leq m,\\ m+1 & \text{if } v = c,\\ m+1+i & \text{if } v = u_i \text{ where } 1 \leq i \leq n,\\ m+2n+2-i & \text{if } v = v_i \text{ where } 1 \leq i \leq n.\end{cases}$$

Note that $S = \sum_{j=1}^{m} f(x_j) = \frac{m(m+1)}{2}$ and therefore for the weight of vertices, we have,

$$\begin{aligned}
w(c) &= S + \sum_{i=1}^{n} \big(f(v_i) + f(u_i)\big)\\
&= \frac{m(m+1)}{2} + \sum_{i=1}^{n} \big(m+1+i+m+2+2n-i\big)\\
&= \frac{m(m+1)}{2} + n(2m+2n+3),\\
\text{for } 1 \le j \le m, \quad w(x_j) &= f(c) + \sum_{i=1}^{n} \big(f(v_i) + f(u_i)\big)\\
&= m+1+\sum_{i=1}^{n} \big(m+1+i+m+2+2n-i\big)\\
&= m+1+\sum_{i=1}^{n} \big(2m+2n+3\big) = m+1+n(2m+2n+3),\\
\text{and for } 1 \le i \le n, \quad w(v_i) &= S + f(c) + f(u_i)\\
&= m+1+m+1+i+\frac{m(m+1)}{2} = \frac{m^2+5m+4}{2} + i,\\
w(u_i) &= S + f(c) + f(v_i)\\
&= m+1+m+2+2n-i+\frac{m(m+1)}{2}\\
&= 2m+2n+3+\frac{m(m+1)}{2} - i.
\end{aligned}$$

It is easy to see that for all $1 \le i \le n$, $w(u_i) \ne w(v_i) \ne w(c)$. Also, clearly, the weight of any vertex of F_n is different from the unique weight of any vertex of O_m. Therefore, f is a local distance antimagic labeling of $F_n \vee O_m$ that assigns $2n+2$ distinct weights.

We now turn to prove the lower bound. Consider any local distance antimagic labeling g of $F_n \vee O_m$. Since the vertices c, u_i, v_i, $1 \le i \le n$ form a clique, the weight of vertices c, u_i, v_i, $1 \le i \le n$ are distinct. Now, as $w(u_i) = g(v_i) + g(c) + \sum_{p=1}^{m} g(x_p)$, we have $w(u_i) \ne w(u_j)$, for $i \ne j$. Similarly, for any $i \ne j$, we have $w(v_i) \ne w(v_j)$. Next, consider the weight of u_i and v_j for $i \ne j$. Suppose $w(u_i) = w(v_j)$ for any $i \ne j$, then we get $g(u_i) = g(v_j)$, which is a contradiction and therefore $w(u_i) \ne w(v_j)$ for any $i \ne j$. Therefore the vertices of F_n receive $2n+1$ distinct weights under g. Also, as all vertices of F_n are adjacent to all vertices of O_m, the weight of the vertices of O_m is distinct from these $2n+1$ weights. Therefore, $\chi_{ld}(F_n \vee O_m) \ge 2n+2$. Hence $\chi_{ld}(F_n \vee O_m) = 2n+2$. □

Theorem 11. *For any positive integer n, $\chi_{ld}(F_n \vee B_{n,n}) = 2n+5$.*

Proof. Let $V(F_n) = \{c\} \cup \{u_i, v_i \ : \ 1 \leq i \leq n\}$ be the vertex set of F_n and $V(B_{n,n}) = \{a, b\} \cup \{x_i, y_i \ : 1 \leq i \leq n\}$ be the vertex set of $B_{n,n}$. We define a bijection $f\colon V(F_n \vee B_{n,n}) \to \{1, 2, \ldots, 2n+1, 2n+2, \ldots, 4n+3\}$ by

$$f(v) = \begin{cases} 1 & \text{if } v = a, \\ 2 & \text{if } v = b, \\ 3 & \text{if } v = c, \\ 3+i & \text{if } v = x_i \text{ where } 1 \leq i \leq n, \\ 3+n+i & \text{if } v = y_i \text{ where } 1 \leq i \leq n, \\ 3+2n+i & \text{if } v = u_i \text{ where } 1 \leq i \leq n, \\ 3+3n+i & \text{if } v = v_i \text{ where } 1 \leq i \leq n. \end{cases}$$

We have,

$$\begin{aligned} T = \sum_{v \in B_{n,n}} f(v) &= f(a) + f(b) + \sum_{i=1}^{n} \big(f(x_i) + f(y_i)\big) \\ &= 1 + 2 + \sum_{i=1}^{n} \big(3 + i + 3 + n + i\big) \\ &= 3 + (6+n)n + n(n+1) = 2n^2 + 7n + 3, \end{aligned}$$

and

$$\begin{aligned} S = \sum_{v \in F_n} f(v) &= f(c) + \sum_{i=1}^{n} \big(f(u_i) + f(v_i)\big) \\ &= 3 + \sum_{i=1}^{n} \big(3 + 2n + i + 3 + 3n + i\big) \\ &= 3 + (6+5n)n + n(n+1) = 6n^2 + 7n + 3. \end{aligned}$$

Note that $S = T + 4n^2$. We use these sums to calculate the weight of vertices. We have the weight of the vertices a, b and c as;

$$\begin{aligned} w(a) &= S + f(b) + \sum_{i=1}^{n} f(x_i) = S + 2 + 3n + \frac{n(n+1)}{2}, \\ w(b) &= S + f(a) + \sum_{i=1}^{n} f(y_i) = S + 1 + (3+n)n + \frac{n(n+1)}{2}, \\ w(c) &= T + \sum_{i=1}^{n} \big(f(u_i) + f(v_i)\big) \\ &= T + \sum_{i=1}^{n} \big(6 + 5n + 2i\big) \\ &= T + (6+5n)n + n(n+1) = T + (6n+7)n. \end{aligned}$$

Further, for $1 \leq i \leq n$,

$$\begin{aligned} w(x_i) &= f(a) + S = 1 + S, \\ w(y_i) &= f(b) + S = 2 + S, \\ w(u_i) &= f(c) + f(v_i) + T = T + 6 + 2n + i, \\ w(v_i) &= f(c) + f(u_i) + T = T + 6 + 3n + i. \end{aligned}$$

It is easy to see that all the vertices of the clique formed by c, u_i, and v_i have distinct weights for all $1 \leq i \leq n$. Clearly, $w(a) \neq w(b)$. Also, note that, $w(a) > w(x_i)$ and $w(b) > w(y_i)$ for all $1 \leq i \leq n$. It is easy to verify that, as $S = T + 4n^2$, the weight of the vertices of F_n is not the same as the weight of any of the vertices of $B_{n,n}$. This ensures that f is a local distance antimagic labeling of $F_n \vee B_{n,n}$ that assigns $2n+5$ distinct weights and hence $\chi_{ld}(F_n \vee B_{n,n}) \leq 2n+5$.
Consider any local distance antimagic labeling g of $H = F_n \vee B_{n,n}$. Let $S = \sum_{v \in F_n} g(v)$ and $T = \sum_{v \in B_{n,n}} g(v)$. Consider the subgraph $B_{n,n}$ of H. Note that as $w(x_i) = g(a) + S$ and $w(y_i) = g(b) + S$, for $1 \leq i \leq n$, all the $x_i's$ have the same weight, which is distinct from the weight of all the $y_i's$ as the labeling g is a bijection. Note that, $w(a) = \sum_{i=1}^{n} g(x_i) + g(b) + S$ and $w(b) = \sum_{i=1}^{n} g(y_i) + g(a) + S$. For $1 \leq i \leq n$ as the vertex a and the vertices x_i are adjacent $w(a) \neq w(x_i)$. Also $w(a) \neq w(y_i)$ as that would imply $\sum_{i=1}^{n} g(x_i) = 0$. Similarly $w(b)$ is distinct from $w(x_i)$ and $w(y_i)$ for $1 \leq i \leq n$. Also, as the vertices a and b are adjacent, $w(a) \neq w(b)$. Therefore the vertices of subgraph $B_{n,n}$ receive 4 distinct weights under f. Now consider the subgraph F_n in H. Since the vertices c, u_i, v_i, $1 \leq i \leq n$ form a clique, the weight of vertices c, u_i, v_i, $1 \leq i \leq n$ are distinct. Now as $w(u_i) = g(v_i) + g(c) + T$, we have $w(u_i) \neq w(u_j)$, for $i \neq j$. Similarly for any $i \neq j$, we have $w(v_i) \neq w(v_j)$. Next, consider the weight of u_i and v_j for $i \neq j$. Suppose $w(u_i) = w(v_j)$ for any $i \neq j$ then we get $g(u_i) = g(v_j)$, which is a contradiction, therefore $w(u_i) \neq w(v_j)$ for any $i \neq j$. Therefore, the vertices of F_n receive $2n+1$ distinct weights under g. Also, these $2n+1$ weights received by vertices of F_n are distinct from the 4 weights received by vertices of $B_{n,n}$ under g, as all the vertices of F_n are adjacent to all the vertices of $B_{n,n}$ in H and g is a local distance antimagic labeling. So in total, the vertices of graph H receive $2n+5$ distinct weights under g. As g is an arbitrary local distance antimagic labeling of H, we have $\chi_{ld}(H) \geq 2n+5$, and the equality follows. ☐

3.2. r-uniform k-colorable graphs

In this subsection, we define and study a new class of graphs called r-uniform k-colorable graphs. The lexicographic product of this graph class with the complement of the complete graph O_n will then be studied in the next subsection.

Definition 1. Let G be a graph with $\chi(G) = k$. If there exists a proper k-vertex coloring of G, having k-color partitions P_1, P_2, ..., P_k, such that every vertex $v \in V(G)$ has exactly r-neighbors in each color partition (except the one in which it is present), then G is called r-uniform k-colorable graph.

Observation 12. An r-uniform k-colorable graph is $(k-1)r$-regular.

Example 1. All r-regular bipartite graphs are r-uniform 2-colorable graphs.

Example 2. The complete k-partite graph $K_{n,n,\dots,n}$ is n-uniform k-colorable graph.

Example 3. The complete graph K_n is 1-uniform n-colorable graph.

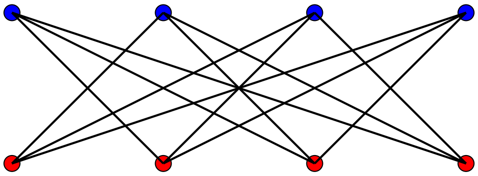

Figure 1. **3-uniform 2-colorable graph.**

The cycle C_n is graph on n-vertices having vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} \ : \ 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. We know that if n is odd then $\chi(C_n) = 3$ and 2 otherwise.

