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Research Article

On the (outer)planarity of the intersection graph of idealization

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Abstract: Let R be a commutative ring with identity. The intersection graph of ideals of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose vertices are in a one-to-one correspondence with nonzero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a nonzero intersection. Let M be a unitary nonzero R-module, and let $R \ltimes M$ be the idealization of M in R. In this paper, we provide a characterization of the planarity of the intersection graph of idealization. We then conclude that the intersection graph of idealization is planar if and only if it is outerplanar.

Keywords: idealization, intersection graph, planarity.

AMS Subject Classification: 13A15, 05C75

1. Introduction

In recent decades, researchers have discussed the importance of associating graphs with algebraic structures. There are a lot of papers on assigning a graph to a ring and various aspects of correspondence of graphs with algebraic structures are seen in [1, 2, 4, 8, 9].

One of the classical topics in the theory of graphs is the intersection graph theory. Let R be a commutative ring and M be a unitary nonzero R-module. The *intersection graph of ideals* of a ring R is an undirected simple graph denoted by $\Gamma(R)$ whose

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vertices are in a one-to-one correspondence with nonzero proper ideals and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a nonzero intersection. The intersection graph of ideals of a ring R was introduced in [10] and studied in several papers, see [1, 2, 12, 13].

Idealization of M in R, denoted by $R \ltimes M$, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r, m)(r', m') := (rr', rm' + r'm)$ for all $r, r' \in R$ and all $m, m' \in M$. This construction was introduced in 1956 by Nagata [14] and has been extensively studied, see [6, 11].

Other area of interest in recent years is the theory of idealization of M in R, see [3, 7]. In [12], authors studied idealization by combinatorial methods and investigated the ideals in $R \ltimes M$ using graph-theoretic concepts. In this paper, we study the graph-theoretic properties of the intersection graph of idealization and we are especially interested in the planarity of the intersection graph of idealization.

Throughout this paper, all graphs are simple with no loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For distinct vertices $x_1, \ldots, x_n \in V(G)$, $x_1 - x_2 - \cdots - x_n$ denotes a *path* from x_1 to x_n. A *cycle* of length n is a path of the form $x_1 - x_2 - \cdots - x_n - x_1$ where $x_i \neq x_j$ when $i \neq j$ and denoted by C_n. A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. By K_n, we mean the complete graph over n vertices. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y such that every edge has one end in X and one end in Y. By $K_{r,s}$, we mean the complete bipartite graph where $|X| = r$ and $|Y| = s$. If $r = 1$, then it is called *star graph*. Let $S \subseteq V$ be any subset of vertices of G. Then the *induced subgraph* by S is the graph whose vertex set is S and edge set contains all edges of G connecting pairs of vertices in S. A *clique* of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G.

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same face. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths.

All over the paper let $S := R \ltimes M$. We denote the set of all *maximal ideals* of R by $\mathrm{Max}(R)$. The nonzero module M is called a *simple module* if the only submodules of M are (0) and M.

Motivated by the work of [12], we study the planarity (outerplanarity) of $\Gamma(S)$ and some properties of idealization $\mathbb{Z}_n$ in $\mathbb{Z}_n$ where $\mathbb{Z}_n$ is the ring of integers modulo n. In section 2, we determine the set of all ideals of the $\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, for distinct prime numbers p and q. In section 3, we characterize all rings R and modules M for which the intersection graph $\Gamma(R \ltimes M)$ is planar. In fact, we prove that $\Gamma(R \ltimes M)$ is planar if and only if R is a field and $\dim_R M \leq 2$, or $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, $\mathfrak{m}_2 M = (0)$ and $\dim_{R/\mathfrak{m}_2} M = 1$, or R has only one nontrivial ideal, and M is a simple module. Finally, we show that $\Gamma(S)$ is outerplanar if and only if it is planar.

2. The ideals of $\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, for distinct prime numbers p and q

It was shown in [12, Lemma 1] that there is a close relation of an ideal L of S with some ideal of the form $I \ltimes N$ where I is an ideal of R and N is a submodule of M. To continue, we will use this lemma frequently in the rest of the paper.

Lemma 1. *Let L be an ideal of S. Then there exist an ideal I_L of R and a submodule N_L of M such that $I_L \ltimes N_L$ is an ideal of S and $L \subseteq I_L \ltimes N_L$.*

Note that in Lemma 1, $I_L = \{r \in R \mid \exists\, m \in M \text{ such that } (r, m) \in L\}$ and $N_L = \{m \in M \mid \exists\, r \in R \text{ such that } (r, m) \in L\}$ by [12, Lemma 1].
The next proposition determines the ideals of the idealization $S = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, for each prime number p.

Proposition 1. *Let $S = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, where p is a prime number. Then the nontrivial ideals of S are $(0) \ltimes p\mathbb{Z}_{p^2}$, $(0) \ltimes \mathbb{Z}_{p^2}$, $p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$, $p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ and $L_s := \{(ips, jp+i) \mid 0 \le i, j \le p-1\}$ for each $1 \le s \le p-1$.*

