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Research Article

On graphs having proper $(1, k)$-dominating sets

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Abstract: For an integer $k \geq 1$, a $(1, k)$-dominating set (or $(1, k)$-dset for short) of a graph $G = (V, E)$ is a set S of vertices having the property that for every vertex $v \in V - S$, there is at least one vertex in S within distance 1 from v and a second vertex in S within distance at most k from v. For $k \geq 2$, a $(1, k)$-dset D of a graph G is a proper $(1, k)$-dset if is not a $(1, k-1)$-dset. More precisely, D is a proper $(1, k)$-dset (or $(1, \overline{k})$-dset for short) if it is $(1, k)$-dset and there is at least a vertex $u \in V - D$ for which there are two distinct vertices t and z in D such that t is adjacent to u, z and u are at distance exactly k apart from each other and no vertex in $D - \{t\}$ is at distance less than k from u. The $(1, k)$-domination number $\gamma_{1,k}(G)$ (resp. $(1, \overline{k})$-domination number $\gamma_{1,\overline{k}}(G)$) of G is the minimum cardinality among all $(1, k)$-dsets (resp. $(1, \overline{k})$-dsets) of G. In this paper, we are interested in the study of $(1, k)$-dsets as well as their existence. We start by giving a necessary and sufficient condition for graphs having $(1, \overline{k})$-dsets for $k \in \{3, 4\}$. Then we consider triangle-free graphs G with equal $\gamma_{1,k}(G)$ and $\gamma_{1,\overline{k}}(G)$ where characterizations are given when $k \in \{3, 4\}$. Finally, we show that the decision problems associated with $(1, 3)$-domination, $(1, \overline{3})$-domination and $(1, \overline{4})$-domination are $\mathcal{NP}$-complete for bipartite graphs.

Keywords: $(1, k)$-domination, $(1, \overline{k})$-domination.

AMS Subject Classification: 05C69

1. Introduction

Let G be a simple graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The *open neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$ and the

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closed neighborhood is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v of G is $d_G(v) = |N_G(v)|$. An *isolated vertex* is a vertex with degree zero. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A *universal vertex* in G is a vertex that is adjacent to all the other vertices in G. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest (u,v)-path in G while the *diameter*, $\operatorname{diam}(G)$, of G is the maximum distance among all pairs of vertices in G. The set of vertices that are within distance k from a vertex $v \in V$ is denoted by $N_G^k[v]$, that is $N_G^k[v] = \{x \in V : d_G(v,x) \le k\}$. For any integer $i = 0, 1, ..., k$, we define $A_G^i(v)$ as the set of all vertices that are at distance exactly i from v, that is $A_G^i(v) = \{x \in V : d_G(v,x) = i\}$. Hence $N_G^k[v] = \bigcup_{i=0}^{k} A_G^i(v)$, where $A_G^0(v) = \{v\}$. For a set $X \subseteq V(G)$, we denote the subgraph induced by X by $G[X]$. As usual, the *path* (*cycle*, *complete graph*, *star*, respectively) of order n is denoted by P_n (C_n, K_n, $K_{1,n-1}$, respectively). Let H be a graph. A graph G is said to be H*-free* if it has no induced subgraph isomorphic to H. In particular, a K_3-free graph will be simply called a *triangle-free* graph. A *tree* is a connected graph with no cycle. A tree T is a *double star*, if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support is denoted $S_{p,q}$. Also a graph is *bipartite* if its vertex set can be partitioned into two independent sets.

A subset D of V is a *dominating set* of G if every vertex is either in D or has a neighbor in D. The cardinality of a minimum dominating set of G is the *domination number* of G, denoted by $\gamma(G)$.

In [7], Hedetniemi et al. generalized the concept of domination to $(1,k)$-domination, by defining a subset $D \subseteq V$ in a graph $G = (V,E)$ to be a $(1,k)$-*dominating set* (abbreviated, $(1,k)$-dset) if for every vertex $v \in V - D$ there are two distinct vertices $u, w \in D$ such that u is adjacent to v, and w is within distance k from v, that is $d_G(v,w) \le k$. In this case the vertex v is said to be $(1,k)$-dominated by D. The $(1,k)$-*domination number* $\gamma_{1,k}(G)$ of G is the minimum cardinality of a $(1,k)$-dominating set in G. A $(1,k)$-dset of cardinality $\gamma_{1,k}(G)$ is called $\gamma_{1,k}(G)$-set. It is worth noting that for $k = 1$, $\gamma_{1,k}(G)$ is the 2-domination number $\gamma_2(G)$ introduced by Fink and Jacobson [6].

It is essential to note that the study of $(1,k)$-dsets in their general way remains open and only particular cases have been considered. Indeed, for $k = 2$, $(1,k)$-dsets were studied by Hedetniemi et al. in [7], and also in [10, 12, 13]. Moreover, the independent version of $(1,k)$-dsets have been considered in [7] for $k = 2$.

Hedetniemi et al. [7] presented the following general result about arbitrary dominating sets in connected graphs.

Theorem 1 ([7]). *Every dominating set in a connected graph G with $\gamma(G) \ge 2$ is a $(1,4)$-dset.*

The following observation is obvious.