Theorem 13. *For n odd, the cycle C_n is 1-uniform 3-colorable graph if and only if $n \equiv 3 \pmod 6$.*

Proof. Suppose $n \equiv 3 \pmod 6$. Let $P_1 = \{v_i \ ; \ i \equiv 1 \pmod 3\}$, $P_2 = \{v_i \ ; \ i \equiv 2 \pmod 3\}$ and $P_3 = \{v_i \ ; \ i \equiv 0 \pmod 3\}$ be the three color partitions of $V(C_n)$ with colors K_1, K_2 and K_3, respectively. By definition, it is clear that any vertex $v_i \in P_1$ is adjacent to v_{i+1} and v_{i-1}, which belong to color partitions P_2 and P_3 respectively. Similarly, we can show that a vertex in any color partition has neighbors in the other two color partitions. Thus, for $n \equiv 3 \pmod 6$, C_n is 1-uniform 3-colorable graph.

Conversely, suppose that for n odd and C_n is a 1-uniform 3-colorable graph. As n is odd, $n \equiv 1, 3, 5 \pmod 6$. If $n \equiv 1 \pmod 6$, and suppose that C_n is a 1-uniform 3-colorable graph with color partitions P_1, P_2 and P_3. Let v_1 belong to the first color partition P_1. As v_1 is adjacent to v_2 and v_n, the vertices v_2 and v_n should belong to the other two color partitions. Without loss of generality, let us assume that the vertex v_2 belongs to P_2 and the vertex v_n belongs to P_3. Further, the vertex v_3 should belong to P_3. Continuing this way, we have $P_1 = \{v_i \ ; \ i \equiv 1 \pmod 3\}$, $P_2 = \{v_i \ ; \ i \equiv 2 \pmod 3\}$ and $P_3 = \{v_i \ ; \ i \equiv 0 \pmod 3\}$. As $n \equiv 1 \pmod 6$, the vertex v_{n-1} must belong to P_3. However, the vertex v_n also belongs to P_3, which leads to a contradiction. Further, if $n \equiv 5 \pmod 6$ and suppose that C_n is a 1-uniform 3-colorable graph with color partitions P_1, P_2 and P_3. Let v_1 belong to P_1. As v_1 is adjacent to v_2 and v_n, the vertices v_2 and v_n should belong to the other two color partitions. Without loss of generality, let us assume that the vertex v_2 belongs to P_2 and the vertex v_n belongs to P_3. In addition, the vertex v_3 should belong to P_3. Continuing this way, we have $P_1 = \{v_i \ ; \ i \equiv 1 \pmod 3\}$, $P_2 = \{v_i \ ; \ i \equiv 2 \pmod 3\}$ and $P_3 = \{v_i \ ; \ i \equiv 0 \pmod 3\}$. As $n \equiv 5 \pmod 6$, the vertex v_{n-2} belongs to P_3, while the vertex v_{n-1} belongs to P_1. Now both neighbors of v_n, namely v_{n-1} and v_1 are in P_1, leading to a contradiction. Thus, for n odd, if C_n is 1-uniform 3-colorable graph, then $n \equiv 3 \pmod 6$. ☐

Problem 2. Characterize r-uniform k-colorable graphs.

3.3. χ_{ld} of lexicographic product of graphs

The lexicographic product of G and H is a graph having vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (x, y) are adjacent if and only if either u is adjacent to x in G or $u = x$ and v is adjacent to y in H. The lexicographic product of two graphs G and H, also called the composition of two graphs, is denoted by $G[H]$. Geller et al. [5], proved that for a bipartite graph G and for any graph H, $\chi(G[H]) = 2.\chi(H)$. In this section, we determine the $\chi_{ld}(G[O_n])$ for various graph classes G and obtain almost similar results as for $\chi(G[O_n])$. We begin by providing an upper bound for $\chi_{ld}(G[O_n])$, where G is a regular graph.

Theorem 14. *Let f be a local distance antimagic labeling of a regular graph G that assigns $\chi_{ld}(G)$ distinct weights to its vertices, then $\chi_{ld}(G[O_n]) \leq \chi_{ld}(G)$ for all $n > 1$.*

Proof. Let $V(G) = \{v_1, v_2, \dots, v_m\}$ and $V(O_n) = \{x_1, x_2, \dots, x_n\}$ be the vertex sets of G and O_n, respectively. For $1 \leq j \leq m$ and $1 \leq i \leq n$, let v_i^j be the vertices of $G[O_n]$ that corresponds with vertices v_j of G. Let f be a local distance antimagic labeling of G with local distance antimagic chromatic number $\chi_{ld}(G)$. Further, let $\deg(v_j) = r$ for $1 \leq j \leq m$. We define a vertex labeling g of $G[O_n]$ by

$$g(v_i^j) = i + (f(v_j) - 1)n.$$

For the weight of the vertex v_i^j; $j = 1, 2, \dots, m$ and $i = 1, 2, \dots, n$ we obtain,

$$\begin{aligned} w_g(v_i^j) &= \sum_{v_p \in N_G(v_j)} \sum_{i=1}^{n} g(v_i^p) \\ &= \sum_{v_p \in N_G(v_j)} \sum_{i=1}^{n} \big(i + (f(v_p) - 1)n\big) \\ &= \sum_{v_p \in N_G(v_j)} \left(\frac{n(n+1)}{2} + (f(v_p) - 1)n^2\right) \\ &= \frac{n(n+1)}{2} r + (w(v_j) - r)n^2. \end{aligned}$$

As f is a local distance antimagic labeling of G, the adjacent vertices of $G[O_n]$ have distinct weights and therefore g is a local distance antimagic labeling of $G[O_n]$ and moreover $\chi_{ld}(G[O_n]) \leq \chi_{ld}(G)$. □

Next, we present constructions of some specific matrices, which we shall use in our proofs in this section.

We present the construction of a matrix $A = (a_{i,j})$ of size $n \times m$ (n even), using the entries $1, 2, \ldots, nm$, as follows: For $1 \leq i \leq n$, define

$$a_{i,j} = \begin{cases} (i-1)m + j & \text{for } i \equiv 1 \pmod 2 \text{ and } 1 \leq j \leq m, \\ im + 1 - j & \text{for } i \equiv 0 \pmod 2 \text{ and } 1 \leq j \leq m. \end{cases} \tag{3.4}$$

Now, let us calculate the sum of entries in j^{th} column of matrix A. In the j^{th} column, the entries at even positions are $2m+1-j,\ 4m+1-j,\ \ldots,\ nm+1-j$ and their sum is

$$\begin{aligned} &= (2+4+\cdots+n)m + \frac{n}{2} - \frac{nj}{2} \\ &= \frac{nm}{4}(n+2) + \frac{n}{2} - \frac{nj}{2}. \end{aligned}$$

Similarly, note that in the j^{th} column, the entries at odd positions are $j,\ 2m+j, 4m+j,\ \ldots,\ (n-2)m+j$ and their sum is

$$\begin{aligned} &= (0+2+4+\cdots+(n-2))m + \frac{nj}{2} \\ &= \frac{nm}{4}(n-2) + \frac{nj}{2}. \end{aligned}$$

Combining these two sums together, we get

$$= \frac{nm}{4}(n+2) + \frac{n}{2} + \frac{nm}{4}(n-2) = \frac{n(nm+1)}{2} = \sigma_A. \tag{3.5}$$

So the sum of entries in every column of A is equal to a constant σ_A.
Now consider an integer $t \geq 1$. For any $1 \leq p \leq t$, we present the construction of t pairs of matrices, $B_p = (b^p_{i,j})$ and $C_p = (c^p_{i,j})$ of size $n \times m$ (n odd) using the entries from $1, 2, \ldots, 2nmt$, such that the sum of entries in any column of B_p is a constant while the sum of entries in any column of C_p is another constant. For $1 \leq i \leq n$, define

$$b^p_{i,j} = \begin{cases} j + 6m(p-1) & \text{for } i = 1 \text{ and } 1 \leq j \leq m, \\ j + m + 6m(p-1) & \text{for } i = 2 \text{ and } 1 \leq j \leq m, \\ 4m - (2j-2) + 6m(p-1) & \text{for } i = 3 \text{ and } 1 \leq j \leq m, \\ 6mt + 2m(p+i-5) + 2j - 1 & \text{for } i \geq 4,\ i \text{ even and } 1 \leq j \leq m, \\ 2imt - 2m(p-1) - (2j-1) & \text{for } i \geq 5,\ i \text{ odd and } 1 \leq j \leq m. \end{cases} \tag{3.6}$$

$$c^p_{i,j} = \begin{cases} 4m+1-2j+6m(p-1) & \text{for } i=1 \text{ and } 1 \le j \le m, \\ 4m+j+6m(p-1) & \text{for } i=2 \text{ and } 1 \le j \le m, \\ 5m+j+6m(p-1) & \text{for } i=3 \text{ and } 1 \le j \le m, \\ 6mt+2m(p+i-5)+2j & \text{for } i \ge 4,\ i \text{ even and } 1 \le j \le m, \\ 2imt-2m(p-1)-(2j-2) & \text{for } i \ge 5,\ i \text{ odd and } 1 \le j \le m. \end{cases} \tag{3.7}$$

It is easy to observe that for a particular p, entries from $6m(p-1)+1$ to $6mp$ are used exactly once in the first three rows of matrices, B_p and C_p. So all the t pairs of matrices B_p and C_p; $1 \le p \le t$, use entries from 1 to $6mt$ in their first three rows exactly once. Also, from the definition of matrices B_p and C_p, it is clear that, for every even row beyond the third row, entries more than $6mt$ are used exactly once, either in matrix B_p or C_p, for $1 \le p \le t$. The same is true for entries in odd rows beyond the third row of matrices B_p and C_p. This shows that all the entries in matrices B_p and C_p are distinct for all $1 \le p \le t$.