Proof. Assume that L is an ideal of S. Then, by Lemma 1, there exist an ideal I_L of $\mathbb{Z}_{p^2}$, and a submodule N_L of $\mathbb{Z}_{p^2}$ such that $L \subseteq I_L \ltimes N_L$. If $I_L = (0)$, then $L = (0) \ltimes N_L = (0) \ltimes (0)$, $L = (0) \ltimes p\mathbb{Z}_{p^2}$ or $L = (0) \ltimes \mathbb{Z}_{p^2}$ by [12, Lemma 2]. Assume now that $I_L \neq (0)$. Hence, $I_L = \mathbb{Z}_{p^2}$ or $I_L = p\mathbb{Z}_{p^2}$. If $I_L = \mathbb{Z}_{p^2}$, then $L = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ by [12, Lemma 2].
So we may assume that $I_L = p\mathbb{Z}_{p^2}$. This gives us an element $m \in \mathbb{Z}_{p^2}$ such that $(p, m) \in L$. Hence, $(0, p) = (p, m)(0, 1) \in L$; so that, one has $(0) \ltimes p\mathbb{Z}_{p^2} \subseteq L \subseteq I_L \ltimes N_L = p\mathbb{Z}_{p^2} \ltimes N_L$. Thus, $p\mathbb{Z}_{p^2} \subseteq N_L$ and we have $N_L = p\mathbb{Z}_{p^2}$ or $N_L = \mathbb{Z}_{p^2}$. Assume first that $N_L = p\mathbb{Z}_{p^2}$. This gives $0 \ltimes p\mathbb{Z}_{p^2} \subseteq L \subseteq p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$ which yields $p \mid |L| \mid p^2$. These imply that $L = (0) \ltimes p\mathbb{Z}_{p^2}$ or $L = p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$.
Assume now that $N_L = \mathbb{Z}_{p^2}$. This in conjunction with $(0) \ltimes p\mathbb{Z}_{p^2} \subset L$ gives $(0) \ltimes p\mathbb{Z}_{p^2} \subset L \subseteq p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$. It yields $p \mid |L| \mid p^3$; so that $|L| = p^2$ or $|L| = p^3$. If $|L| = p^3$, then $L = p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$. In the rest of the proof, we assume that $|L| = p^2$. Since $N_L = \mathbb{Z}_{p^2}$, one has an element $r \in \mathbb{Z}_{p^2}$ such that $(r, 1) \in L$ and then $r \in I_L = p\mathbb{Z}_{p^2}$. If $r = 0$, then $(0, 1) \in L$. Thus $(0) \ltimes \mathbb{Z}_{p^2} \subset L$ and $L = I_L \ltimes \mathbb{Z}_{p^2} = p\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, by [12, Lemma 2], a contradiction. So we can assume that $0 \neq r \in p\mathbb{Z}_{p^2}$ and $r = ps$, for some $1 \le s \le p-1$. Hence $(ps, 1) \in L$. If there is $1 \le s' \le p-1$ such that $s \neq s'$ and $(ps', 1) \in L$, then $(p(s-s'), 0) \in L$. Since $s - s'$ is invertible element in $\mathbb{Z}_{p^2}$, one has $((s-s')^{-1}, 0)(p(s-s'), 0) = (p, 0) \in L$. Thus, L contains the set $p\mathbb{Z}_{p^2} \ltimes (0)$. On the other hand, we know that $(0) \ltimes p\mathbb{Z}_{p^2} \subset L$. It follows that $p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2} \subsetneq L$ which implies $L = p\mathbb{Z}_{p^2} \ltimes p\mathbb{Z}_{p^2}$.
Therefore, we can assume that there is only one element $1 \le s \le p-1$ such that $(ps, 1) \in L$. On the other hand, we know that $(0) \ltimes p\mathbb{Z}_{p^2} \subset L$, i.e., $(0, jp) \in L$, for all $1 \le j \le p-1$. By adding the element $(ps, 1)$ to these elements, we get

$\{(ips, jp+i) \mid 0 \leq i, j \leq p-1\} \subseteq L$. It follows from $|L|= p^2$ that $L = \{(ips, jp+i) \mid 0 \leq i, j \leq p-1\} = L_s$. Note that one can easily check that L_s is an ideal of S. Assume that $(ps, 1) = (ips', jp+i)$ for $1 \leq s \neq s' \leq p-1$. So we have $1 \equiv jp+i \pmod{p^2}$ and this implies that $i = 1$. Also, $ps \equiv ips' \pmod{p^2}$ in conjunction with $i = 1$ yields that $p \mid s - s'$, a contradiction. Therefore $(ps, 1) \in L_s \setminus L_{s'}$ for $1 \leq s \neq s' \leq p-1$. This completes the proof. □

It was shown in [12, Lemma 8] that $\Gamma(\mathbb{Z}_p \ltimes \mathbb{Z}_p) \cong K_1$, for each prime number p. The following corollary which is an immediate result of Proposition 1, gives us the structure of the graph $\Gamma(\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2})$.

Corollary 1. *Let $S = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$, where p is a prime number. Then $\Gamma(\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}) \cong K_{p+3}$.*

The following proposition determines the ideals of $S = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, for distinct prime numbers p and q.

Proposition 2. *Let $S = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, where p and q are distinct prime numbers. Then the nontrivial ideals of S are $(0) \ltimes p\mathbb{Z}_{pq}$, $(0) \ltimes q\mathbb{Z}_{pq}$, $(0) \ltimes \mathbb{Z}_{pq}$, $p\mathbb{Z}_{pq} \ltimes p\mathbb{Z}_{pq}$, $p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, $q\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq} \ltimes q\mathbb{Z}_{pq}$.*

