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Research Article

Augmented Sombor index of graph operations

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Abstract: The augmented Sombor index (ASO) is a vertex-degree-based topological index defined as $ASO(G) = \sum_{uv\in E(G)} \sqrt{(d_u^2 + d_v^2)/(d_u + d_v - 2)}$. In this paper, we derive explicit formulas for the ASO index of graphs resulting from various graph operations, including the Cartesian product, join, corona product, and wreath product. We establish rigorous bounds for the ASO index under these operations and provide Nordhaus-Gaddum type results. Our work extends the mathematical theory of this recently introduced index and provides computational tools for its application in chemical graph theory.

Keywords: Augmented Sombor index, topological index, graph operation, bounds, Cartesian product, join, corona product, wreath product.

AMS Subject Classification: 05C50, 05C90

1. Introduction

Chemical graph theory provides a fundamental bridge between mathematics and chemistry, where molecular structures are modeled as graphs with atoms represented by vertices and chemical bonds by edges. Within this framework, *topological indices* play a crucial role as numerical descriptors derived from a graph's structure, designed to correlate with physicochemical properties of corresponding molecules [6].

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The predictive power of these indices in Quantitative Structure-Property Relationship (QSPR) and Quantitative Structure-Activity Relationship (QSAR) studies has established their importance in chemical compound design and analysis [3].
Among various topological indices, degree-based variants have attracted significant attention due to their simplicity and strong correlative ability. A notable recent addition is the *Sombor index*, introduced by [7], defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}.$$

Extensive research has led to investigations of its extremal values across various graph classes [4, 10] and the introduction of several variants including the elliptic Sombor index [1], Euler Sombor index [2], and diminished Sombor index [11].
Building upon this foundation, [5] proposed a refined variant termed the *augmented Sombor index* (ASO), defined for a graph G with no P_2 component as:

$$ASO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u^2 + d_v^2}{d_u + d_v - 2}}.$$

The authors demonstrated its superior performance over the original Sombor index in correlating with key thermodynamic properties of octane isomers, such as entropy and acentric factor. Furthermore, they established fundamental mathematical properties by characterizing extremal graphs with respect to the ASO index for trees, connected graphs, quasi-trees, and unicyclic graphs.
Despite these promising developments, the mathematical theory of the ASO index remains underdeveloped. A well-established and productive line of inquiry in topological index theory involves analyzing behavior under standard graph operations [8]. Determining the index of complex graphs from simpler components is both theoretically elegant and computationally valuable. Operations like Cartesian product, join, and corona product enable construction of complex networks and nanostructures from basic building blocks.
This paper addresses these research gaps by establishing a comprehensive mathematical framework for the ASO index under graph operations. Our main contributions include:

- Deriving explicit analytical formulas for the ASO index of graphs formed through Cartesian products, joins, corona products, and wreath products;
- Establishing various bounds for the ASO index under these operations;
- Providing Nordhaus-Gaddum type results for the ASO index.

2. Preliminaries

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, its degree is denoted by $d(v)$. We recall several fundamental graph operations that provide mechanisms for constructing complex structures from simpler components.

Definition 1 ([9]). The *Cartesian product* of two graphs G and H, denoted by $G\square H$, has vertex set $V(G\square H) = V(G) \times V(H)$ and two vertices (u, x) and (v, y) are adjacent if either:

1. $u = v$ and $xy \in E(H)$, or
2. $x = y$ and $uv \in E(G)$.

In $G\square H$, each vertex (u, v) has degree $d_G(u) + d_H(v)$.

Definition 2 ([9]). The *join* of two graphs G and H, denoted by $G + H$, has vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. In $G+H$, vertices in G have degree $d_G(u) + |V(H)|$ and vertices in H have degree $d_H(v) + |V(G)|$.

Definition 3 ([9]). The *corona product* of two graphs G and H, denoted by $G \circ H$, is obtained by taking one copy of G and $|V(G)|$ copies of H, and joining each vertex of the i-th copy of H to the i-th vertex of G. In $G \circ H$, vertices in G have degree $d_G(u) + |V(H)|$ and vertices in the attached copies have degree $d_H(v) + 1$.

Definition 4 ([9]). The *wreath product* (lexicographic product) of two graphs G and H, denoted by $G \wr H$, has vertex set $V(G \wr H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either:

1. $u_1u_2 \in E(G)$, or
2. $u_1 = u_2$ and $v_1v_2 \in E(H)$.

In $G \wr H$, each vertex (u, v) has degree $d_G(u) \cdot |V(H)| + d_H(v)$.

The augmented Sombor index (ASO) is defined as:

$$ASO(G) = \sum_{uv \in E(G)} \sqrt{h(d(u), d(v))}$$

where

$$h(x, y) = \frac{x^2 + y^2}{x + y - 2}.$$

We establish fundamental properties of the function h that will be used throughout our analysis.