Observation 2. Let G be a connected graph with $\gamma(G) \geq 2$, and let k be an integer with $1 \leq k \leq 4$. Then every $(1,j)$-dominating set is a $(1,k)$-dominating set for $j \leq k$, and thus

$$\gamma(G) = \gamma_{1,4}(G) \leq \gamma_{1,3}(G) \leq \gamma_{1,2}(G) \leq \gamma_{1,1}(G) = \gamma_2(G) \tag{1.1}$$

Later, in [9], Michalski introduced the concept of proper $(1,2)$-dominating sets. A $(1,2)$-dominating set D is a *proper* $(1,2)$*-dominating* set (or simply $(1,\overline{2})$-dset) of a graph G if D is a $(1,2)$-dset but it is not a $(1,1)$-dset. In other words, D is a proper $(1,2)$-dominating set if it is $(1,2)$-dominating and there exists a vertex $x \in V - D$ such that x has exactly one adjacent vertex in D. The cardinality of a minimum $(1,\overline{2})$-dset in a graph G is the *proper* $(1,2)$*-domination number* $\gamma_{1,\overline{2}}(G)$ of G.

In [11], Michalski et al. studied some properties of $(1,\overline{2})$-dsets. In particular a necessary and sufficient condition for the existence of such sets in a connected graph was given. In [8], Kosiorowska et al. studied intersections of the $(1,1)$-sets and $(1,\overline{2})$-dsets, where they introduced the $(1,\overline{2})$-intersection index defined as the minimum possible cardinality of such intersections. The authors [8] determined the exact value of $(1,\overline{2})$-intersection index for some classes of graphs.

We generalize the concept of proper $(1,2)$-dominating sets to proper $(1,k)$-dominating sets. For $k \geq 2$, a $(1,k)$-dset D of a graph G is a *proper* $(1,k)$*-dset* if is not a $(1,k-1)$-dset. In other words, D is a proper $(1,k)$-dset, abbreviated $(1,\overline{k})$-dset, if it is $(1,k)$-dset and there is at least a vertex $u \in V - D$ for which there are two distinct vertices t and z in D such that t is adjacent to u, $d_G(u,z) = k$ and no vertex in $D - \{t\}$ is at distance less than k from u. In this case the vertex u is said to be $(1,\overline{k})$-dominated by D. The cardinality of a minimum $(1,\overline{k})$-dset in a graph G is *the proper* $(1,k)$*-domination number* $\gamma_{1,\overline{k}}(G)$ of G. A $(1,\overline{k})$-dset of cardinality $\gamma_{1,\overline{k}}(G)$ is called $\gamma_{1,\overline{k}}(G)$-set. Since every $(1,\overline{k})$-dset of G is a $(1,k)$-dset of G, we have $\gamma_{1,k}(G) \leq \gamma_{1,\overline{k}}(G)$. Moreover, by Theorem 1, the following result is provided.

Corollary 1. *If G is a connected graph having a $(1,\overline{k})$-dset, then $k \leq 4$.*

From this Corollary, we deduce that a proper $(1,k)$-dominating set does not exist, for $k \geq 5$.

It is worth mentioning that an inequality chain similar to (1.1) cannot apply to $\gamma_{1,\overline{k}}(G)$, as can be seen by considering the tree $S^*_{p,q}$ obtained from a double star $S_{p,q}$, with $p \leq q$, by subdividing once the edge joining the support vertices of $S_{p,q}$. One can see that $\gamma_{1,\overline{4}}\left(S^*_{p,q}\right) = p+1$, $\gamma_{1,\overline{3}}\left(S^*_{p,q}\right) = 2$ and $\gamma_{1,\overline{2}}\left(S^*_{p,q}\right) = 3$.

In this paper, we are interested in the study and the existence of $(1,\overline{k})$-dsets. We start by giving a characterization of connected graphs having such sets for $k \in \{3,4\}$. Then we study connected triangle-free graphs G with $\gamma_{1,k}(G) = \gamma_{1,\overline{k}}(G)$ for $k \in \{3,4\}$. Finally, we show that decision problem corresponding to the problem of computing $\gamma_{1,k}(G)$ and $\gamma_{1,\overline{k}}(G)$ are $\mathcal{NP}$-complete even when restricted to bipartite graphs.

2. Graphs having $(1, \overline{k})$-dsets

First, let us recall a result of Michalski et al. [11] providing a characterization of connected graphs having a $(1, \overline{2})$-dset.

Theorem 3 ([11]). *A connected graph G has a $(1, \overline{2})$-dset if and only if G is not a complete graph.*

In what follows we investigate the cases $k = 3$ and $k = 4$ for connected graphs.

Theorem 4. *Let G be a connected graph. Then G has a $(1, \overline{3})$-dset if and only if* $\mathrm{diam}(G) \geq 3$ *and there exist two adjacent vertices t and u such that $N_G(u) \subset N_G[t]$ and for all $y \in A_G^2(u)$ we have $N_G(y) \cap A_G^3(u) = \emptyset$, $y \in N_G(t)$.*