We now calculate the sum of the entries in any column of matrices B_p and C_p, where $1 \le p \le t$.
The sum of the first three entries in any column of $B_p = 18mp - 13m + 2$ while the sum of the first three entries in any column of $C_p = 18mp - 5m + 1$.
The sum of the remaining entries in any column of B_p

$$\begin{aligned} &= (6mt+2mp-1)\left(\frac{n-3}{2}\right) + 2m(-1+1+3+\cdots+n-6) + 2mt(5+7+\cdots+n) \\ &\quad + (1-2mp+2m)\left(\frac{n-3}{2}\right) \\ &= (6mt+2m)\left(\frac{n-3}{2}\right) + m(n-7)\left(\frac{n-3}{2}\right) + mt(5+n)\left(\frac{n-3}{2}\right) \\ &= \big((11+n)mt - 5m + mn\big)\left(\frac{n-3}{2}\right) = S_B \text{ (say)}. \end{aligned}$$

The sum of the remaining entries in any column of C_p

$$\begin{aligned} &= (6mt+2mp)\left(\frac{n-3}{2}\right) + 2m(-1+1+3+\cdots+n-6) + 2mt(5+7+\cdots+n) \\ &\quad + (2-2mp+2m)\left(\frac{n-3}{2}\right) \\ &= (6mt+2m+2)\left(\frac{n-3}{2}\right) + m(n-7)\left(\frac{n-3}{2}\right) + mt(5+n)\left(\frac{n-3}{2}\right) \\ &= \big((11+n)mt - 5m + mn + 2\big)\left(\frac{n-3}{2}\right) = S_C \text{ (say)}. \end{aligned}$$

$$\text{So the sum of all entries in any column of } B_p = S_B + 18mp - 13m + 2 = S_1^p \text{ (say)}, \tag{3.8}$$

$$\text{and the sum of all entries in any column of } C_p = S_C + 18mp - 5m + 1 = S_2^p \text{ (say)}. \tag{3.9}$$

Note that, for any $1 \leq p \leq t$, $S_1^p < S_2^p$. Also, for $p \neq q$, we have $S_1^p \neq S_1^q$ and $S_2^p \neq S_2^q$. Now, if for some $p \neq q$, we have, $S_1^p = S_2^q$, then

$$\begin{gathered} S_B + 18mp - 13m + 2 = S_C + 18mq - 5m + 1 \\ 18m(p-q) = 8m + n - 4. \end{gathered}$$

This is not possible as the left-hand side is even while the right-hand side is odd, and hence $S_1^p \neq S_2^q$. Hence, we can conclude that the column sums of B^p and C^p are distinct, for all $1 \leq p \leq t$.

In particular, when $t = 1$, we get two matrices B and C having column sums as $mn^2 - 4m + 2$ and $mn^2 + 4m + n - 2$, respectively.

Below, we present examples of constructions discussed above when $t = 3$, $n = 3$ and $m = 4$.

$$B_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 16 & 14 & 12 & 10 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 15 & 13 & 11 & 9 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix}$$

The sum of entries in every column of B_1 and C_1 are 22 and 53, respectively.

$$B_2 = \begin{bmatrix} 25 & 26 & 27 & 28 \\ 29 & 30 & 31 & 32 \\ 40 & 38 & 36 & 34 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 39 & 37 & 35 & 33 \\ 41 & 42 & 43 & 44 \\ 45 & 46 & 47 & 48 \end{bmatrix}$$

The sum of entries in every column of B_2 and C_2 are 94 and 125, respectively.

$$B_3 = \begin{bmatrix} 49 & 50 & 51 & 52 \\ 53 & 54 & 55 & 56 \\ 64 & 62 & 60 & 58 \end{bmatrix} \qquad C_3 = \begin{bmatrix} 63 & 61 & 59 & 57 \\ 65 & 66 & 67 & 68 \\ 69 & 70 & 71 & 72 \end{bmatrix}$$

The sum of entries in every column of B_3 and C_3 are 166 and 197, respectively.

Using the constructions discussed above, we prove results about the lexicographic product of some graphs with the complement of the complete graph O_n, in the following subsections.

The lexicographic product of a graph G with the complement of a complete graph O_n consists of $|G|$ copies of O_n. Basically, its vertices can be arranged into n rows and $|G|$ columns. Depending upon the structure of G, we label the vertices in a particular column using matrices A, B^p and C^p, such that the adjacent vertices receive distinct weights.

3.3.1. χ_{ld} of lexicographic product of r-uniform k-colorable graphs with O_n

In this subsection, we study the lexicographic product of r-uniform k-colorable graphs defined in section 3.2 with O_n.

Theorem 15. *Let n be even and G be an r-uniform k-colorable graph of order m. Then $\chi_{ld}(G[O_n]) = k$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of G, where $P_1 = \{v_1, v_2, \ldots, v_{n_1}\}$, $P_2 = \{v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2}\}$, $\ldots$, $P_k = \{v_{n_1+n_2+\cdots+n_{k-1}+1}, v_{n_1+n_2+\cdots+n_{k-1}+2}, \ldots, v_{n_1+n_2+\cdots+n_{k-1}+n_k}\}$ are its color partitions with $n_1 + n_2 + \cdots + n_k = m$. Note that each of n_1, n_2, $\ldots$, n_k is at least r. Further, let $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of O_n. For $1 \le j \le m$ and $1 \le i \le n$, let v_i^j be the vertices of $G[O_n]$ that correspond with vertices v_j in the color partition of G. Using the definition of Matrix 3.4, we construct k matrices, $A_p = (a_{i,j}^p)$ of size $n \times n_p$; $1 \le p \le k$. Setting $n_0 = 0$, we use these k matrices to define a bijection $f\colon V(G[O_n]) \to \{1, 2, \ldots, mn\}$ as follows: for $1 \le i \le n$ and $1 \le j \le m$,

$$f(v_i^j) = a^p_{i,j-(n_1+n_2+\cdots+n_{p-1})} + n\left(\sum_{t=0}^{p-1} n_t\right) \quad \text{for } v_j \in P_p \text{ where } 1 \le p \le k.$$

Now we calculate the weight of the vertex $v_i^j \in V(G[O_n])$ where $1 \le i \le n$ and $1 \le j \le m$. By definition of the lexicographic product, we have,

$$w(v_i^j) = \sum_{v_q \in N_G(v_j)} \sum_{i=1}^{n} f(v_i^q).$$

Each vertex of G belongs to some color partition. Without loss of generality, assume that the vertex v_j belongs to color partition P_l, for some $1 \le l \le k$. If the vertex v_q belongs to color partition P_s of G, then $\sum_{i=1}^{n} f(v_i^q)$ is the column sum of Matrix A_s along with the term $n\left(\sum_{t=0}^{s-1} n_t\right)$ added n times. From Equation 3.5 the column sum of Matrix A_s is $\frac{n(nn_s+1)}{2}$, and hence

$$\sum_{i=1}^{n} f(v_i^q) = \frac{n(nn_s+1)}{2} + n^2\left(\sum_{t=0}^{s-1} n_t\right).$$

Since the vertex v_j has r-neighbors in every other color partition, we have

$$\begin{aligned}
w(v_i^j) &= \sum_{s=1, s\neq l}^{k} r\left[\frac{n(nn_s+1)}{2} + n^2\left(\sum_{t=0}^{s-1} n_t\right)\right] \\
&= \sum_{s=1, s\neq l}^{k} \frac{rn(nn_s+1)}{2} + rn^2 \sum_{s=1, s\neq l}^{k} \left(\sum_{t=0}^{s-1} n_t\right) \\
&= \frac{rn}{2} \sum_{s=1, s\neq l}^{k} (nn_s+1) \\
&+ rn^2\Big(n_0 + (n_0+n_1) + \cdots + (n_0+n_1+\cdots+n_{k-1}) - (n_0+n_1+\cdots+n_{l-1})\Big) \\
&= \frac{rn}{2}\left(n \sum_{s=1, s\neq l}^{k} n_s + (k-1)\right) + rn^2\left(\sum_{t=0}^{k-1}(k-t)n_t - \sum_{t=0}^{l-1} n_t\right) \\
&= \frac{rn}{2}\big(n(m-n_l) + (k-1)\big) + rn^2\left(\sum_{t=0}^{k-1}(k-t)n_t - \sum_{t=0}^{l-1} n_t\right).
\end{aligned}$$

Observe that, vertices in copies of G due to vertices belonging to a particular color partition of G have the same weight, while vertices in copies of G due to vertices belonging to different color partitions have distinct weights. Hence, f is a local distance antimagic labeling of $G[O_n]$ that assigns at most k distinct weights to its vertices. Hence $\chi_{ld}(G[O_n]) \leq k$. By definition, we know that $\chi_{ld}(G) \geq \chi(G)$ and as $\chi(G[O_n]) = k$, the equality follows. □

Theorem 16. *Let $n > 1$ be odd, and G be an r-uniform k-colorable graph of order m having each partite set of odd size. Then $\chi_{ld}(G[O_n]) = k$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of G, where $P_1 = \{v_1, v_2, \ldots, v_{n_1}\}$, $P_2 = \{v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2}\}$, …, $P_k = \{v_{n_1+n_2+\cdots+n_{k-1}+1}, v_{n_1+n_2+\cdots+n_{k-1}+2}, \ldots, v_{n_1+n_2+\cdots+n_{k-1}+n_k}\}$ are the color partitions with $n_1 + n_2 + \cdots + n_k = m$. Note that each of n_1, n_2, …, n_k is at least r. Further, let $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of O_n. For $1 \leq j \leq m$ and $1 \leq i \leq n$, let v_i^j be the vertices of $G[O_n]$ that correspond with vertices v_j in the color partition of G. For each $1 \leq p \leq k$, as n_p is odd, by Theorem 8, there exists a magic rectangle $A_p = (a_{i,j}^p)$ of size $n \times n_p$. Setting $n_0 = 0$, we use these magic rectangles to define a bijection $f\colon V(G[O_n]) \to \{1, 2, \ldots, mn\}$ as follows: for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$f(v_i^j) = a_{i,j-(n_1+n_2+\cdots+n_{p-1})}^p + n\left(\sum_{t=0}^{p-1} n_t\right) \quad \text{for } v_j \in P_p \text{ where } 1 \leq p \leq k.$$

Now we calculate the weight of the vertex $v_i^j \in V(G[O_n])$ where $1 \leq i \leq n$ and

$1 \leq j \leq m$. By definition of the lexicographic product, we have

$$w(v_i^j) = \sum_{v_q \in N_G(v_j)} \sum_{i=1}^{n} f(v_i^q).$$

Each vertex of G belongs to some color partition. Without loss of generality, assume that the vertex v_j belongs to color partition P_l, for some $1 \leq l \leq k$. If the vertex v_q belongs to color partition P_s of G, then $\sum_{i=1}^{n} f(v_i^q)$ is the column sum of magic rectangle A_s along with the term $n\left(\sum_{t=0}^{s-1} n_t\right)$ added n times. Since the column sum of magic rectangle A_s is $\dfrac{n(nn_s+1)}{2}$, we have

$$\sum_{i=1}^{n} f(v_i^q) = \frac{n(nn_s+1)}{2} + n^2\left(\sum_{t=0}^{s-1} n_t\right).$$

Since the vertex v_j has r-neighbors in every other color partition, using calculations as done in Theorem 15, we have

$$w(v_i^j) = \frac{rn}{2}\big(n(m-n_l) + (k-1)\big) + rn^2\left(\sum_{t=0}^{k-1}(k-t)n_t - \sum_{t=0}^{l-1} n_t\right).$$

Observe that, vertices in copies of G due to vertices belonging to a particular color partition of G have the same weight, while vertices in copies of G due to vertices belonging to different color partitions have distinct weights. Hence, f is a local distance antimagic labeling of $G[O_n]$ that assigns k distinct weights to its vertices. Hence $\chi_{ld}(G[O_n]) \leq k$. As $\chi(G[O_n]) = k$, and by definition it is known that $\chi_{ld}(G) \geq \chi(G)$, hence the equality follows. □