Proof. Assume that L is an ideal of S. Then by Lemma 1, there exist an ideal I_L of $\mathbb{Z}_{pq}$ and a submodule N_L of $\mathbb{Z}_{pq}$ such that $L \subseteq I_L \ltimes N_L$. If $I_L = (0)$, then $L = (0) \ltimes N_L = (0) \ltimes (0)$, $L = (0) \ltimes p\mathbb{Z}_{pq}$, $L = (0) \ltimes q\mathbb{Z}_{pq}$ or $L = (0) \ltimes \mathbb{Z}_{pq}$, by [12, Lemma 2]. If $I_L = \mathbb{Z}_{pq}$, then $L = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, by [12, Lemma 2]. So we may assume that $I_L = p\mathbb{Z}_{pq}$ or $I_L = q\mathbb{Z}_{pq}$ and, by symmetry, it is sufficient to determine the ideal L for which $I_L = p\mathbb{Z}_{pq}$. Since $p \in I_L$, there is an element $x \in \mathbb{Z}_{pq}$ such that $(p, x) \in L$. So $(0, p) = (p, x)(0, 1) \in L$ and, hence one has $(0) \ltimes p\mathbb{Z}_{pq} \subseteq L \subseteq I_L \ltimes N_L = p\mathbb{Z}_{pq} \ltimes N_L$. Thus $p\mathbb{Z}_{pq} \subseteq N_L$ and this implies that $N_L = p\mathbb{Z}_{pq}$ or $N_L = \mathbb{Z}_{pq}$.
Assume first that $N_L = p\mathbb{Z}_{pq}$. Then we have $q| \ |L| \ |q^2$ and so $|L|= q$ or $|L|= q^2$. If $|L|= q$, then $L = (0) \ltimes p\mathbb{Z}_{pq}$ which contradicts $I_L = p\mathbb{Z}_{pq}$. If $|L|= q^2$, then $L = p\mathbb{Z}_{pq} \ltimes p\mathbb{Z}_{pq}$.
Assume now that $N_L = \mathbb{Z}_{pq}$. In this case, we have $(0) \ltimes p\mathbb{Z}_{pq} \subseteq L \subseteq p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$. Hence $q| \ |L| \ |pq^2$. If $|L|= q$, then $L = (0) \ltimes p\mathbb{Z}_{pq}$.
Assume next that $|L|= pq^2$. Since $N_L = \mathbb{Z}_{pq}$, one has an element $r \in I_L = p\mathbb{Z}_{pq}$ such that $(r, 1) \in L$. If $r = 0$, then $(0, 1) \in L$; so $(0) \ltimes \mathbb{Z}_{pq} \subset L$ and $L = I_L \ltimes \mathbb{Z}_{pq} = p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, by [12, Lemma 2]. If $0 \neq r \in p\mathbb{Z}_{pq}$, then $r = ps$, for some $1 \leq s \leq pq-1$. Thus we have $(ps, 1) \in L$. On the other hand, we know that $(0) \ltimes p\mathbb{Z}_{pq} \subset L$, i.e., $(0, jp) \in L$, for all $1 \leq j \leq q-1$. By adding the element $(ps, 1)$ to these elements, we have $\{(ips, jp+i) \mid 0 \leq i \leq pq-1, \ 0 \leq j \leq q-1\} \subseteq L$. By counting the number of elements, this implies that $L = \{(ips, jp+i) \mid 0 \leq i \leq pq-1, \ 0 \leq j \leq q-1\} = p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$.
Assume finally that $|L|= pq$. Since $N_L = \mathbb{Z}_{pq}$, one has an element $r \in p\mathbb{Z}_{pq}$ such that $(r, 1) \in L$. If $r = 0$, then $(0, 1) \in L$; so $(0) \ltimes \mathbb{Z}_{pq} \subseteq L$ and $L = I_L \ltimes \mathbb{Z}_{pq} = p\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, by [12, Lemma 2], which is a contradiction. If $0 \neq r \in p\mathbb{Z}_{pq}$, then $r = ps$, for some

$1 \leq s \leq pq-1$. Arguing as above, we can see $\{(ips, jp+i) \mid 0 \leq i \leq pq-1,\ 0 \leq j \leq q-1\} \subseteq L$, which is a contradiction, since $|L| = pq$.
Similarly, for $I_L = q\mathbb{Z}_{pq}$, we have $L = q\mathbb{Z}_{pq} \ltimes q\mathbb{Z}_{pq}$, $L = (0) \ltimes q\mathbb{Z}_{pq}$ or $L = q\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$ and we are done. □

As an immediate result, we obtain the structure of the graph $\Gamma(\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq})$.

Corollary 2. *Let $S = \mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$, where p, q are distinct prime numbers. Then we have $\Gamma(\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}) \cong K_7 \setminus C_4$, where $C_4 := (0) \ltimes p\mathbb{Z}_{pq} - q\mathbb{Z}_{pq} \ltimes q\mathbb{Z}_{pq} - p\mathbb{Z}_{pq} \ltimes p\mathbb{Z}_{pq} - (0) \ltimes q\mathbb{Z}_{pq} - (0) \ltimes p\mathbb{Z}_{pq}$.*

A simple characterization of planar graphs was given by Kuratowski in 1930 [15, Theorem 6.2.2].

Theorem 1. *A graph is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5.*

Corollaries 1 and 2 with Theorem 1 show that the intersection graphs of the idealizations $\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq} \ltimes \mathbb{Z}_{pq}$ are not planar (and hence not outerplanar). A natural question that arises here is that under which conditions the intersection graph of idealization is planar (outerplanar)? We study this question in the next section.

3. Planarity (outerplanarity) of the intersection graph of the idealization

In this section, we characterize when the intersection graph of the idealization is a planar graph or an outerplanar graph.
The next theorem provides a characterization for the planarity of the intersection graph $R \ltimes M$ when R is a field.

Theorem 2. *Let $S = F \ltimes M$, where F is a field. Then $\Gamma(S)$ is planar if and only if $\dim_F M \leq 2$.*

Proof. Assume first that $\dim_F M = 1$, hence, by [12, Lemma 8], we have $\Gamma(S) = K_1$. If $\dim_F M = 2$, then it follows from [12, Theorem 5] that $\Gamma(S)$ is a star graph and it is planar. Suppose now that $\dim_F M = 3$ and consider two following cases.