Proof. Assume that G has a $(1, \overline{3})$-dset D. By the definition of a $(1, \overline{3})$-dset, D is a $(1, 3)$-set of G for which there exist a vertex $u \in V - D$ and two vertices t and z in D such that $d_G(u, t) = 1$, $d_G(u, z) = 3$ and no vertex $y \in D - \{t\}$ satisfies $d_G(u, y) \leq 2$. Clearly $\mathrm{diam}(G) \geq 3$ and the vertices u and t are adjacent. Moreover, since D is a dominating set, every neighbor y of u must be adjacent to t, for otherwise some neighbor of y in D will be at distance 2 from u, contradicting the definition of $(1, 3)$-sets. Hence $N_G(u) \subset N_G[t]$. Now let $y \in A_G^2(u)$ and $N_G(y) \cap A_G^3(u) = \emptyset$. Since D is a dominating set of G and there is no vertex $y \in D - \{t\}$ such that $d_G(u, y) = 2$, $y \in N_G(t)$.
Conversely, assume that $\mathrm{diam}(G) \geq 3$, and there exist two adjacent vertices t and u such that $N_G(u) \subset N_G[t]$ where for every $y \in A_G^2(u)$ such that $N_G(y) \cap A_G^3(u) = \emptyset$, $y \in N_G(t)$. Then $A_G^3(u) \neq \emptyset$, otherwise t is a universal vertex; meaning that $\mathrm{diam}(G) \leq 2$ or G is not connected, a contradiction. Now let u-y_1-y_2-z be a shortest path between u and z such that $y_1 \in A_G^1(u)$, $y_2 \in A_G^2(u)$ and $z \in A_G^3(u)$. We shall show that either $D = \left(V - N_G^2[u]\right) \cup \{t\}$ or $D' = D \cup \{y_2\}$, is a $(1, \overline{3})$-dset of G. Note that $A_G^3(u) \subset D$. We distinguish between three cases.

Case 1. Every vertex in $A_G^1(u) - \{t\}$ has a neighbor in $A_G^2(u)$ and every vertex in $A_G^2(u)$ has a neighbor in $A_G^3(u)$. Then u is adjacent to t and since $z \in A_G^3(u)$, u is $(1, \overline{3})$-dominated by D. Moreover, since $N_G(u) \subset N_G[t]$, every vertex y in $A_G^1(u) - \{t\}$ is adjacent to t and there exists $z' \in A_G^3(u)$ such that $d_G(y, z') \leq 2$. So, y is $(1, 3)$-dominated by D. Also, since $N_G^2[u] - \{t\} \subset V - D$ and $A_G^3(u) \subset D$, every vertex y' in $A_G^2(u) - \{t\}$ has a neighbor in $A_G^3(u)$ and since $d_G(y', t) \leq 2$, y' is $(1, 3)$-dominated by D. Hence D is a $(1, \overline{3})$-dset of G.

Case 2. There exists a vertex $y \in A_G^2(u)$ such that $N_G(y) \cap A_G^3(u) = \emptyset$, $y \in N_G(t)$ and y is not $(1, 3)$-dominated by D. Then $yy_1, yy_2, y_2t \notin E$. Moreover $y'y_2 \notin E$ for every vertex $y' \in A_G^1(u) \cap N_G(y)$. We will show that $D' = D \cup \{y_2\}$ is a $(1, \overline{3})$-dset of G. Clearly $y \in N_G(t)$ and the path y-t-y_1-y_2 is a shortest path between y and y_2. Hence y is $(1, \overline{3})$-dominated by D'. Thus, it remains to show that every vertex x of $(V - D') - \{u\}$ is $(1, 3)$-dominated by D'. Note that $N_G[y] \subset N_G^2(u)$. If

$x \in A_G^1(u) - \{t\}$, then x is adjacent to t, because $N_G(u) \subset N_G[t]$, and it is easy to see that $d_G(x, y_2) \leq 3$. Now assume that $x \in A_G^2(u)$. Then either x has no neighbor in $A_G^3(u)$, and thus $x \in N_G(t)$ and $d_G(x, y_2) \leq 3$, or x has a neighbor in $A_G^3(u)$, and thus x has a neighbor in $A_G^3(u)$ and $d_G(x,t) \leq 2$. In both cases, x is $(1,3)$-dominated by D'.

Case 3. There exists a vertex $y \in A_G^1(u) - \{t\}$ such that y has no neighbor in $A_G^2(u)$ and y is not $(1,3)$-dominated by D. Then $yy_1, yy_2, y_2t \notin E$. In this case, the same argument applied in Case 2 shows that $D' = D \cup \{y_2\}$ is a $(1,\overline{3})$-dset of G, and this completes the proof. □

Theorem 5. *Let G be a connected graph. Then G has a $(1,\overline{4})$-dset if and only if* $\mathrm{diam}(G) \geq 4$*, there exist two adjacent vertices t and u such that $N_G^2(u) \subset N_G[t]$ and for all $y \in A_G^3(u)$, $N_G(y) \cap A_G^4(u) \neq \emptyset$.*