Theorem 17. *Let $n > 1$ be odd, and G be an r-uniform $2k$-colorable graph of order m having all color partitions of the same size. Then $\chi_{ld}(G[O_n]) = 2k$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of G, where $P_1 = \{v_1, v_2, \ldots, v_s\}$, $P_2 = \{v_{s+1}, v_{s+2}, \ldots, v_{2s}\}$, $P_3 = \{v_{2s+1}, v_{2s+2}, \ldots, v_{3s}\}, \ldots$, $P_{2k} = \{v_{(2k-1)s+1}, v_{(2k-1)s+2}, \ldots, v_{2ks}\}$ are its color partitions with $2ks = m$. Note that $s \geq r$. Further, let $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of O_n. For $1 \leq j \leq m$ and $1 \leq i \leq n$, let v_i^j be the vertices of $G[O_n]$ that correspond with vertices v_j in the color partition of G. Using the definition of Matrix 3.6 and Matrix 3.7, we define k pairs of matrices $B_p = (b_{i,j}^p)$ and $C_p = (c_{i,j}^p)$ of size $n \times s$. We use these k pairs of matrices to define a bijection $f\colon V(G[O_n]) \to \{1,2,\dots,mn\}$ as follows: for $1 \le i \le n$ and $1 \le j \le m$,

$$f(v_i^j) = \begin{cases} b^p_{i,j-(p-1)s} & \text{for } v_j \in P_p, \text{ where } 1 \le p \le k, \\ c^{p-k}_{i,j-(p-1)s} & \text{for } v_j \in P_p, \text{ where } k+1 \le p \le 2k. \end{cases}$$

Now we calculate the weight of the vertex $v_i^j \in V(G[O_n])$ where $1 \le i \le n$ and $1 \le j \le m$. By definition of the lexicographic product we have,

$$w(v_i^j) = \sum_{v_q \in N_G(v_j)} \sum_{i=1}^{n} f(v_i^q).$$

Each vertex of G belongs to some color partition. Without loss of generality, assume that the vertex v_j belongs to color partition P_l for some $1 \le l \le 2k$. If the vertex v_q belongs to the color partition P_s of G, where $1 \le s \le k$, then $\sum\limits_{i=1}^{n} f(v_i^q)$ is the column sum of the matrix B_s, while if $k+1 \le s \le 2k$, then $\sum\limits_{i=1}^{n} f(v_i^q)$ is the column sum of the matrix C_s. From Equations 3.8 and 3.9, the column sum of matrix B_s is S_1^s and the column sum of matrix C_s is S_2^s. Since the vertex v_j belongs to color partition P_l and has exactly r-neighbors in every other color partition, we have for $1 \le i \le n$,

$$w(v_i^j) = \begin{cases} r\left(\sum_{a=1,a\neq p}^{k} S_1^a + \sum_{b=1}^{k} S_2^b\right) & \text{if } 1 \le l \le k, \\ \\ r\left(\sum_{a=1}^{k} S_1^a + \sum_{b=1,b\neq p}^{k} S_2^b\right) & \text{if } k+1 \le l \le 2k. \end{cases}$$

As seen before, for any $1 \le p \le k$, $S_1^p \neq S_2^p$. Also, for $p \neq q$, $S_1^p \neq S_1^q$, $S_2^p \neq S_2^q$ and $S_1^p \neq S_2^q$. Observe that, vertices in copies of G due to vertices belonging to a particular color partition of G have the same weight, while vertices in copies of G due to vertices belonging to different color partitions have distinct weights. Hence, f is a local distance antimagic labeling of $G[O_n]$ that assigns $2k$ distinct weights to its vertices. Hence $\chi_{ld}(G[O_n]) \le 2k$. As $\chi(G[O_n]) = 2k$, the equality follows. ☐

Corollary 4. *For integers $n > 1$, $m > 1$ and an r-regular bipartite graph G of order m, we have $\chi_{ld}(G[O_n]) = 2$.*

Proof. The result follows from Theorem 15 and Theorem 17. ☐

Corollary 5. *For integers $n > 1$ and m odd with $m \equiv 0 \pmod 3$, $\chi_{ld}(C_m[O_n]) = 3$.*

Proof. The result follows from Theorem 15 and Theorem 17. ☐

3.3.2. χ_{ld} of lexicographic product of some specific graphs with O_n

We have seen that for a regular bipartite graph G, $\chi_{ld}(G[O_n]) = 2$. Our next goal is to investigate a similar result for non-regular bipartite graphs. The following theorems state that the result is true if all the vertices in a partite set have the same degree.

Lemma 1. *For integers $n \geq 2$ even and $m > 1$ and a non-regular bipartite graph G of order m with all the vertices in a partite set having the same degree, we have $\chi_{ld}(G[O_n]) = 2$.*

Proof. Let $V(G) = A \cup B$, where $A = \{u_1, u_2, \ldots, u_s\}$ and $B = \{v_1, v_2, \ldots, v_r\}$ are the two partite sets, be the vertex set of G with $s + r = m$. Also, let $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of O_n. Let $deg(u_p) = d_1$, for all $1 \leq p \leq s$ and $deg(v_q) = d_2$, for all $1 \leq q \leq r$. As G is a non-regular graph, we have $d_1 \neq d_2$. For $1 \leq i \leq n$, let u_i^j and v_i^j be the vertices of $G[O_n]$ that correspond with vertices u_j for $1 \leq j \leq s$ and v_j for $1 \leq j \leq r$ respectively of G. As n is even, we construct a matrix $A = (a_{i,j})$ of size $n \times m$ using the definition of Matrix 3.4. We use this matrix to label $G[O_n]$. Define $f\colon V(G[O_n]) \to \{1, 2, \ldots, mn\}$ by $f(u_i^j) = a_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq s$ and $f(v_i^j) = a_{i,s+j}$ for $1 \leq i \leq n$ and $1 \leq j \leq r$. As the sum of the entries of elements in any column of Matrix 3.4 is $\frac{n(nm+1)}{2}$, for the weight of vertices we have,

$$w(u_i^j) = \sum_{v_j \in N(u_j)} \sum_{i=1}^{n} f(v_i^j) = \frac{d_1 n(nm+1)}{2} \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq s,$$

$$w(v_i^j) = \sum_{u_j \in N(v_j)} \sum_{i=1}^{n} f(u_i^j) = \frac{d_2 n(nm+1)}{2} \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq r.$$

As $d_1 \neq d_2$, the two weights obtained are distinct, and hence f is a local distance antimagic labeling of $G[O_n]$, and the result follows. □

Lemma 2. *For $n \geq 3$ odd and $m > 1$ odd and a non-regular bipartite graph G of order m with all the vertices in a partite set having the same degree, we have, $\chi_{ld}(G[O_n]) = 2$.*

Proof. Let $V(G) = A \cup B$, where $A = \{u_1, u_2, \ldots, u_s\}$ and $B = \{v_1, v_2, \ldots, v_r\}$ are the two partite sets, be the vertex set of G with $s+r = m$. Also, let $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of O_n. Let $deg(u_p) = d_1$, for all $1 \leq p \leq s$ and $deg(v_q) = d_2$, for all $1 \leq q \leq r$. As G is a non-regular graph, we have $d_1 \neq d_2$. For $1 \leq i \leq n$, let u_i^j and v_i^j be the vertices of $G[O_n]$ that correspond with vertices u_j for $1 \leq j \leq s$ and v_j for $1 \leq j \leq r$ respectively of G. As n and m are odd, by Theorem 8, we have a magic rectangle A of size $n \times m$ with entries $(a_{i,j})$, having the sum of entries in each column as $\frac{n(nm+1)}{2}$. We use this magic rectangle to label $G[O_n]$. Define $f\colon V(G[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(u_i^j) = a_{i,j} \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq s,$$
$$f(v_i^j) = a_{i,s+j} \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq r.$$

For the weight of the vertices we have,

$$
\begin{aligned}
w(u_i^j) &= \sum_{v_j \in N(u_j)} \sum_{i=1}^{n} f(v_i^j) = \frac{d_1 n(nm+1)}{2} \quad \text{for } 1 \le i \le n \text{ and } 1 \le j \le s,\\
w(v_i^j) &= \sum_{u_j \in N(v_j)} \sum_{i=1}^{n} f(u_i^j) = \frac{d_2 n(nm+1)}{2} \quad \text{for } 1 \le i \le n \text{ and } 1 \le j \le r.
\end{aligned}
$$

As $d_1 \neq d_2$, the two weights obtained are distinct, and hence f is a local distance antimagic labeling of $G[O_n]$, and the result follows.

□

Theorem 18. *For integers $n > 1$ and $a \neq b$, $\chi_{ld}(K_{a,b}[O_n]) = 2$.*

Proof. Let $V(K_{a,b}) = \{u_1, u_2, \dots, u_a\} \cup \{v_1, v_2, \dots, v_b\}$ and $V(O_n) = \{x_1, x_2, \dots, x_n\}$ be the vertex sets of $K_{a,b}$ and O_n respectively. For $1 \le i \le n$, let u_i^j and v_i^j be the vertices of $K_{a,b}[O_n]$ that correspond with vertices u_j for $1 \le j \le a$ and v_j for $1 \le j \le b$ respectively of $K_{a,b}$. If n is even, the result follows from Lemma 1. If n and $a+b$ are odd, the result follows from Lemma 2. We shall now prove the result for the case; n is odd and $a+b$ is even. Without loss of generality, let us assume $a < b$. Define $f\colon V(K_{a,b}[O_n]) \to \{1, 2, \dots, (a+b)n\}$ by

$$
\begin{aligned}
f(u_i^j) &= (i-1)a + j \quad \text{for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, a,\\
f(v_i^j) &= an + (i-1)b + j \quad \text{for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, b.
\end{aligned}
$$

For the weight of the vertices we have,

$$
\begin{aligned}
w(u_i^j) &= \sum_{j=1}^{b} \sum_{i=1}^{n} f(v_i^j)\\
&= \sum_{j=1}^{b} \sum_{i=1}^{n} \big(an + (i-1)b + j\big)\\
&= \sum_{j=1}^{b} \left(an^2 + jn + \frac{bn(n-1)}{2}\right) = \frac{bn(bn+1)}{2} + abn^2 \qquad \text{for } 1 \le i \le n \text{ and } j = 1, 2, \dots, a,\\
w(v_i^j) &= \sum_{j=1}^{a} \sum_{i=1}^{n} f(u_i^j)\\
&= \sum_{j=1}^{a} \sum_{i=1}^{n} \big((i-1)a + j\big)\\
&= \sum_{j=1}^{a} \left(\frac{an(n-1)}{2} + jn\right) = \frac{an(an+1)}{2} \qquad \text{for } 1 \le i \le n \text{ and } j = 1, 2, \dots, b.
\end{aligned}
$$

Clearly, both the weights are distinct and therefore, f is a local distance antimagic labeling. Hence, $\chi_{ld}(K_{a,b}[O_n]) = 2$.