Case 1. Assume first that F is an infinite field. Then $M \cong F^3$ and M has infinite subspaces with nonzero intersection. On the other hand, by [12, Lemma 4], the proper ideals of S have the form $(0) \ltimes N$ where N is a subspace of M. Therefore $\Gamma(S)$ contains the subgraph K_5 and it is not planar, by Theorem 1.

Case 2. Assume now that F is a finite field and $|F| = q$, where q is a power of a positive prime integer. Let $\{W_1, \ldots, W_{q^2+q+1}\}$ be the set of all 2-dimensional subspaces of M. One has $\dim_F M = \dim_F W_i + \dim_F W_j - \dim_F W_i \cap W_j$, for

$1 \leq i \neq j \leq q^2+q+1$. Thus $W_i \cap W_j \neq (0)$, for $1 \leq i \neq j \leq q^2+q+1$. Therefore $\Gamma(S)$ contains the subgraph K_5 and it is not planar, by Theorem 1 and we are done in this case.
Suppose next that $\dim_F M = 4$. Choose $x_1, x_2, x_3, x_4 \in M$ such that $M = \langle x_1, x_2, x_3, x_4 \rangle$. Denote by N_i the subspace generated by x_i, for $1 \leq i \leq 4$. It is not hard to see that the induced subgraph by the $X = \{(0) \ltimes N_1 + N_2 + N_3, (0) \ltimes N_2 + N_3, (0) \ltimes N_1 + N_3 + N_4\}$ and $Y = \{(0) \ltimes N_1 + N_3, (0) \ltimes N_2 + N_4, (0) \ltimes N_3\}$ has $K_{3,3}$ as a subgraph, hence $\Gamma(S)$ it is not planar, by Theorem 1.
Suppose finally that $\dim_F M \geq 5$, choose independent elements x_1, x_2, x_3, x_4 of M, and set $N_i = \langle x_1, \ldots, x_i \rangle$, for $1 \leq i \leq 4$. Clearly, the induced subgraph by the vertices $(0) \ltimes N_1, (0) \ltimes N_2, (0) \ltimes N_3, (0) \ltimes N_4, (0) \ltimes M$ forms a K_5 and $\Gamma(S)$ is not planar, by Theorem 1. □

A maximal ideal of S is of the form $\mathfrak{m} \ltimes M$ for a maximal ideal $\mathfrak{m}$ of R [3, Theorem 3.2]. It follows that all maximal ideals of S make a clique with $(0) \ltimes M$ and one has the following lemma.

Lemma 2. *Let $S = R \ltimes M$. Then $\omega(\Gamma(S)) \geq |\operatorname{Max}(R)| + 1$.*

Using the above lemma, one can show that $\Gamma(S)$ is not planar whenever $|\operatorname{Max}(R)| \geq 3$.

Lemma 3. *Let $S = R \ltimes M$ and assume that $|\operatorname{Max}(R)| \geq 3$. Then $\Gamma(S)$ is not planar.*

Proof. It follows from Lemma 2 and Theorem 1 that $\Gamma(S)$ is not planar whenever $|\operatorname{Max}(R)| \geq 4$. So we may assume that $|\operatorname{Max}(R)| = 3$ and $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$. We claim that $\mathfrak{m}_i \cap \mathfrak{m}_j \neq (0)$ for all distinct $i, j \in \{1,2,3\}$. Suppose by contrary that $\mathfrak{m}_i \cap \mathfrak{m}_j = (0)$ for some distinct $i, j \in \{1,2,3\}$. Let $k \in \{1,2,3\} \setminus \{i,j\}$. Then $\mathfrak{m}_k \supset (0) = \mathfrak{m}_i \cap \mathfrak{m}_j$. So $\mathfrak{m}_k \supseteq \mathfrak{m}_i$ or $\mathfrak{m}_k \supseteq \mathfrak{m}_j$, a contradiction. Hence, $\mathfrak{m}_i \cap \mathfrak{m}_j \neq (0)$ for all distinct $i, j \in \{1,2,3\}$. Thus, the induced subgraph by vertices $\mathfrak{m}_1 \ltimes M$, $\mathfrak{m}_2 \ltimes M$, $\mathfrak{m}_3 \ltimes M$, $(\mathfrak{m}_1 \cap \mathfrak{m}_2) \ltimes M$ and $(0) \ltimes M$ forms a K_5. Therefore $\Gamma(S)$ is not planar. □

In the next two lemmas, we consider the case where $|\operatorname{Max}(R)| = 2$

Lemma 4. *Let $S = R \ltimes M$ and assume $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ such that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$. Then $\Gamma(S)$ is not planar.*

Proof. Assume first that the module M is not simple and N is a nontrivial submodule. Then the vertices $\mathfrak{m}_1 \ltimes M, \mathfrak{m}_2 \ltimes M, (\mathfrak{m}_1 \cap \mathfrak{m}_2) \ltimes M, (0) \ltimes N$ and $(0) \ltimes M$ in $\Gamma(S)$ form a subgraph K_5, so $\Gamma(S)$ is not planar. Suppose now that M is a simple module and without loss of generality, let $M \cong R/\mathfrak{m}_1$. Then the induced subgraph by the vertices $\mathfrak{m}_1 \ltimes R/\mathfrak{m}_1, \mathfrak{m}_2 \ltimes R/\mathfrak{m}_1, (\mathfrak{m}_1 \cap \mathfrak{m}_2) \ltimes R/\mathfrak{m}_1, \mathfrak{m}_1 \ltimes (0)$ and $(\mathfrak{m}_1 \cap \mathfrak{m}_2) \ltimes (0)$ in $\Gamma(S)$ form a subgraph K_5, so $\Gamma(S)$ is not planar. □

Remark 1. Assume that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$. It follows that $R = \mathfrak{m}_1 + \mathfrak{m}_2$ and there are idempotent elements e_1, e_2 such that $\mathfrak{m}_i = Re_i$ for $i = 1, 2$ and $e_1 e_2 = 0$. Thus one can easily see that $M = \mathfrak{m}_1 M \oplus \mathfrak{m}_2 M$.