Proof. Assume that G contains a $(1,\overline{4})$-dset, say D. By the definition of a $(1,\overline{4})$-dset, D is a $(1,4)$-dset of G, there exists a vertex $u \in V - D$ and there exist two distinct vertices t and $z \in D$ such that $N_G(u) \cap D = \{t\}$, $d_G(u,z) = 4$ and there is no vertex $z' \in D - \{t\}$ such that $d_G(u,z') \leq 3$. Therefore, $\mathrm{diam}(G) \geq 4$, u and t are adjacent and $N_G[y] \cap (D - \{t\}) = \emptyset$, for all $y \in N_G^2[u] - \{t\}$. Moreover, since D is a dominating set of G, $N_G[y] \cap D = \{t\}$, for all $y \in N_G^2[u] - \{t\}$. Hence $N_G^2(u) \subset N_G[t]$. Now let $y \in A_G^3(u)$. Clearly $y \in V - D$, and y is not adjacent to t, for otherwise $y \in A_G^2(u)$. Since D is a dominating set of G, $N_G(y) \cap D \neq \emptyset$, and thus we have $N_G(y) \cap A_G^4(u) \neq \emptyset$ for all $y \in A_G^3(u)$.
Conversely, assume that $\mathrm{diam}(G) \geq 4$, and there exist two adjacent vertices t and u such that $N_G^2(u) \subset N_G[t]$, where for every $y \in A_G^3(u)$, $N_G(y) \cap A_G^4(u) \neq \emptyset$. Then $A_G^4(u) \neq \emptyset$, otherwise t is a universal vertex; meaning that $\mathrm{diam}(G) \leq 2$ or G is not connected, a contradiction. We shall show that the set $D = \left(V - N_G^3[u]\right) \cup \{t\}$ is a $(1,\overline{4})$-dset of G. First we prove that u is $(1,\overline{4})$-dominated by D. Clearly u is adjacent to t, and since $A_G^4(u) \subset D$ and $N_G(y) \cap A_G^4(u) \neq \emptyset$ for every $y \in A_G^3(u)$, $N_G(y) \cap A_G^4(u) \subset D$. Hence, u is $(1,\overline{4})$-dominated by D. Now, it remains to show that every vertex y of $N_G^3(u) - \{t\}$ is $(1,4)$-dominated by D. Observe that $N_G^3(u) - \{t\} = (N_G^2(u) - \{t\}) \cup A_G^3(u)$. Now, let u-y_1-y_2-y_3-z be a shortest path such that $y_i \in A_G^i(u)$ for $i = 1,2,3$ and $z \in A_G^4(u)$. Since $N_G^2(u) \subset N_G[t]$, $y_i \in N_G(t)$ for $i = 1,2$. Moreover $y_3 \notin N_G(t)$, for otherwise $y_3 \in A_G^2(u)$. Now, if $y \in N_G^2(u) - \{t\}$, then obviously $N_G(y) \cap D = \{t\}$. Moreover, we have the path y-t-y_2-y_3-z with $z \in D$ and $yz \notin E$. Hence $2 \leq d_G(y,z) \leq 4$, which means that y is $(1,4)$-dominated by D. Finally, assume that $y \in A_G^3(u)$. Then $N_G(y) \cap A_G^4(u) \neq \emptyset$ and $d_G(y,t) = 2$. Therefore there exists a vertex $w \in A_G^4(u) \cap D$ such that $yw \in E$. Hence y is $(1,4)$-dominated by D, and this completes the proof. □

Since a cycle C_n of order n has no vertices t and u satisfying conditions in Theorems 4 and 5, we deduce the following result.

Corollary 2. *Let C_n be a cycle of order n. Then C_n has neither a $(1, \overline{3})$-dset nor a $(1, \overline{4})$-dset.*

Restricted to the class of triangle-free graphs, we have the following necessary condition for the existence of $(1, \overline{k})$-dsets for $k \in \{3, 4\}$.

Proposition 1. *Let $k \in \{3, 4\}$ and let G be a connected triangle-free graph with $\mathrm{diam}(G) \geq k$. If G has a $(1, \overline{k})$-dset D, then every vertex which is $(1, \overline{k})$-dominated by D is a leaf.*

Proof. Assume that G has a $(1, \overline{k})$-dset D, and let u be a vertex of $V - D$ which is $(1, \overline{k})$-dominated by D. Then there exist two distinct vertices $t, z \in D$ such that $N_G(u) \cap D = \{t\}$, $d_G(u, z) = k$ and no vertex $y \in D - \{t\}$ is at distance at most $k - 1$ from u . Thus, if u is not a leaf, then u has another neighbor v which belongs to $V - D$. Since D is a dominating set and $d_G(u, y) \geq k$ for every vertex $y \in D - \{t\}$, v has to be adjacent to t. Hence $\{u, t, v\}$ induces a triangle, a contradiction. □

The following corollaries are easily deduced from Proposition 1.

Corollary 3. *Let $k \in \{3, 4\}$ and let G be a connected bipartite graph with $\mathrm{diam}(G) \geq k$. If G has a $(1, \overline{k})$-dset D, then every vertex that is $(1, \overline{k})$-dominated by D is a leaf.*

Corollary 4. *Let $k \in \{3, 4\}$ and let T be a tree with $\mathrm{diam}(T) \geq k$. If T has a $(1, \overline{k})$-dset D, then every vertex that is $(1, \overline{k})$-dominated by D is a leaf.*

Corollary 5. *If G is a connected triangle-free graph and without leaf, then G has neither a $(1, \overline{3})$-dset nor a $(1, \overline{4})$-dset.*

Therefore, it is interesting to study triangle-free graphs G with minimum degree one having $(1, \overline{k})$-dsets for $k \in \{3, 4\}$. For this purpose, we begin by providing a necessary and sufficient condition for the existence of $(1, \overline{3})$-dsets, for connected triangle-free graph with $\mathrm{diam}(G) \geq 3$.