□

Above, we have seen examples of non-regular bipartite graphs G for which $\chi_{ld}(G[O_n]) = 2$. However, there are some non-regular bipartite graphs for which the result is not true. Below, we present examples of such graphs.

Theorem 19. *Let G be a bipartite graph having two vertices x and y belonging to the same partite set, such that $N(x) \subset N(y)$. Then $\chi_{ld}(G[O_n]) \geq 3$.*

Proof. For $1 \leq i \leq n$, let x_i and y_i be the vertices of $G[O_n]$ that correspond with vertices x and y of G. As $N(x) \subset N(y)$, by the construction of $G[O_n]$, $N(x_i) \subset N(y_i)$ for all $1 \leq i \leq n$. Consider the vertices x_1 and y_1 of $G[O_n]$. If $\chi_{ld}(G[O_n]) = 2$, then as x and y belong to the same partite set, we should have $w(x_1) = w(y_1)$. But as $N(x_1) \subset N(y_1)$, this is not possible. Therefore $\chi_{ld}(G[O_n]) \geq 3$. □

Theorem 20. *Let G be a tree of order $m \geq 3$. Then, $\chi_{ld}(G[O_n]) = 2$ if and only if G is a star.*

Proof. Suppose $G = K_{1,\ m-1}$, then from Theorem 18, $\chi_{ld}(G[O_n]) = 2$. Conversely suppose G is a non-star graph with $\chi_{ld}(G[O_n]) = 2$. Then, G has at least two non-leaf vertices. Let x be a support vertex and y be its adjacent non-leaf vertex. Consider a leaf u attached to the vertex x. As vertices u and y are adjacent to vertex x, they belong to the same partite set, and in addition, $N(u) \subset N(y)$. Therefore by Theorem 19, $\chi_{ld}(G[O_n]) \geq 3$. This is a contradiction; hence, the graph G must be a star. □

Corollary 6. *For integers $n > 1$ and $m \geq 3$, $\chi_{ld}(P_m[O_n]) \geq 3$.*

After obtaining the lower bound for $\chi_{ld}(P_m[O_n])$, we now turn to give an upper bound for the same.

Theorem 21. *For $n > 1$ even and $m \geq 3$,* $\chi_{ld}(P_m[O_n]) \leq \begin{cases} 3 & \text{if } m \text{ is odd,} \\ 4 & \text{if } m \text{ is even.} \end{cases}$

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex sets of P_m and O_n respectively. Let x_i^j be the vertices of $P_m[O_n]$, where $1 \leq j \leq m$ and $1 \leq i \leq n$. We shall deal with the proof in two cases depending upon the parity of m.

Case 1. m is odd.

Let $m = 2k + 1$ for some positive integer k. Using the definition of Matrix 3.4, we construct two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of size $n \times (k + 1)$ and $n \times k$, respectively. As shown in Equation 3.5, the sum of entries in any column of A is $\frac{n(n(k+1)+1)}{2}$ and sum of entries in any column of B is $\frac{n(nk+1)}{2}$. We use these two matrices to label $P_m[O_n]$. Define $f\colon V(P_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(x_i^j) = \begin{cases} a_{i,(\frac{j+1}{2})} & \text{for } j \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n, \\ b_{i,(\frac{j}{2})} + n(k+1) & \text{for } j \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n. \end{cases}$$

For the weight of the vertices, we have, for $1 \leq i \leq n$,

$$
\begin{aligned}
w(x_i^1) &= \sum_{i=1}^{n} f(x_i^2) = \tfrac{n^2k+n}{2} + n^2k + n^2 = \sum_{i=1}^{n} f(x_i^{m-1}) = w(x_i^m),\\
w(x_i^j) &= \sum_{i=1}^{n} \left(f(x_i^{j-1}) + f(x_i^{j+1})\right)\\
&= \begin{cases} \sum_{i=1}^{n} \left(b_{i,(\frac{j-1}{2})} + b_{i,(\frac{j+1}{2})} + 2n(k+1)\right) & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, m,\\ \\ \sum_{i=1}^{n} \left(a_{i,(\frac{j}{2})} + a_{i,(\frac{j+2}{2})}\right) & \text{for } j \equiv 0 \pmod 2, \end{cases}\\
\\
&= \begin{cases} \frac{2n(nk+1)}{2} + 2n^2(k+1) & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, m,\\ \\ \frac{2n(n(k+1)+1)}{2} & \text{for } j \equiv 0 \pmod 2, \end{cases}\\
\\
&= \begin{cases} 3n^2k + 2n^2 + n & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, m,\\ \\ n^2k + n^2 + n & \text{for } j \equiv 0 \pmod 2. \end{cases}
\end{aligned}
$$

Note that as $n^2k + n^2 + n < \frac{n^2k+n}{2} + n^2k + n^2 < 3n^2k + 2n^2 + n$, all the three weights are distinct. For fixed j, the vertices x_i^j receive the same weight for all $1 \leq i \leq n$. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$, f is a local distance antimagic labeling of $P_m[O_n]$ that assigns three distinct weights.

Case 2. m is even.

Let $m = 2k$ for some positive integer $k > 1$. Using the definition of Matrix 3.4 we construct a matrix $A = (a_{i,j})$ of size $n \times k$ and use it to label $P_m[O_n]$. From Equation 3.5, we know that the sum of the entries in any column of A is $\frac{n(nk+1)}{2}$. Define $f\colon V(P_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$
f(x_i^j) = \begin{cases} a_{i,(\frac{j+1}{2})} & \text{for } j \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n,\\ a_{i,(\frac{j}{2})} + nk & \text{for } j \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n. \end{cases}
$$

For the weight of the vertices, we have, for $1 \leq i \leq n$,

$$
\begin{aligned}
w(x_i^1) &= \sum_{i=1}^{n} f(x_i^2) = \frac{n^2k+n}{2} + n^2k,\\
w(x_i^m) &= \sum_{i=1}^{n} f(x_i^{m-1}) = \frac{n^2k+n}{2},
\end{aligned}
$$

$$\begin{aligned} w(x_i^j) &= \sum_{i=1}^{n} \left(f(x_i^{j-1}) + f(x_i^{j+1})\right) \\ &= \begin{cases} \sum_{i=1}^{n} \left(a_{i,(\frac{j-1}{2})} + a_{i,(\frac{j+1}{2})} + 2nk\right) & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, \\ \sum_{i=1}^{n} \left(a_{i,(\frac{j}{2})} + a_{i,(\frac{j+2}{2})}\right) & \text{for } j \equiv 0 \pmod 2,\ j \neq m, \end{cases} \\ &= \begin{cases} 3n^2k + n & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, \\ n^2k + n & \text{for } j \equiv 0 \pmod 2,\ j \neq m. \end{cases} \end{aligned}$$

Note that as $\frac{n^2k+n}{2} < n^2k + n < \frac{n^2k+n}{2} + n^2k < 3n^2k + n$, all four weights are distinct. For fixed j, the vertices x_i^j receive the same weight for all $1 \leq i \leq n$. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$, f is a local distance antimagic labeling of $P_m[O_n]$ that assigns four distinct weights. □

Theorem 22. *For $n > 1$ odd and $m \geq 3$,* $\chi_{ld}(P_m[O_n]) \leq \begin{cases} 3 & \textit{for } m \equiv 3 \pmod 4, \\ 4 & \textit{otherwise.} \end{cases}$

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex sets of P_m and O_n respectively. Let x_i^j be the vertices of $P_m[O_n]$, where $1 \leq j \leq m$ and $1 \leq i \leq n$. We shall deal with the proof in three cases depending upon the nature of m.

Case 1. $m \equiv 1 \pmod 4$.
Let $m = 4k + 1$ for some positive integer k. Since n and $2k + 1$ are odd, according to Theorem 8, there is a magic rectangle A with entries $a_{i,j}$ of size $n \times (2k+1)$ that has the sum of entries in any column as $\frac{n(n(2k+1)+1)}{2}$. Using the definition of Matrix 3.6 and Matrix 3.7 with $t = 1$, we construct two matrices $B = (b_{i,j})$ and $C = (c_{i,j})$,respectively of size $n \times k$. The sum of the entries in any column of B is $kn^2 - 4k + 2$, while the sum of the entries in any column of C is $kn^2 + 4k + n - 2$. We use these constructions to label $P_m[O_n]$. Define $f\colon V(P_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(x_i^j) = \begin{cases} a_{i,(\frac{j+1}{2})} + 2nk & \text{for } j \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n, \\ b_{i,(\frac{j+2}{2})} & \text{for } j \equiv 2 \pmod 4 \text{ and } 1 \leq i \leq n, \\ c_{i,(\frac{j}{4})} & \text{for } j \equiv 0 \pmod 4 \text{ and } 1 \leq i \leq n. \end{cases}$$

For the weight of the vertices we have, for $1 \leq i \leq n$,

$$w(x_i^1) = \sum_{i=1}^{n} f(x_i^2) = \sum_{i=1}^{n} b_{i,2} = kn^2 - 4k + 2,$$
$$w(x_i^m) = \sum_{i=1}^{n} f(x_i^{m-1}) = \sum_{i=1}^{n} c_{i,(\frac{m-1}{4})} = kn^2 + 4k + n - 2,$$

$$
\begin{aligned}
w(x_i^j) &= \sum_{i=1}^{n} \big(f(x_i^{j-1}) + f(x_i^{j+1})\big) \\
&= \begin{cases}
\sum_{i=1}^{n} \big(c_{i,(\frac{j-1}{4})} + b_{i,(\frac{j+3}{2})}\big) & \text{for } j \equiv 1 \pmod 4,\ j \neq 1, m, \\
\sum_{i=1}^{n} \big(c_{i,(\frac{j+1}{4})} + b_{i,(\frac{j+1}{2})}\big) & \text{for } j \equiv 3 \pmod 4, \\
\sum_{i=1}^{n} \big(a_{i,(\frac{j}{2})} + 4nk + a_{i,(\frac{j+2}{2})}\big) & \text{for } j \equiv 0, 2 \pmod 4,
\end{cases} \\
&= \begin{cases}
2n^2k + n & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, m, \\
n^2(2k+1) + n + 4n^2k & \text{for } j \equiv 0 \pmod 2.
\end{cases}
\end{aligned}
$$

Note that as $kn^2 - 4k + 2 < kn^2 + 4k + n - 2 < 2n^2k + n < n^2(2k+1) + n + 4n^2k$, all four weights are distinct. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$, f is a local distance antimagic labeling of $P_m[O_n]$ that assigns four distinct weights.

Case 2. $m \equiv 3 \pmod 4$.