Let I be an ideal of the ring R such that $IM = (0)$. It's clear that M is a module over the quotient ring R/I using the natural projection $R \to R/I$.

Lemma 5. *Let $S = R \ltimes M$ and assume $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ such that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$. Then the following statements hold.*

1. *If $\mathfrak{m}_1 M \neq (0)$ and $\mathfrak{m}_2 M \neq (0)$, then $\Gamma(S)$ is not planar.*
2. *If $\mathfrak{m}_1 M \neq (0)$, $\mathfrak{m}_2 M = (0)$ and $\dim_{R/\mathfrak{m}_2} M \geq 2$, then $\Gamma(S)$ is not planar.*
3. *If $\mathfrak{m}_1 M \neq (0)$, $\mathfrak{m}_2 M = (0)$ and $\dim_{R/\mathfrak{m}_2} M = 1$, then $\Gamma(S)$ is planar.*

Proof. (1) Assume that $\mathfrak{m}_1 M \neq (0)$ and $\mathfrak{m}_2 M \neq (0)$. Observe that the nonzero proper ideals $\mathfrak{m}_1 \ltimes \mathfrak{m}_1 M$, $\mathfrak{m}_2 \ltimes \mathfrak{m}_2 M$, $\mathfrak{m}_1 \ltimes M$, $\mathfrak{m}_2 \ltimes M$, $(0) \ltimes M$ of S are pairwise distinct, and the subgraph of $\Gamma(S)$ induced by these ideals forms a K_5. So, $\Gamma(S)$ is not planar.
(2) Assume $\mathfrak{m}_1 M \neq (0)$, $\mathfrak{m}_2 M = (0)$, and $\dim_{R/\mathfrak{m}_2} M \geq 2$. Note that $M = \mathfrak{m}_1 M$ and we can find $x, y \in M$ such that $\{x, y\}$ is linearly independent over $R/\mathfrak{m}_2$. Hence, Rx, Ry are nonzero proper submodules of M and $Rx \neq Ry$. Observe that the nonzero proper ideals $\mathfrak{m}_1 \ltimes M$, $\mathfrak{m}_2 \ltimes M$, $\mathfrak{m}_2 \ltimes Rx$, $\mathfrak{m}_2 \ltimes Ry$, $(0) \ltimes M$ of S are pairwise distinct, and the induced subgraph forms a K_5. So, $\Gamma(S)$ is not planar.
(3) Assume $\mathfrak{m}_1 M \neq (0)$, $\mathfrak{m}_2 M = (0)$, and $\dim_{R/\mathfrak{m}_2} M = 1$. Note that $M = \mathfrak{m}_1 M$ is a simple R-module, and let $M = Rx$ for some nonzero $x \in M$. Observe that the set of nonzero proper ideals of R equals $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ and (0) is the only proper submodule of M. We claim that the set of nonzero proper ideals of S equals $\{\mathfrak{m}_1 \ltimes M, \mathfrak{m}_2 \ltimes (0), \mathfrak{m}_2 \ltimes M, (0) \ltimes M\}$. To this, assume that L is an ideal of S, and consider the ideal I_L of R. We have $I_L = (0), \mathfrak{m}_1, \mathfrak{m}_2$, or R. If $I_L = (0)$, then $L = (0) \ltimes N_L$ by [12, Lemma 2], so that, $L = (0) \ltimes (0)$, or $L = (0) \ltimes M$ since M is simple. If $I_L = R$, then $L = R \ltimes M$ by [12, Lemma 2].
Assume now that $I_L = \mathfrak{m}_1$. Then, $\mathfrak{m}_1 M \subseteq N_L$ since $\mathfrak{m}_1 \ltimes N_L$ is an ideal of S by Lemma 1. Hence, $M = \mathfrak{m}_1 M \subseteq N_L$. This means $N_L = M$. Adopting the notation of Remark 1, we are going to show $\mathfrak{m}_1 \ltimes M = S(e_1, x)$. To see this, let an arbitrary element $(re_1, sx) \in \mathfrak{m}_1 \ltimes M$, where $r, s \in R$. Considering $s - r \in R = \mathfrak{m}_1 + \mathfrak{m}_2$, one can find $a \in R$ such that $s - r - ae_1 \in \mathfrak{m}_2$. Then, $(s - r - ae_1)x = 0$; so that, $sx = rx + ae_1 x$. This implies that $(re_1, sx) = (r, ax)(e_1, x) \in S(e_1, x)$. It follows that $\mathfrak{m}_1 \ltimes M = S(e_1, x)$. Next, $e_1 \in \mathfrak{m}_1 = I_L$ yields an elements $s \in R$ such that $(e_1, sx) \in L$. Considering $1 - s \in R = \mathfrak{m}_1 + \mathfrak{m}_2$, one can find $b \in R$ such that $1 - s - be_1 \in \mathfrak{m}_2$. Then, $(1 - s - be_1)x = 0$; so, $x = sx + be_1 x$. This shows that $(e_1, x) = (1, bx)(e_1, sx) \in L$. Hence, $\mathfrak{m}_1 \ltimes M = S(e_1, x) \subseteq L$. Therefore, $L = \mathfrak{m}_1 \ltimes M$.
Assume finally that $I_L = \mathfrak{m}_2$. It follows from $e_2 \in \mathfrak{m}_2 = I_L$ that there is $s \in R$ such that $(e_2, sx) \in L$. If $(e_2 \ltimes M) \cap L = \{(e_2, 0)\}$, then $(re_2, 0) = (r, 0)(e_2, 0) \in L$