Theorem 6. *Let G be a connected triangle-free graph with $\mathrm{diam}(G) \geq 3$. Then G has a $(1, \overline{3})$-dset if and only if G contains a leaf.*

Proof. The necessary condition follows from Proposition 1. For the sufficient condition, let u be a leaf of G and let t its support vertex. Since $\mathrm{diam}(G) \geq 3$, u and t satisfy the condition of Theorem 4. Hence G has a $(1, \overline{3})$-dset. □

Corollary 6. *Let G be a bipartite graph with $\mathrm{diam}(G) \geq 3$. Then G has a $(1, \overline{3})$-dset if and only if G contains a leaf.*

Corollary 7. *If T is a tree with* $\mathrm{diam}(T) \geq 3$*, then T has a* $(1, \overline{3})$*-dset.*

It is worth noting that a connected triangle-free graph G with $\mathrm{diam}(G) \geq 4$ containing a leaf may not have a $(1, \overline{4})$-dset. This can be seen by considering the tree T obtained from two copies of P_4 by adding a new vertex attached to exactly one support vertex of each copy of P_4. It is easy to see that T does not have a $(1, \overline{4})$-dset.

The next result provides a necessary and sufficient condition for the existence of $(1, \overline{4})$-dsets, for trees T with $\mathrm{diam}(T) \geq 4$.

Theorem 7. *Let T be a tree with* $\mathrm{diam}(T) \geq 4$*. Then T has a* $(1, \overline{4})$*-dset if and only if T contains a leaf u such that $d_T(u, y) \neq 3$ for every leaf y of T.*

Proof. Assume that T has a $(1, \overline{4})$-dset. Suppose, for a contradiction, that for every leaf u there exists a leaf y such that $d_T(u, y) = 3$. If t denotes the support vertex adjacent to u, then u and t do not satisfy the condition of Theorem 5, since $y \in A_T^3(u)$, $N_T(y) \cap A_T^4(u) = \emptyset$ and $y \notin N_T(t)$, a contradiction.
Conversely, assume that T contains a leaf u such that $d_T(u, y) \neq 3$ for every leaf y of T, and let t be the support vertex of u. Clearly $N_T^2(u) \subset N_T[t]$ and for all $y \in A_T^3(u)$, $N_G(y) \cap A_T^4(u) \neq \emptyset$, which means that u and t satisfy the condition of Theorem 5. Hence T has a $(1, \overline{4})$-dset. □

As an example of graph verifying the conditions of Theorem 6 (Theorem 7, respectively), we consider the tree T obtained from the star $K_{1,p}$ where $p \geq 2$, by subdividing all its edges once. Then T has a $(1, \overline{3})$-dset (resp., a $(1, \overline{4})$-dset). Even though any $(1, \overline{3})$-dset is different from every $(1, \overline{4})$-dset, we still have equality $\gamma_{1,\overline{3}}(T) = \gamma_{1,\overline{4}}(T) = p$.

3. Graphs G with $\gamma_{1,k}(G) = \gamma_{1,\overline{k}}(G)$

In this section we consider relationships between the proper$(1, k)$-domination and $(1, k)$-domination numbers of a graph G. As mentioned above, $\gamma_{1,k}(G) \leq \gamma_{1,\overline{k}}(G)$ for every connected graph with $\mathrm{diam}(G) \geq k$. In [11], Michalski et al. showed that for $k = 2$ the equality holds in the previous inequality for any connected noncomplete graph.

Theorem 8 ([11]). *For an arbitrary connected noncomplete graph G, we have*

$$\gamma_{1,\overline{2}}(G) = \gamma_{1,2}(G).$$

However, for $k = \{3, 4\}$, inequality $\gamma_{1,k}(G) \leq \gamma_{1,\overline{k}}(G)$ can be strict, as can be seen by the graphs in Figure 1. Indeed, for the graph G_1, the set $\{t, a_2\}$ is a $\gamma_{1,3}$-set which is not a $\gamma_{1,\overline{3}}$-set, while the set $\left\{t, a_3^{'}, a_3^{''}\right\}$ is a $\gamma_{1,\overline{3}}$-set. Also, for the graph G_2, the set $\{t, a_3\}$ is a $\gamma_{1,4}$-set which is not a $\gamma_{1,\overline{4}}$-set, while the set $\left\{t, a_4^{'}, a_4^{''}\right\}$ is a $\gamma_{1,\overline{4}}$-set.

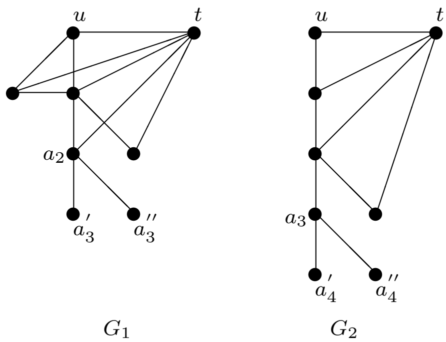

Figure 1. **Graphs with** $\gamma_{1,3}(G_1) < \gamma_{1,\overline{3}}(G_1)$ **and** $\gamma_{1,4}(G_2) < \gamma_{1,\overline{4}}(G_2)$.

As we have seen, by Corollary 5, connected triangle-free graphs with no leaf have no $\gamma_{1,\overline{k}}$-set for $k \in \{3,4\}$. Accordingly, it is interesting to study equality between the proper $(1,k)$-domination and $(1,k)$-domination numbers for connected triangle-free graphs having a leaf. For this purpose, we begin by paths to show that $\gamma_{1,k}(P_n) = \gamma_{1,\overline{k}}(P_n)$ for $k \in \{3,4\}$. It is well-known that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. On the other hand, Raczek [13] proved the following result, for $\gamma_{1,2}(P_n)$.