Let $m = 4k+3$ for some positive integer k. Since n and $2k+1$ are odd, by Theorem 8, there is a magic rectangle A with entries $a_{i,j}$ of size $n \times (2k+1)$ having the sum of the entries in any column of A as $\frac{n(n(2k+1)+1)}{2}$. Using the definition of Matrix 3.6 and Matrix 3.7 with $t = 1$, we construct two matrices $B = (b_{i,j})$ and $C = (c_{i,j})$ respectively of size $n \times (k+1)$. The sum of the entries in any column of B is $kn^2 + n^2 - 4k - 2$ while the sum of the entries in any column of C is $kn^2 + n^2 + 4k + n + 2$. We use these constructions to label $P_m[O_n]$. Define $f\colon V(P_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$
f(x_i^j) = \begin{cases}
a_{i,(\frac{j}{2})} + 2n(k+1) & \text{for } j \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n, \\
b_{i,(\frac{j+3}{4})} & \text{for } j \equiv 1 \pmod 4 \text{ and } 1 \leq i \leq n, \\
c_{i,(\frac{j+1}{4})} & \text{for } j \equiv 3 \pmod 4 \text{ and } 1 \leq i \leq n.
\end{cases}
$$

For the weight of the vertices, we have, for $1 \leq i \leq n$,

$$
\begin{aligned}
w(x_i^1) &= \sum_{i=1}^{n} f(x_i^2) = \sum_{i=1}^{n} \big(a_{i,1} + 2n(k+1)\big) = \frac{n^2(2k+1)+n}{2} + 2n^2(k+1), \\
w(x_i^m) &= \sum_{i=1}^{n} f(x_i^{m-1}) = \sum_{i=1}^{n} \big(a_{i,(\frac{m-1}{2})} + 2n(k+1)\big) = \frac{n^2(2k+1)+n}{2} + 2n^2(k+1), \\
w(x_i^j) &= \sum_{i=1}^{n} \big(f(x_i^{j-1}) + f(x_i^{j+1})\big) \\
&= \begin{cases}
\sum_{i=1}^{n} \big(a_{i,(\frac{j-1}{2})} + a_{i,(\frac{j+1}{2})} + 4n(k+1)\big) & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, m, \\
\sum_{i=1}^{n} \big(c_{i,(\frac{j}{4})} + b_{i,(\frac{j+4}{4})}\big) & \text{for } j \equiv 0 \pmod 4, \\
\sum_{i=1}^{n} \big(c_{i,(\frac{j+2}{2})} + b_{i,(\frac{j+2}{2})}\big) & \text{for } j \equiv 2 \pmod 4,
\end{cases} \\
&= \begin{cases}
n^2(6k+5) + n & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, m, \\
n^2(2k+2) + n & \text{for } j \equiv 0 \pmod 2.
\end{cases}
\end{aligned}
$$

Note that as $n^2(2k+2)+n < \frac{n^2(2k+1)+n}{2} + 2n^2(k+1) < n^2(6k+5)+n$, all three weights are distinct. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$, f is a local distance antimagic labeling of $P_m[O_n]$ that assigns three distinct weights.

Case 3. m is even.

Let $m = 2k$ for some positive integer $k > 1$. Using the definition of Matrix 3.6 and Matrix 3.7 with $t = 1$, we construct matrices $B = (b_{i,j})$ and $C = (c_{i,j})$ respectively of size $n \times k$ and use them to label $P_m[O_n]$. The sum of the entries in any column of B is $kn^2 - 4k + 2$, and the sum of the entries in any column of C is $kn^2 + 4k + n - 2$. Define $f \colon V(P_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(x_i^j) = \begin{cases} b_{i,(\frac{j+1}{2})} & \text{for } j \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n, \\ c_{i,(\frac{j}{2})} & \text{for } j \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq n. \end{cases}$$

For the weight of the vertices, we have, for $1 \leq i \leq n$,

$$\begin{aligned} w(x_i^1) &= \sum_{i=1}^{n} f(x_i^2) = \sum_{i=1}^{n} c_{i,1} = kn^2 + 4k + n - 2, \\ w(x_i^m) &= \sum_{i=1}^{n} f(x_i^{m-1}) = \sum_{i=1}^{n} b_{i,(\frac{m}{2})} = kn^2 - 4k + 2, \\ w(x_i^j) &= \sum_{i=1}^{n} \left(f(x_i^{j-1}) + f(x_i^{j+1})\right) \\ &= \begin{cases} \sum_{i=1}^{n} \left(c_{i,(\frac{j-1}{2})} + c_{i,(\frac{j+1}{2})}\right) & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, \\ \sum_{i=1}^{n} \left(b_{i,(\frac{j}{2})} + b_{i,(\frac{j+2}{2})}\right) & \text{for } j \equiv 0 \pmod 2,\ j \neq m, \end{cases} \\ &= \begin{cases} 2(kn^2 + 4k + n - 2) & \text{for } j \equiv 1 \pmod 2,\ j \neq 1, \\ 2(kn^2 - 4k + 2) & \text{for } j \equiv 0 \pmod 2,\ j \neq m. \end{cases} \end{aligned}$$

Note that as $kn^2 - 4k + 2 < kn^2 + 4k + n - 2 < 2(kn^2 - 4k + 2) < 2(kn^2 + 4k + n - 2)$, all four weights are distinct. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$, f is a local distance antimagic labeling of $P_m[O_n]$ that assigns four distinct weights. $\square$

In Corollary 5, we see that if n is odd and $m \equiv 0 \pmod 3$, then $\chi_{ld}(C_m[O_n]) = 3$. In the following Theorem, we generalize this result for any form of odd integer m and any integer n.

Theorem 23. *For integers $m \geq 5$ odd and $n \geq 3$, $\chi_{ld}(C_m[O_n]) = 3$.*

Proof. Let $V(C_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(O_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex sets of C_m and O_n respectively. For $1 \leq j \leq m$ and $1 \leq i \leq n$, let x_i^j be the vertices of $C_m[O_n]$. We prove the result in two main cases depending upon the parity of n.

Case 1. n is even.
We study this case in three more cases depending on the nature of m.
Subcase 1.1. $m \equiv 1 \pmod 6$.
Using the definition of Matrix 3.4, we construct three matrices $A = (a_{i,j})$, $B = (b_{i,j})$ and $C = (c_{i,j})$. Matrix A is of size $n \times (\frac{m+2}{3})$, while the matrices B and C are of size $n \times (\frac{m-1}{3})$. The sum of the entries in any column of Matrix A is $\frac{n\left(n(\frac{m+2}{3})+1\right)}{2}$. Further, the sum of the entries in any column of Matrices B or C is $\frac{n\left(n(\frac{m-1}{3})+1\right)}{2}$. We use these three matrices to label $C_m[O_n]$. Define $f\colon V(C_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(x_i^j) = \begin{cases} a_{i,(\frac{j+2}{3})} & \text{for } j \equiv 1 \pmod 3 \text{ and } 1 \le i \le n, \\ b_{i,(\frac{j+1}{3})} + \frac{n(m+2)}{3} & \text{for } j \equiv 2 \pmod 3 \text{ and } 1 \le i \le n, \\ c_{i,(\frac{j}{3})} + \frac{n(2m+1)}{3} & \text{for } j \equiv 0 \pmod 3 \text{ and } 1 \le i \le n. \end{cases}$$

For the weight of the vertices, we have, for $1 \le i \le n$ and $1 \le j \le m$,

$$w(x_i^j) = \sum_{i=1}^{n} \left(f(x_i^{j-1}) + f(x_i^{j+1})\right),$$

where $j-1$ and $j+1$ are taken modulo m and therefore,

$$w(x_i^j) = \begin{cases} \sum\limits_{i=1}^{n} \left(b_{i,1} + \frac{n(m+2)}{3} + a_{i,(\frac{m+2}{3})}\right) & \text{for } j = 1, \\ \sum\limits_{i=1}^{n} \left(c_{i,(\frac{m-1}{3})} + \frac{n(2m+1)}{3} + a_{i,1}\right) & \text{for } j = m, \\ \sum\limits_{i=1}^{n} \left(b_{i,(\frac{j}{3})} + a_{i,(\frac{j+3}{3})} + \frac{n(m+2)}{3}\right) & \text{for } j \equiv 0 \pmod 3, \\ \sum\limits_{i=1}^{n} \left(a_{i,(\frac{j+1}{3})} + c_{i,(\frac{j+1}{3})} + \frac{n(2m+1)}{3}\right) & \text{for } j \equiv 2 \pmod 3, \\ \sum\limits_{i=1}^{n} \left(c_{i,(\frac{j-1}{3})} + b_{i,(\frac{j+2}{3})} + \frac{n(m+2)}{3} + \frac{n(2m+1)}{3}\right) & \text{for } j \equiv 1 \pmod 3,\ j \ne 1, m. \end{cases}$$

$$w(x_i^j) = \begin{cases} \frac{n^2(4m+5)}{6} + n & \text{for } j = 1,\ j \equiv 0 \pmod 3, \\ \frac{n^2(6m+3)}{6} + n & \text{for } j = m,\ j \equiv 2 \pmod 3, \\ \frac{n^2(8m+4)}{6} + n & \text{for } j \equiv 1 \pmod 3,\ j \ne 1, m. \end{cases}$$

$$w(x_i^j)=\begin{cases}\frac{n}{2}\big(\frac{n(m+2)}{3}+1\big)+\frac{n^2(m+2)}{3} & \\ +\frac{n}{2}\big(\frac{n(m-1)}{3}+1\big) & \text{for } j=1,\ j\equiv 0 \pmod 3,\\ & \\ \frac{n}{2}\big(\frac{n(m+2)}{3}+1\big)+\frac{n^2(2m+1)}{3} & \\ +\frac{n}{2}\big(\frac{n(m-1)}{3}+1\big) & \text{for } j=m,\ j\equiv 2 \pmod 3,\\ & \\ n\big(\frac{n(m-1)}{3}+1\big)+\frac{n^2(m+2)}{3} & \\ +\frac{n^2(2m+1)}{3} & \text{for } j\equiv 1 \pmod 3,\ j\neq 1,m.\end{cases}$$

Clearly, all the three weights are distinct. As $w(x_i^j)\neq w(x_i^{j+1})$, for all $1\leq i\leq n$ and $1\leq j\leq m$, where j and $j+1$ are taken modulo m, f is a local distance antimagic labeling of $C_m[O_n]$ that assigns three distinct weights to its vertices.

Subcase 1.2. $m\equiv 3 \pmod 6$.
The result for this case follows from Corollary 5.

Subcase 1.3. $m\equiv 5 \pmod 6$.
If $m\equiv 5 \pmod 6$ then either $m\equiv 5 \pmod{12}$ or $m\equiv 11 \pmod{12}$. So we shall deal with this case in two special cases.