for all $r \in R$, which shows $\mathfrak{m}_2 \ltimes (0) \subseteq L$. Here, we have $L = \mathfrak{m}_2 \ltimes (0)$. Suppose then that $(e_2 \ltimes M) \cap L \neq \{(e_2, 0)\}$, and fix $s \in R$ with $(e_2, sx) \in L$ and $sx \neq 0$. Hence, $s \notin \mathfrak{m}_2$; so that $R = Rs + \mathfrak{m}_2$. Thus, there is $a \in R$ such that $1 - as \in \mathfrak{m}_2$. This means $(1 - as)x = 0$; so $x = asx$. It shows that $(0, x) = (0, sx)(a, 0) \in L$, which implies that $(0, rx) = (0, x)(r, 0) \in L$ for every $r \in R$. Thus, $(0) \ltimes M \subseteq L$. On the other hand, $(e_2, 0) = (e_2, sx)(e_2, 0) \in L$ yields that $\mathfrak{m}_2 \ltimes (0) \subseteq L$. Hence, $\mathfrak{m}_2 \ltimes M = \mathfrak{m}_2 \ltimes (0) + (0) \ltimes M \subseteq L$. Therefore, $L = \mathfrak{m}_2 \ltimes M$. This completes the proof of claim.

Observe that the subgraph H_1 of $\Gamma(S)$ induced by $\{\mathfrak{m}_1 \ltimes M, \mathfrak{m}_2 \ltimes M, (0) \ltimes M\}$ forms a K_3. Note that $\Gamma(S) = H_1 \cup H_2$, where H_2 consists of a pendant edge, joining $\mathfrak{m}_2 \ltimes M$ and $\mathfrak{m}_2 \ltimes (0)$. It is easy to verify that $\Gamma(S)$ is planar. □

Example 1. Let $S = \mathbb{Z}_{pq} \ltimes M$ where $M = \mathbb{Z}_{pq}/p\mathbb{Z}_{pq}$. We have $\mathrm{Max}(\mathbb{Z}_{pq}) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ where $\mathfrak{m}_1 = q\mathbb{Z}_{pq}$ and $\mathfrak{m}_2 = p\mathbb{Z}_{pq}$. It is obvious that $\mathfrak{m}_1 M \neq (0), \mathfrak{m}_2 M = (0)$ and $\dim_{\mathbb{Z}_{pq}/p\mathbb{Z}_{pq}} M = 1$. Then S has four nontrivial ideals and it is a planar graph, by Lemma 5.

In the sequel, we consider $|\mathrm{Max}(R)| = 1$.

Lemma 6. *Let $S = R \ltimes M$ and assume that R has only one nonzero proper ideal $\mathfrak{m}$ and that M is a simple R-module. Then $\Gamma(S)$ is planar.*

Proof. By [12, Theorem 11], $\Gamma(S)$ is a star graph; so it is planar. □

The minimal number of generators of a module M will be denoted by $\mu(M)$. In the following lemmas, we consider the values of $\mu(\mathfrak{m})$ carefully.

Lemma 7. *Let $S = R \ltimes M$ and assume that $(R, \mathfrak{m})$ is a local ring such that $\mu(\mathfrak{m}) \geq 3$. Then $\Gamma(S)$ is not planar.*

Proof. First, assume that $\mu(\mathfrak{m}) \geq 4$ and choose distinct elements r_1, r_2, r_3, r_4 of a generating set of $\mathfrak{m}$. Let $I_1 = \langle r_1 \rangle, I_2 = \langle r_1, r_2 \rangle, I_3 = \langle r_1, r_2, r_3 \rangle$ and $I_4 = \langle r_1, r_2, r_3, r_4 \rangle$. The vertices $I_1 \ltimes M, I_2 \ltimes M, I_3 \ltimes M, I_4 \ltimes M$ and $(0) \ltimes M$ in $\Gamma(S)$ form a subgraph K_5; so that $\Gamma(S)$ is not planar. Assume now that $\mu(\mathfrak{m}) = 3$ and let $\mathfrak{m} = \langle r_1, r_2, r_3 \rangle$. The distinct vertices $\mathfrak{m} \ltimes M, \langle r_1, r_2 \rangle \ltimes M, \langle r_1, r_3 \rangle \ltimes M, \langle r_2, r_3 \rangle \ltimes M$ and $(0) \ltimes M$ in $\Gamma(S)$ form a subgraph K_5; so that $\Gamma(S)$ is not planar. □

Lemma 8. *Let $S = R \ltimes M$ and assume that $(R, \mathfrak{m})$ is a local ring such that $\mu(\mathfrak{m}) = 2$. Then $\Gamma(S)$ is not planar.*

Proof. By assumption, $\mu(\mathfrak{m}) = 2$. Let $\{r_1, r_2\}$ be a minimal set of generators for $\mathfrak{m}$. Note that $\mathfrak{m} = Rr_1 + Rr_2 = Rr_1 + R(r_1 + r_2) = Rr_2 + R(r_1 + r_2)$. Hence, $Rr_1, Rr_2, R(r_1 + r_2)$ are pairwise distinct nonzero proper ideals of R, and each of them is distinct from $\mathfrak{m}$. Note that the subgraph of $\Gamma(R \ltimes M)$ induced by $\{Rr_1 \ltimes$

M, $Rr_2 \ltimes M$, $R(r_1+r_2) \ltimes M$, $\mathfrak{m} \ltimes M$, $(0) \ltimes M\}$ forms a K_5. So, $\Gamma(R \ltimes M)$ is not planar. □

We use the following simple lemma in the proof of Lemma 10.