Theorem 9 ([13]). *For any integer* $n \geq 2$*, we have* $\gamma_{1,2}(P_n) = \lceil \frac{n+2}{3} \rceil$.

The next result provides the exact value of the $(1,3)$-domination number for paths.

Theorem 10. *For any integer* $n \geq 2$*, we have* $\gamma_{1,3}(P_n) = \lceil \frac{n+2}{3} \rceil$.

Proof. Let P_n be a path of order $n \geq 2$, with vertices labeled in order $x_1, x_2, ..., x_n$. Since the result is obvious for $n = 2$, let $n \geq 3$ and consider the following cases.
Case 1. $n = 3p$ for some integer $p \geq 1$. It is easy to see that the only minimum dominating set is $D = \{x_2, x_5, \ldots, x_{3p-1}\}$. But D is not $(1,3)$-dset, since x_1 is not $(1,3)$-dominated by D. It means that $\gamma_{1,3}(P_n) > \gamma(P_n) = \frac{n}{3} = p$. Moreover, the set $D_1 = \{x_1, x_4, \ldots, x_{3p-5}\} \cup \{x_{3p-3}, x_{3p-1}\}$ is a $(1,3)$-dset of cardinality $p+1 = \frac{n+3}{3}$. Hence $\gamma_{1,3}(P_n) = \frac{n+3}{3} = \lceil \frac{n+2}{3} \rceil$.
Case 2. $n = 3p+1$ for some integer $p \geq 1$. It is easy to see that the set $D_2 = \{x_1, x_4, \ldots, x_{3p-2}\} \cup \{x_{3p}\}$ is a $(1,3)$-dset of cardinality $p+1 = \frac{n+2}{3}$. Since $\gamma_{1,3}(P_n) \geq \gamma(P_n) = \frac{n+2}{3}$, we have $\gamma_{1,3}(P_n) = \frac{n+2}{3} = \lceil \frac{n+2}{3} \rceil$.
Case 3. $n = 3p+2$ for some integer $p \geq 1$. It is easy to check that every minimum dominating set of P_n is not a $(1,3)$-dset leading to $\gamma_{1,3}(P_n) > \gamma(P_n) = \frac{n+1}{3} = p+1$.

Moreover, the set $D_3 = \{x_1, x_4, \ldots, x_{3p+1}\} \cup \{x_{3p-1}\}$ is a $(1,3)$-dset of cardinality $p+2$. Hence $\gamma_{1,3}(P_n) = \frac{n+4}{3} = \lceil \frac{n+2}{3} \rceil$. □

Now, we are ready to show that $\gamma_{1,k}(P_n) = \gamma_{1,\overline{k}}(P_n)$ for $k \in \{3,4\}$.

Proposition 2. *For any integer $n \geq 4$, we have $\gamma_{1,3}(P_n) = \gamma_{1,\overline{3}}(P_n)$.*

Proof. Let P_n be a path of order $n \geq 4$. Since $\gamma_{1,3}(P_n) \leq \gamma_{1,\overline{3}}(P_n)$, it is enough to show that P_n contains a $\gamma_{1,3}$-set which is a $(1,\overline{3})$-dset. For that, it is easy to see, according to the values of n, that the sets D_1, D_2 and D_3 given in the proof of Theorem 10 are also $(1,\overline{3})$-dsets of P_n, which completes the proof. □

Proposition 3. *For any integer $n \geq 5$, we have $\gamma_{1,4}(P_n) = \gamma_{1,\overline{4}}(P_n)$.*

Proof. Let P_n be a path of order $n \geq 5$. Since $\gamma(G) = \gamma_{1,4}(P_n) \leq \gamma_{1,\overline{4}}(P_n)$, it is enough to show that P_n contains a γ-set which is a $(1,\overline{4})$-dset. It is easy to see that $D_1 = \{x_2, x_5, \ldots, x_{3p-1}\}$, $D_2 = \{x_2, x_5, \ldots, x_{3p-1}\} \cup \{x_{3p+1}\}$ and $D_3 = \{x_2, x_5, \ldots, x_{3p+2}\}$ are minimum dominating sets of P_n for $n = 3p$, $n = 3p+1$ and $n = 3p+2$, respectively. Moreover, one can see that x_1 is $(1,\overline{4})$-dominated by each of these three sets. Therefore, D_i is a $(1,\overline{4})$-dset of P_n for each $i \in \{1,2,3\}$, leading to the desired equality. □

From Theorems 9, 10, and Propositions 2 and 3, we deduce the following.

Corollary 8. *For any integer $n \geq 5$, we have:*

1. $\gamma_{1,\overline{3}}(P_n) = \gamma_{1,3}(P_n) = \gamma_{1,2}(P_n) = \lceil \frac{n+2}{3} \rceil$.
2. $\gamma_{1,\overline{4}}(P_n) = \gamma_{1,4}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.
3. $\gamma_{1,\overline{3}}(P_n) = \gamma_{1,\overline{4}}(P_n)$ *if and only if* $n \equiv 1 \,(\mathrm{mod} 3)$.