Subcase 1.3.1. $m\equiv 5 \pmod{12}$.
Using the definition of Matrix 3.4, we construct three matrices $A=(a_{i,j})$, $B=(b_{i,j})$ and $C=(c_{i,j})$. Matrix A is of size $n\times(\frac{m+1}{2})$, while the matrices B and C are of size $n\times(\frac{m-1}{4})$. The sum of the entries in any column of Matrix A is $\frac{n\big(n(\frac{m+1}{2})+1\big)}{2}$. Further, the sum of the entries in any column of Matrices B or C is $\frac{n\big(n(\frac{m-1}{4})+1\big)}{2}$. We use these three matrices to label $C_m[O_n]$. Define $f\colon V(C_m[O_n])\to\{1,2,\ldots,mn\}$ by

$$f(x_i^j)=\begin{cases} b_{i,(\frac{j+3}{4})}+\frac{n(m+1)}{2} & \text{for } j\equiv 1 \pmod 4,\ j\neq m \text{ and } 1\leq i\leq n,\\ c_{i,(\frac{j+2}{4})}+\frac{n(3m+1)}{4} & \text{for } j\equiv 2 \pmod 4 \text{ and } 1\leq i\leq n,\\ a_{i,(\frac{j+1}{4})} & \text{for } j\equiv 3 \pmod 4 \text{ and } 1\leq i\leq n,\\ a_{i,(\frac{m-1}{4}+\frac{j}{4})} & \text{for } j\equiv 0 \pmod 4 \text{ and } 1\leq i\leq n,\\ a_{i,(\frac{m+1}{2})} & \text{for } j=m \text{ and } 1\leq i\leq n.\end{cases}$$

For the weight of the vertices, we have, for $1\leq i\leq n$ and $1\leq j\leq m$,

$$w(x_i^j)=\sum_{i=1}^{n}\big(f(x_i^{j-1})+f(x_i^{j+1})\big),$$

where $j-1$ and $j+1$ are taken modulo m and therefore,

$$w(x_i^j) = \begin{cases} \sum\limits_{i=1}^{n} \big(a_{i,(\frac{m+j-2}{4})} + c_{i,(\frac{j+3}{4})} + \frac{n(3m+1)}{4}\big) & \text{for } j \equiv 1 \pmod 4,\ j \neq m, \\ \sum\limits_{i=1}^{n} \big(a_{i,(\frac{m+j}{4})} + c_{i,(\frac{j+1}{4})} + \frac{n(3m+1)}{4}\big) & \text{for } j \equiv 3 \pmod 4, \\ \sum\limits_{i=1}^{n} \big(b_{i,(\frac{j+4}{4})} + a_{i,(\frac{j}{4})} + \frac{n(m+1)}{2}\big) & \text{for } j \equiv 0 \pmod 4, \\ \sum\limits_{i=1}^{n} \big(b_{i,(\frac{j+2}{4})} + a_{i,(\frac{j+2}{4})} + \frac{n(m+1)}{2}\big) & \text{for } j \equiv 2 \pmod 4, \\ \sum\limits_{i=1}^{n} \big(b_{i,1} + a_{i,(\frac{m-1}{2})} + \frac{n(m+1)}{2}\big) & \text{for } j = m, \\ \sum\limits_{i=1}^{n} \big(a_{i,(\frac{m-1}{4})} + a_{i,(\frac{m+1}{2})}\big) & \text{for } j = m-1. \end{cases}$$

$$w(x_i^j) == \begin{cases} \frac{n}{2}\big(\frac{n(m-1)}{4} + 1\big) + \frac{n^2(3m+1)}{4} & \\ +\frac{n}{2}\big(\frac{n(m+1)}{2} + 1\big) & \text{for } j \equiv 1 \pmod 2,\ j \neq m, \\ & \\ \frac{n}{2}\big(\frac{n(m-1)}{4} + 1\big) + \frac{n^2(m+1)}{2} & \\ +\frac{n}{2}\big(\frac{n(m+1)}{2} + 1\big) & \text{for } j \equiv 0 \pmod 2,\ j = m,\ j \neq m-1, \\ & \\ n\big(\frac{n(m+1)}{2} + 1\big) & \text{for } j = m-1. \end{cases}$$

$$w(x_i^j) = \begin{cases} \frac{n^2(9m+3)}{8} + n & \text{for } j \equiv 1 \pmod 2,\ j \neq m, \\ & \\ \frac{n^2(7m+5)}{8} + n & \text{for } j \equiv 0 \pmod 2,\ j = m,\ j \neq m-1, \\ & \\ \frac{n^2(4m+4)}{8} + n & \text{for } j = m-1. \end{cases}$$

Clearly, all three weights are distinct. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, where j and $j+1$ are taken modulo m, f is a local distance antimagic labeling of $C_m[O_n]$ that assigns three distinct weights to its vertices.

Subcase 1.3.2. $m \equiv 11 \pmod{12}$.

Using the definition of Matrix 3.4, we construct three matrices $A = (a_{i,j})$, $B = (b_{i,j})$ and $C = (c_{i,j})$. Matrix A is of size $n \times (\frac{m-1}{2})$, while the matrices B and C are of size $n \times (\frac{m+1}{4})$. The sum of the entries in any column of Matrix A is $\frac{n\left(n(\frac{m-1}{2})+1\right)}{2}$.

Further, the sum of the entries in any column of Matrices B or C is $\frac{n\left(n(\frac{m+1}{4})+1\right)}{2}$. We use these three matrices to label $C_m[O_n]$. Define $f\colon V(C_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(x_i^j) = \begin{cases} b_{i,(\frac{j+3}{4})} + \frac{n(m-1)}{2} & \text{for } j \equiv 1 \pmod 4\text{, and } 1 \leq i \leq n, \\ c_{i,(\frac{j+2}{4})} + \frac{n(3m-1)}{4} & \text{for } j \equiv 2 \pmod 4 \text{ and } 1 \leq i \leq n, \\ a_{i,(\frac{j+1}{4})} & \text{for } j \equiv 3 \pmod 4 \text{ and } 1 \leq i \leq n, \\ a_{i,(\frac{m+1}{4}+\frac{j}{4})} & \text{for } j \equiv 0 \pmod 4 \text{ and } 1 \leq i \leq n. \end{cases}$$

For the weight of the vertices, we have, for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$w(x_i^j) = \sum_{i=1}^{n} \left(f(x_i^{j-1}) + f(x_i^{j+1})\right),$$

where $j-1$ and $j+1$ are taken modulo m and therefore,

$$w(x_i^j) = \begin{cases} \sum_{i=1}^{n} \left(c_{i,(\frac{j+3}{4})} + a_{i,(\frac{m+j}{4})} + \frac{n(3m-1)}{4}\right) & \text{for } j \equiv 1 \pmod 4\text{, } j \neq m, \\ \sum_{i=1}^{n} \left(a_{i,(\frac{j+2}{4})} + b_{i,(\frac{j+2}{4})} + \frac{n(m-1)}{2}\right) & \text{for } j \equiv 2 \pmod 4, \\ \sum_{i=1}^{n} \left(c_{i,(\frac{j+1}{4})} + a_{i,(\frac{m+j+2}{4})} + \frac{n(3m-1)}{4}\right) & \text{for } j \equiv 3 \pmod 4, \\ \sum_{i=1}^{n} \left(b_{i,(\frac{j+4}{4})} + a_{i,(\frac{j}{4})} + \frac{n(m-1)}{2}\right) & \text{for } j \equiv 0 \pmod 4, \\ \sum_{i=1}^{n} \left(b_{i,1} + c_{i,(\frac{m+1}{4})} + \frac{n(m-1)}{2} + \frac{n(3m-1)}{4}\right) & \text{for } j = m. \end{cases}$$

$$w(x_i^j) = \begin{cases} \frac{n}{2}\left(\frac{n(m+1)}{4} + 1\right) + \frac{n}{2}\left(\frac{n(m-1)}{2} + 1\right) + \\ \frac{n^2(3m-1)}{4} & \text{for } j \equiv 1 \pmod 2\text{, } j \neq m, \\ \\ \frac{n}{2}\left(\frac{n(m+1)}{4} + 1\right) + \frac{n}{2}\left(\frac{n(m-1)}{2} + 1\right) + \\ \frac{n^2(m-1)}{2} & \text{for } j \equiv 0 \pmod 2, \\ \\ n\left(\frac{n(m+1)}{4} + 1\right) + \frac{n^2(3m-1)}{4} + \frac{n^2(m-1)}{2} & \text{for } j = m. \end{cases}$$

$$w(x_i^j) = \begin{cases} \frac{n^2(9m-3)}{8} + n & \text{for } j \equiv 1 \pmod 2\text{, } j \neq m, \\ \\ \frac{n^2(7m-5)}{8} + n & \text{for } j \equiv 0 \pmod 2, \\ \\ \frac{n^2(12m-4)}{8} + n & \text{for } j = m. \end{cases}$$

We see that all three weights are distinct. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, where j and $j+1$ are taken modulo m, f is a local distance antimagic labeling of $C_m[O_n]$ that assigns three distinct weights to its vertices.

Case 2. n is odd.

We shall prove the result in three cases depending on the nature of m.

Subcase 2.1. $m \equiv 1 \pmod 6$.

As $m \equiv 1 \pmod 6$, $\left(\frac{m+2}{3}\right)$ is an odd positive integer. By Theorem 8 we have a magic rectangle A with entries $(a_{i,j})$ of size $n \times (\frac{m+2}{3})$, having constant column sum as $\frac{n^2(m+2)}{6} + \frac{n}{2}$. Further using the definition of Matrix 3.6 and Matrix 3.7 with $t = 1$, we construct two matrices $B = (b_{i,j})$ and $C = (c_{i,j})$ respectively of size $n \times (\frac{m-1}{3})$. The sum of the entries in any column of Matrix B is $(\frac{m-1}{3})n^2 - 4(\frac{m-1}{3}) + 2$, while the sum of the entries in any column of Matrix C is $(\frac{m-1}{3})n^2 + 4(\frac{m-1}{3}) + n - 2$. We use these constructions to label $C_m[O_n]$. Define $f \colon V(C_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(x_i^j) = \begin{cases} a_{i,(\frac{j+2}{3})} & \text{for } j \equiv 1 \pmod 3 \text{ and } 1 \leq i \leq n, \\ b_{i,(\frac{j+1}{3})} + \frac{n(m+2)}{3} & \text{for } j \equiv 2 \pmod 3 \text{ and } 1 \leq i \leq n, \\ c_{i,(\frac{j}{3})} + \frac{n(2m+1)}{3} & \text{for } j \equiv 0 \pmod 3 \text{ and } 1 \leq i \leq n. \end{cases}$$

For the weight of the vertices, we have, for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$w(x_i^j) = \sum_{i=1}^{n} \big(f(x_i^{j-1}) + f(x_i^{j+1})\big),$$

where $j-1$ and $j+1$ are taken modulo m and therefore by similar calculations of vertex weights as discussed in Subcase 1.1, we have

$$w(x_i^j) = \begin{cases} \frac{n}{2}\left(\frac{n(m+2)}{3} + 1\right) + (\frac{m-1}{3})n^2 - 4(\frac{m-1}{3}) + \\ n^2(\frac{m+2}{3}) + 2 & \text{for } j = 1,\ j \equiv 0 \pmod 3, \\ \\ \frac{n}{2}\left(\frac{n(m+2)}{3} + 1\right) + (\frac{m-1}{3})n^2 + 4(\frac{m-1}{3}) + \\ n^2(\frac{2m+1}{3}) + n - 2 & \text{for } j = m,\ j \equiv 2 \pmod 3, \\ \\ 2(\frac{m-1}{3})n^2 + n^2(m+1) + n & \text{for } j \equiv 1 \pmod 3,\ j \neq 1, m. \end{cases}$$

$$w(x_i^j) = \begin{cases} \frac{n^2(5m+4)}{6} + \frac{n}{2} - \frac{4m}{3} + \frac{10}{3} & \text{for } j = 1,\ j \equiv 0 \pmod 3, \\ \\ \frac{n^2(7m+2)}{6} + \frac{3n}{2} + \frac{4m}{3} - \frac{10}{3} & \text{for } j = m,\ j \equiv 2 \pmod 3, \\ \\ \frac{n^2(10m+2)}{6} + n & \text{for } j \equiv 1 \pmod 3,\ j \neq 1, m. \end{cases}$$

Note that, $\frac{n^2(5m+2)}{6}+\frac{n}{2}-\frac{4m}{3}+\frac{10}{3}<\frac{n^2(7m+2)}{6}+\frac{3n}{2}+\frac{4m}{3}-\frac{10}{3}<\frac{n^2(10m+2)}{6}+n$ and hence all the three weights are distinct. As $w(x_i^j)\neq w(x_i^{j+1})$, for all $1\leq i\leq n$ and $1\leq j\leq m$, where j and $j+1$ are taken modulo m, f is a local distance antimagic labeling of $C_m[O_n]$ that assigns three distinct weights to its vertices.