Lemma 9. *Assume $(R, \mathfrak{m})$ is a local ring such that $\mathfrak{m} = Rr \neq 0$ and $r^4 = 0$, but $r^3 \neq 0$. Then all nontrivial ideals of R are $\mathfrak{m}, \mathfrak{m}^2$ and $\mathfrak{m}^3$.*

Proof. Let I be a nontrivial ideal of R and $0 \neq a \in I$. Thus there is an element $s \in R$ such that $a = sr$. If $s \notin \mathfrak{m}$, then $r = s^{-1}a \in I$ and so $I = \mathfrak{m}$. So, suppose that $s \in \mathfrak{m}$. Hence we have $a = s'r^2$, for some $s' \in R$ and then $I \subseteq \mathfrak{m}^2$. This implies that $I = \mathfrak{m}^2$ or $I \subset \mathfrak{m}^2$. Assume that $I \subset \mathfrak{m}^2$. Thus $a = sr^2$, for some $s \in R$. If $s \notin \mathfrak{m}$, then $r^2 = s^{-1}a \in I$ and so $I = \mathfrak{m}^2$ which is a contradiction. Hence $s \in \mathfrak{m}$ and one has $a = s'r^3 \in \mathfrak{m}^3$, for some $s' \in R$. Thus we have $I \subseteq \mathfrak{m}^3$. Therefore one has $a = sr^3$, for some $s \in R$. Since $r^4 = 0$, $s \in \mathfrak{m}$ implies that $I = (0)$ which is a contradiction. So we have $s \notin \mathfrak{m}$ and $\mathfrak{m}^3 = I$. □

Lemma 10. *Let $S = R \ltimes M$ and assume that $(R, \mathfrak{m})$ is a local ring such that $\mathfrak{m} = Rr \neq (0)$. Then $\Gamma(S)$ is not planar unless that $\mathfrak{m}^2 = (0)$ and M is a simple module.*

Proof. By hypothesis, $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m} = Rr$ for some $r \in R \setminus (0)$, and M is a nonzero R-module. Assume that $S = R \ltimes M$ is such that $\Gamma(S)$ is planar. First note that M is finitely generated. If M is not finitely generated, then there exists a strictly increasing sequence of nonzero proper submodules $N_1 \subset N_2 \subset N_3 \subset \cdots$ of M. Note that the subgraph of $\Gamma(S)$ induced by $\{(0) \ltimes N_n \mid n \in \mathbb{N}\}$ is an infinite clique, a contradiction. So, M is finitely generated. Observe that $M/\mathfrak{m}M$ is a finite dimensional vector space over the field $R/\mathfrak{m}$. By Nakayama's lemma, $M \neq \mathfrak{m}M$. So, $1 \leq \dim_{R/\mathfrak{m}} M/\mathfrak{m}M$. We claim that $\dim_{R/\mathfrak{m}} M/\mathfrak{m}M = 1$. Suppose that $\dim_{R/\mathfrak{m}} M/\mathfrak{m}M \geq 2$. Then, we can find $x, y \in M$ such that $\{x+\mathfrak{m}M, y+\mathfrak{m}M\}$ is linearly independent over $R/\mathfrak{m}$. So, the nonzero proper submodules $\mathfrak{m}M+Rx, \mathfrak{m}M+Ry, \mathfrak{m}M+R(x+y)$ of M are pairwise distinct. Observe that the subgraph of $\Gamma(S)$ induced by $\mathfrak{m} \ltimes M$, $\mathfrak{m} \ltimes \mathfrak{m}M + Rx$, $\mathfrak{m} \ltimes \mathfrak{m}M + Ry$, $\mathfrak{m} \ltimes \mathfrak{m}M + R(x+y)$, $(0) \ltimes M$ forms a K_5, a contradiction. Hence, $\dim_{R/\mathfrak{m}} M/\mathfrak{m}M = 1$. Let $x \in M \setminus \mathfrak{m}M$ be such that $\{x + \mathfrak{m}M\}$ is a basis of $M/\mathfrak{m}M$ as a vector space over the field $R/\mathfrak{m}$. Then, we obtain from [5, Proposition 2.8] that $M = Rx$. Suppose that $\mathfrak{m}^2 \neq (0)$. Then, the nonzero proper ideals $\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}M, \mathfrak{m}^2 \ltimes \mathfrak{m}M, (0) \ltimes M$ of S are pairwise distinct. Note that the ideal $S(r, x)$ of S is nonzero and proper. We assert that $(r, 0) \notin S(r, x)$. For if $(r, 0) \in S(r, x)$, then we can find $a, b \in R$ such that $(r, 0) = (r, x)(a, bx)$. Hence, $r = ar$ and $ax + rbx = 0$. From $r \neq 0$ and $r(1-a) = 0$, we obtain that a is a unit in R. Hence, $a + rb$ is a unit in R. From $(a + rb)x = 0$, we get that $x = 0$, a contradiction. Thus, $(r, 0) \notin S(r, x)$. Therefore, $S(r, x) \notin \{\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}M\}$. It is clear that $S(r, x) \notin \{\mathfrak{m}^2 \ltimes \mathfrak{m}M, (0) \ltimes M\}$. Hence, the nonzero proper ideals $\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}M, \mathfrak{m}^2 \ltimes \mathfrak{m}M, (0) \ltimes M, S(r, x)$ are pairwise distinct. Note that the subgraph of $\Gamma(S)$ induced by $\{\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}M, \mathfrak{m}^2 \ltimes \mathfrak{m}M, (0) \ltimes M, S(r, x)\}$