Restricted to the class of trees T, the equality $\gamma_{1,k}(T) = \gamma_{1,\overline{k}}(T)$ is possible, and can be seen by considering the trees $S^*_{p,q}$ and $S^{**}_{p,q}$ both obtained from a double star $S_{p,q}$ by subdividing the edge joining the support vertices once and twice, respectively. Then $\gamma_{1,3}(S^*_{p,q}) = \gamma_{1,\overline{3}}(S^*_{p,q}) = 2$ and $\gamma_{1,4}(S^{**}_{p,q}) = \gamma_{1,\overline{4}}(S^{**}_{p,q}) = 2$. However, the difference $\gamma_{1,\overline{k}}(T) - \gamma_{1,k}(T)$ can be arbitrary large as can be seen by $S_{p,q}$ and $S^*_{p,q}$. Indeed, $\gamma_{1,3}(S_{p,q}) = \gamma_{1,4}(S^*_{p,q}) = 2$, while $\gamma_{1,\overline{3}}(S_{p,q}) = \gamma_{1,\overline{4}}(S^*_{p,q}) = p+1$.

The next result provides a necessary and sufficient condition for connected triangle-free graphs G such that $\gamma_{1,k}(G) = \gamma_{1,\overline{k}}(G)$ for $k \in \{3,4\}$.

Theorem 11. *Let G be a connected triangle-free graph with $\mathrm{diam}(G) \geq 3$. Then $\gamma_{1,3}(G) = \gamma_{1,\overline{3}}(G)$ if and only if G has a $\gamma_{1,3}$-set D containing a support vertex which is isolated in $G[D]$.*

Proof. Assume that $\gamma_{1,3}(G)=\gamma_{1,\overline{3}}(G)$. Then G has a $(1,3)$-dset D which is also a $(1,\overline{3})$-dset. By Proposition 1 there is a leaf $u \notin D$ whose support vertex, say t, belongs to D and $d_G(u,z)=3$ for some $z \in D$. Since there is no vertex $y \in D-\{t\}$ such that $d_G(u,y)=2$, we deduce that t is isolated in $G[D]$.
Conversely, assume that G has a $\gamma_{1,3}$-set D containing a support vertex t which is isolated in $G[D]$, and let u be a leaf neighbor of t. Since t is isolated in $G[D]$ and $\operatorname{diam}(G) \geq 3$, there is a vertex $z \in D$ such that $d_G(u,z)=3$. Therefore u is adjacent to t and $d_G(u,z)=3$, which means that D is a $(1,\overline{3})$-dset of G. Hence $\gamma_{1,\overline{3}}(G) \leq |D| = \gamma_{1,3}(G)$, and the equality follows from $\gamma_{1,3}(G) \leq \gamma_{1,\overline{3}}(G)$. □

Theorem 12. *Let G be a connected triangle-free graph with $\operatorname{diam}(G) \geq 4$. Then $\gamma_{1,4}(G)=\gamma_{1,\overline{4}}(G)$ if and only if G has a $\gamma_{1,4}(G)$-set D containing a support vertex t such that $d_G(t,y) \geq 3$ for every $y \in D-\{t\}$.*

Proof. Assume that $\gamma_{1,4}(G)=\gamma_{1,\overline{4}}(G)$. Thus G has a $(1,4)$-dset D which is also a $(1,\overline{4})$-dset. By Proposition 1, there is a leaf u that is adjacent to a support vertex t and $d_G(u,z)=4$ for some vertex z in D. Therefore $d_G(t,y) \geq 3$ for every $y \in D-\{t\}$.
Conversely, assume that G has a $\gamma_{1,4}(G)$-set D containing a support vertex t such that $d_G(t,y) \geq 3$ for every $y \in D-\{t\}$, and let u be a leaf adjacent to t. Clearly $u \in V-D$, $N_G(u) \cap D = \{t\}$ and $d_G(u,y) \geq 4$ for every $y \in D-\{t\}$. Then there exists a vertex z in D such that $d_G(u,z)=4$ and there is no vertex $y \in D-\{t\}$ with $d_G(u,y) \leq 3$. Therefore u is adjacent to t and $d_G(u,z)=4$, meaning that D is $(1,\overline{4})$-set of G. Hence $\gamma_{1,\overline{4}}(G) \leq |D| = \gamma_{1,4}(G)$, and the equality follows from $\gamma_{1,4}(G) \leq \gamma_{1,\overline{4}}(G)$. □

The following problem can be raised for trees.
Problem. Give a constructive characterization of trees T with $\gamma_{1,k}(G)=\gamma_{1,\overline{k}}(G)$, for $k \in \{3,4\}$.

4. Complexity results

In [13], Raczek showed that the decision problem corresponding to the problem of computing the $(1,2)$-dominating number is $\mathcal{NP}$-complete for bipartite and split graphs. We recall that for $k=4$ (respectively, $k=1$), $(1,k)$-dset coincide with dominating sets (respectively, 2-dominating sets) for which the corresponding decision problems are $\mathcal{NP}$-complete for bipartite and chordal graphs (see [1–5]).
Our aim in this section is to study the complexity of the $(1,3)$-domination number, the proper $(1,3)$-domination number and the proper $(1,4)$-domination number to which we shall refer as $(1,3)$-DOMINATING SET, $(1,\overline{3})$-DOMINATING SET and $(1,\overline{4})$-DOMINATING SET, respectively.

$(1,3)$-DOMINATING SET
Instance: A graph G and a positive integer p.
Question: Does G have a $(1,3)$-dset of size at most p?