Subcase 2.2. $m\equiv 3 \pmod 6$.
The result follows from Corollary 5.

Subcase 2.3. $m\equiv 5 \pmod 6$.
If $m\equiv 5 \pmod 6$ then either $m\equiv 5 \pmod{12}$ or $m\equiv 11 \pmod{12}$. So we shall deal with this case in these two special cases.

Subcase 2.3.1. $m\equiv 5 \pmod{12}$.
As $m\equiv 5 \pmod{12}$, $\left(\frac{m+1}{2}\right)$ is an odd positive integer. By Theorem 8 we have a magic rectangle A with entries $(a_{i,j})$ of size $n\times(\frac{m+1}{2})$, having constant column sum as $\frac{n^2(m+1)}{4}+\frac{n}{2}$. Further using the definition of Matrix 3.6 and Matrix 3.7 with $t=1$, we construct two matrices $B=(b_{i,j})$ and $C=(c_{i,j})$ respectively of size $n\times(\frac{m-1}{4})$. The sum of the entries in any column of Matrix B is $(\frac{m-1}{4})n^2-4(\frac{m-1}{4})+2$, while the sum of the entries in any column of Matrix C is $(\frac{m-1}{4})n^2+4(\frac{m-1}{4})+n-2$. We use these constructions to label $C_m[O_n]$. Define $f\colon V(C_m[O_n])\to\{1,2,\ldots,mn\}$ by

$$f(x_i^j)=\begin{cases} b_{i,(\frac{j+3}{4})}+\frac{n(m+1)}{2} & \text{for } j\equiv 1 \pmod 4,\ j\neq m \text{ and } 1\leq i\leq n,\\ c_{i,(\frac{j+2}{4})}+\frac{n(3m+1)}{4} & \text{for } j\equiv 2 \pmod 4 \text{ and } 1\leq i\leq n,\\ a_{i,(\frac{j+1}{4})} & \text{for } j\equiv 3 \pmod 4 \text{ and } 1\leq i\leq n,\\ a_{i,(\frac{m-1}{4}+\frac{j}{4})} & \text{for } j\equiv 0 \pmod 4 \text{ and } 1\leq i\leq n,\\ a_{i,(\frac{m+1}{2})} & \text{for } j=m \text{ and } 1\leq i\leq n. \end{cases}$$

For the weight of the vertices, we have, for $1\leq i\leq n$ and $1\leq j\leq m$,

$$w(x_i^j)=\sum_{i=1}^{n}\big(f(x_i^{j-1})+f(x_i^{j+1})\big),$$

where $j-1$ and $j+1$ are taken modulo m and therefore by similar calculations of vertex weights as discussed in Subcase 1.3.1, we have,

$$w(x_i^j)=\begin{cases} \frac{n}{2}\big(\frac{n(m+1)}{2}+1\big)+(\frac{m-1}{4})n^2+4(\frac{m-1}{4})+ \\ n^2(\frac{3m+1}{4})+n-2 & \text{for } j\equiv 1 \pmod 2,\ j\neq m,\\ \\ \frac{n}{2}\big(\frac{n(m+1)}{2}+1\big)+(\frac{m-1}{4})n^2-4(\frac{m-1}{4})+ \\ n^2(\frac{m+1}{2})+2 & \text{for } j\equiv 0 \pmod 2,\ j=m,\ j\neq m-1,\\ \\ n\big(\frac{n(m+1)}{2}+1\big) & \text{for } j=m-1. \end{cases}$$

$$w(x_i^j) = \begin{cases} \frac{n^2(5m+1)}{4} + \frac{3n}{2} + m - 3 & \text{for } j \equiv 1 \pmod 2,\ j \neq m, \\ \frac{n^2(4m+2)}{4} + \frac{n}{2} - m + 3 & \text{for } j \equiv 0 \pmod 2,\ j = m,\ j \neq m-1, \\ \frac{n^2(2m+2)}{4} + n & \text{for } j = m-1. \end{cases}$$

Note that as $\frac{n^2(2m+2)}{4} + n < \frac{n^2(4m+2)}{4} + \frac{n}{2} - m + 3 < \frac{n^2(5m+1)}{4} + \frac{3n}{2} + m - 3$, all the three weights are distinct. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, where j and $j+1$ are taken modulo m, f is a local distance antimagic labeling of $C_m[O_n]$ that assigns three distinct weights to its vertices.

Subcase 2.3.2: $m \equiv 11 \pmod{12}$.

As $m \equiv 11 \pmod{12}$, $\left(\frac{m-1}{2}\right)$ is an odd positive integer. By Theorem 8 we have a magic rectangle A with entries $(a_{i,j})$ of size $n \times (\frac{m-1}{2})$, having constant column sum as $\frac{n^2(m-1)}{4} + \frac{n}{2}$. Further, using the definition of Matrix 3.6 and Matrix 3.7 with $t = 1$, we construct two matrices $B = (b_{i,j})$ and $C = (c_{i,j})$ respectively of size $n \times (\frac{m+1}{4})$. The sum of the entries in any column of Matrix B is $(\frac{m+1}{4})n^2 - 4(\frac{m+1}{4}) + 2$, while the sum of the entries in any column of Matrix C is $(\frac{m+1}{4})n^2 + 4(\frac{m+1}{4}) + n - 2$. We use these constructions to label $C_m[O_n]$. Define $f\colon V(C_m[O_n]) \to \{1, 2, \ldots, mn\}$ by

$$f(x_i^j) = \begin{cases} b_{i,(\frac{j+3}{4})} + \frac{n(m-1)}{2} & \text{for } j \equiv 1 \pmod 4 \text{ and } 1 \leq i \leq n, \\ c_{i,(\frac{j+2}{4})} + \frac{n(3m-1)}{4} & \text{for } j \equiv 2 \pmod 4 \text{ and } 1 \leq i \leq n, \\ a_{i,(\frac{j+1}{4})} & \text{for } j \equiv 3 \pmod 4 \text{ and } 1 \leq i \leq n, \\ a_{i,(\frac{m+1}{4}+\frac{j}{4})} & \text{for } j \equiv 0 \pmod 4 \text{ and } 1 \leq i \leq n. \end{cases}$$

For the weight of the vertices, we have, for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$w(x_i^j) = \sum_{i=1}^{n} \left(f(x_i^{j-1}) + f(x_i^{j+1})\right),$$

where $j-1$ and $j+1$ are taken modulo m and therefore by similar calculations of vertex weights as discussed in Subcase 1.3.2, we have,

$$w(x_i^j) = \begin{cases} \frac{n}{2}\left(\frac{n(m-1)}{2} + 1\right) + (\frac{m+1}{4})n^2 + 4(\frac{m+1}{4}) + \\ n^2(\frac{3m-1}{4}) + n - 2 & \text{for } j \equiv 1 \pmod 2,\ j \neq m, \\ \frac{n}{2}\left(\frac{n(m-1)}{2} + 1\right) + (\frac{m+1}{4})n^2 - 4(\frac{m+1}{4}) + \\ n^2(\frac{m-1}{2}) + 2 & \text{for } j \equiv 0 \pmod 2, \\ \frac{n^2(5m-3)}{4} + \frac{n^2(m+1)}{2} + n & \text{for } j = m. \end{cases}$$

$$w(x_i^j) = \begin{cases} \frac{n^2(5m-1)}{4} + \frac{3n}{2} + m - 1 & \text{for } j \equiv 1 \pmod 2,\ j \neq m, \\ \frac{n^2(4m-2)}{4} + \frac{n}{2} - m + 1 & \text{for } j \equiv 0 \pmod 2, \\ \frac{n^2(7m-1)}{4} + n & \text{for } j = m. \end{cases}$$

Note that as, $\frac{n^2(4m-2)}{4} + \frac{n}{2} - m + 1 < \frac{n^2(5m-1)}{4} + \frac{3n}{2} + m - 1 < \frac{n^2(7m-1)}{4} + n$, all the three weights are distinct. As $w(x_i^j) \neq w(x_i^{j+1})$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, where j and $j+1$ are taken modulo m, f is a local distance antimagic labeling of $C_m[O_n]$ that assigns three distinct weights to its vertices.

In each case, we get three distinct weights and therefore $\chi_{ld}(C_m[O_n]) \leq 3$. As $\chi(C_m[O_n]) = 3$, we have $\chi_{ld}(C_m[O_n]) = 3$. □

4. Conclusion

In this paper, we provided two examples of classes of graphs G of order n having $\chi_{ld}(G) = p$, where $2 \leq p \leq n$. Further in Section 3.1, we studied the local distance antimagic chromatic number for the join of graphs and presented examples of classes of graphs G and H for which $\chi_{ld}(G \vee H) = \chi_{ld}(G) + \chi_{ld}(H)$.

In Section 3.2 we introduced a new class of graphs called as r-uniform k-colorable graphs. Section 3.3 studied the lexicographic product of some graphs with the complement of the complete graph. We found several classes of graphs G for which $\chi_{ld}(G[O_n]) = \chi(G)$. We also found classes of graphs for which the equality does not hold.

The following problems naturally arise:

Problem 3. Characterize graphs G and H for which $\chi_{ld}(G \vee H) = \chi_{ld}(G) + \chi_{ld}(H)$.

Problem 4. Characterize graphs G for which $\chi_{ld}(G[O_n]) = \chi(G)$.

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