forms a K_5, a contradiction. Therefore, $\mathfrak{m}^2 = (0)$. Note that we can show as in the proof of Lemma 9 that $\mathfrak{m}$ is the only nonzero proper ideal of R. We claim that M is simple. It is enough to show that $\mathfrak{m}\, M = (0)$. Suppose that $\mathfrak{m}\, M \neq (0)$. Observe that the nonzero proper ideals $\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}\, M, (0) \ltimes M, (0) \ltimes \mathfrak{m}\, M$ of S are pairwise distinct. Note that the ideal $S(r, x)$ of S is nonzero and proper. It is clear that $S(r, x) \notin \{(0) \ltimes M, (0) \ltimes \mathfrak{m}\, M\}$ and $(r, 0) \in \mathfrak{m} \ltimes \mathfrak{m}\, M$. We have already verified that $(r, 0) \notin S(r, x)$. Thus, the nonzero proper ideal $S(r, x) \notin \{\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}\, M, (0) \ltimes M, (0) \ltimes \mathfrak{m}\, M\}$. As $(0, rx) = (r, 0)(0, x)$ and $(0, rx) = (r, 0)(r, x)$, we get that $(0, rx)$ belongs to each member of $\{\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}\, M, (0) \ltimes M, (0) \ltimes \mathfrak{m}\, M, S(r, x)\}$. So, the subgraph of $\Gamma(S)$ induced by $\{\mathfrak{m} \ltimes M, \mathfrak{m} \ltimes \mathfrak{m}\, M, (0) \ltimes M, (0) \ltimes \mathfrak{m}\, M, S(r, x)\}$ forms a K_5, a contradiction. Therefore, M is simple. Thus, if $\Gamma(S)$ is planar, then $\mathfrak{m}^2 = (0)$ and M is a simple R-module.

Assume that $\mathfrak{m}^2 = (0)$ and M is a simple R-module. Then, $\mathfrak{m}$ is the only nonzero proper ideal of R, (0) is the only proper submodule of M. We obtain from Lemma 1 that the set of nonzero proper ideals os S equals $\{(0) \ltimes M, \mathfrak{m} \ltimes (0), \mathfrak{m} \ltimes M\}$. Note that $\mathfrak{m} \ltimes M$ is adjacent to $(0) \ltimes M$ and $\mathfrak{m} \ltimes (0)$ in $\Gamma(S)$, but $(0) \ltimes M$ and $\mathfrak{m} \ltimes (0)$ are not adjacent in $\Gamma(S)$. So, $\Gamma(S)$ is a star graph. Hence $\Gamma(S)$ is planar. ☐

Combining the above lemmas, we get the following theorem.

Theorem 3. *Let $S = R \ltimes M$. Then $\Gamma(S)$ is planar if and only if one of the following conditions hold:*

1. *R is a field, and $\dim_R M \leq 2$,*
2. *$\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, $\mathfrak{m}_2\, M = (0)$, and $\dim_{R/\mathfrak{m}_2} M = 1$,*
3. *R has only one nontrivial ideal, and M is a simple module.*

Proof. Suppose that $\Gamma(S)$ is planar. If R is a field, then $\dim_R M \leq 2$ by Theorem 2. So we may assume that R is not a field. Then, by Lemma 3, $|\operatorname{Max}(R)| \leq 2$. Assume first that $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$; so that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, by Lemma 4. Hence one has $\mathfrak{m}_2\, M = (0)$ and $\dim_{R/\mathfrak{m}_2} M = 1$, by Lemma 5. Assume now that $|\operatorname{Max}(R)| = 1$ and $\mathfrak{m}$ is the maximal ideal of R. We have $\mu(\mathfrak{m}) = 1$, by Lemmas 7 and 8. Noting Lemma 10, we get the assertion. The converse is obvious by Theorem 2 and Lemmas 6 and 5. ☐

Chartrand and Harary also proved an analogue of Kuratowski's theorem for outerplanar graphs [15, Exercise 6.2.7].

Theorem 4. *A graph G is outerplanar if and only if G contains no subgraph that is a subdivision of K_4 or $K_{2,3}$.*

Obviously, an outerplanar graph is a planar graph, but, the converse is not true necessarily. The next corollary shows that this is true for the intersection graph of idealization.

Corollary 3. *Let $S = R \ltimes M$. Then $\Gamma(S)$ is outerplanar if and only if it is planar.*

Proof. Assume that $\Gamma(S)$ is planar. Hence, by Theorem 3, one of the following cases holds.

1. R is a field and $\dim_R M \leq 2$,
2. $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, $\mathfrak{m}_2 M = (0)$ and $\dim_{R/\mathfrak{m}_2} M = 1$,
3. R has only one nontrivial ideal, and M is a simple module.

In case 1, the graph $\Gamma(S)$ is outerplanar by the proof of Theorem 2. In case 2, note that the graph $\Gamma(S)$ has been completely obtained in the proof of Lemma 5(3). It is clear that $\Gamma(S)$ is not isomorphic to K_4. Thus, $\Gamma(S)$ is outerplanar by Theorem 4 in this case. In case 3, $\Gamma(S)$ is a star graph by [12, Theorem 11]; so, it is outerplanar. Therefore, in each case, $\Gamma(S)$ is outerplanar. □

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