$(1, \overline{3})$-DOMINATING SET
Instance: A graph G and a positive integer p.
Question: Does G have a $(1, \overline{3})$-dset of size at most p?

$(1, \overline{4})$-DOMINATING SET
Instance: A graph G and a positive integer p.
Question: Does G have a $(1, \overline{4})$-dset of size at most p?

We will show that the three decision problems mentioned above are $\mathcal{NP}$-complete for bipartite graphs by reducing the well-known $\mathcal{NP}$-complete problem Exact-3-Cover (X3C) to them.

EXACT COVER BY 3-SETS (X3C)
Instance: Given a set of elements X with $|X| = 3q$ and a collection C of 3-element subsets of X.
Question: Does C contain an exact cover for X, i.e. Can we find a subcollection C' of C such that every element of X occurs in exactly one member of C'?

Theorem 13. $(1, 3)$-*DOMINATING SET is $\mathcal{NP}$-complete for bipartite graphs.*

Proof. $(1, 3)$-DOMINATING SET is members of $\mathcal{NP}$, since we can check in polynomial time whether a set S has cardinality at most p and is a $(1, 3)$-dset.
Next, we show how to construct a bipartite graph $G = (V, E)$ and a positive integer p from any instance X and C of X3C so that, X3C has a solution if and only if G has a $(1, 3)$-dset of cardinality at most p.
We are given an instance $X = \{x_1, ..., x_{3q}\}$ and $C = \{C_1, ..., C_m\}$ of X3C, where C_j are subsets of X of size $|C_j| = 3$ for $1 \leq j \leq m$, with $m \geq 2$. For each $x_i \in X$, we create a vertex x_i. For each $C_j \in C$, we create a vertex c_j and add a path $P_4 : u$-t-y-z. To complete the construction, we add all edges $x_i c_j$ if $x_i \in C_j$ and all edges $z c_j$ for $j \in \{1, 2, ..., m\}$. Clearly G is bipartite (see Figure 2). Set $p = q + 2$.
Suppose that the instance X and C of X3C has a solution C'. We construct a $(1, 3)$-dset D as follows: put in D, the vertices t, z and c_i for each $C_i \in C'$. Clearly y and u are $(1, 3)$-dominated by D. Now, observe that since C' is a solution for X3C, each x_i has exactly one neighbor in D and exactly q vertices of $Y = \{c_1, c_2, ..., c_m\}$ are in D. Moreover, each x_i is at distance two from z. The remaining vertices of Y are adjacent to z and are $(1, 3)$-dominated by D. Therefore, D is a $(1, 3)$-dset for G of cardinality $q + 2 = p$.
Conversely, assume that G has a $(1, 3)$-dset D, of cardinality at most $p = q + 2$. Clearly for the path u-t-y-z at least two vertices are in D. Now, let $n_1 = |D \cap X|$ and $q_1 = |D \cap Y|$. Then $|D| \geq n_1 + q_1 + 2$. Since the vertices of $X - D$ have to be dominated by $D \cap Y$, $|X - D| \leq 3\,|D \cap Y|$. So, we have $3q - n_1 \leq 3q_1$, leading to $n_1 \geq 3q - 3q_1$. Therefore $|D| \geq n_1 + q_1 + 2 \geq 3q - 2q_1 + 2$. Combining this with the fact that $|D| \leq p = q + 2$, we deduce that $q_1 \geq q$. It follows that $|D \cap Y| = q$, and hence every vertex x_i of X has exactly one neighbor in $D \cap Y$, meaning that $C' = \{C_i : c_i \in D \cap Y\}$ is an exact cover for X. □

Looking at the proof of Theorem 13, we can observe that D is also a $(1, \overline{3})$-dset, since u is $(1, \overline{3})$-dominated by D. Therefore we can state the following.

Corollary 9. $(1, \overline{3})$-*DOMINATING SET is* $\mathcal{NP}$-*complete for bipartite graphs.*

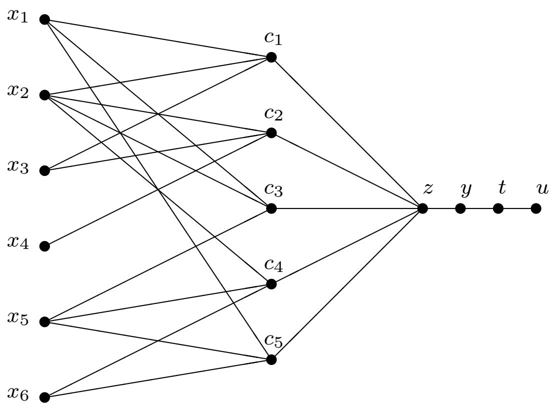

Figure 2. **A construction of bipartite graph G for $q = 2$.**

Finally to show the $\mathcal{NP}$-completeness of $(1, \overline{4})$-DOMINATING SET, we construct a bipartite graph G' in the same way as G in the proof of Theorem 13, but instead of adding the path u-t-y-z, we add the path u-t-y_1-y_2-z. It is not difficult to see that $\{t, z\} \cup \{c_i : C_i \in C'\}$ is $(1, \overline{4})$-dset of G', since u is $(1, \overline{4})$-dominated by D. Therefore we have the following result.

Theorem 14. $(1, \overline{4})$-*DOMINATING SET is* $\mathcal{NP}$-*complete for bipartite graphs.*

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