Lemma 1. *For any real numbers $x, y \geq 2$, the function $h(x, y)$ satisfies:*

1. *$h(x,y) = h(y,x)$ (symmetric)*
2. *$h(x,x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$*
3. *$h(x,y) \geq 4$ with equality if and only if $x = y = 2$*

Proof. (1) Symmetry follows immediately from the definition.
(2) When $x = y$, we have:

$$h(x,x) = \frac{2x^2}{2x-2} = \frac{x^2}{x-1} = \frac{(x-1)^2 + 2x - 1}{x-1} = x + 1 + \frac{1}{x-1}.$$

(3) By the Cauchy-Schwarz inequality:

$$h(x,y) = \frac{x^2+y^2}{x+y-2} \geq \frac{(x+y)^2}{2(x+y-2)}.$$

Let $t = x + y \geq 4$. Then we need to show $\frac{t^2}{2(t-2)} \geq 4$, which simplifies to $t^2 \geq 8t - 16$ or $(t-4)^2 \geq 0$, with equality when $t = 4$, i.e., $x = y = 2$. $\square$

Lemma 2 ([5]). *For any pendent edge $uv \in E(G)$ with $d(u) > d(v) = 1$, we have:*

$$5 \leq h(d(u), 1) \leq n + \frac{2}{n-2}$$

with left equality if and only if $d(u) \in \{2, 3\}$, and right equality if and only if $d(u) = n - 1$.

Lemma 3 ([5]). *For any non-pendent edge $uv \in E(G)$ with $d(u), d(v) \geq 2$, we have:*

$$4 \leq h(d(u), d(v)) \leq n + \frac{1}{n-2}$$

with left equality if and only if $d(u) = d(v) = 2$, and right equality if and only if $d(u) = d(v) = n - 1$.

3. ASO index of graph operations

3.1. Cartesian product

Theorem 1. *Let G and H be two connected graphs. Then the ASO index of their Cartesian product $G \square H$ is:*

$$ASO(G \square H) = \sum_{u \in V(G)} ASO_H(u) + \sum_{v \in V(H)} ASO_G(v)$$

where

$$ASO_H(u) = \sum_{vw \in E(H)} \sqrt{h(d_G(u) + d_H(v), d_G(u) + d_H(w))}$$

and

$$ASO_G(v) = \sum_{uw \in E(G)} \sqrt{h(d_G(u) + d_H(v), d_G(w) + d_H(v))}.$$

Proof. The edge set of $G \square H$ partitions into edges from H (with a fixed vertex in G) and edges from G (with a fixed vertex in H). For each vertex $u \in V(G)$, the edges $\{((u,v),(u,w)) : vw \in E(H)\}$ contribute exactly $ASO_H(u)$. Similarly, for each vertex $v \in V(H)$, the edges $\{((u,v),(w,v)) : uw \in E(G)\}$ contribute exactly $ASO_G(v)$. The result follows by summing over all vertices. $\square$

Corollary 1. *If G is r-regular and H is s-regular, then:*

$$\begin{aligned} ASO(G \square H) &= n_G \cdot m_H \cdot \sqrt{\frac{2(r+s)^2}{2(r+s)-2}} + n_H \cdot m_G \cdot \sqrt{\frac{2(r+s)^2}{2(r+s)-2}} \\ &= (n_G m_H + n_H m_G)\sqrt{\frac{(r+s)^2}{r+s-1}} \end{aligned}$$

where n_G, n_H are the orders and m_G, m_H are the sizes of G and H, respectively.

Proof. In a regular graph, all vertices have the same degree. For r-regular G and s-regular H, every vertex in $G \square H$ has degree $r+s$. For any edge in $G \square H$, both endpoints have degree $r+s$, so:

$$h(r+s, r+s) = \frac{2(r+s)^2}{2(r+s)-2} = \frac{(r+s)^2}{r+s-1}.$$

The number of edges in $G \square H$ is $n_G m_H + n_H m_G$, which gives the result. $\square$

3.2. Join of two graphs

Theorem 2. *Let G and H be two graphs. Then the ASO index of their join $G + H$ is:*

$$ASO(G + H) = ASO_G^+ + ASO_H^+ + ASO_{GH}$$

where

$$\begin{aligned} ASO_G^+ &= \sum_{uv \in E(G)} \sqrt{h(d_G(u) + n_H, d_G(v) + n_H)} \\ ASO_H^+ &= \sum_{uv \in E(H)} \sqrt{h(d_H(u) + n_G, d_H(v) + n_G)} \\ ASO_{GH} &= \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{h(d_G(u) + n_H, d_H(v) + n_G)}. \end{aligned}$$

Proof. The join $G+H$ contains three types of edges: edges within G, edges within H, and edges between G and H. In $G+H$, the degree of a vertex $u \in V(G)$ becomes $d_G(u)+n_H$, and the degree of a vertex $v \in V(H)$ becomes $d_H(v)+n_G$. The three terms correspond to the contributions from these three edge types. □

Corollary 2. *If G is r-regular and H is s-regular, then:*

$$\begin{aligned} ASO(G+H) =& \frac{n_G r}{2}\sqrt{\frac{(r+n_H)^2}{r+n_H-1}} + \frac{n_H s}{2}\sqrt{\frac{(s+n_G)^2}{s+n_G-1}} \\ &+ n_G n_H \sqrt{\frac{(r+n_H)^2+(s+n_G)^2}{r+n_H+s+n_G-2}}. \end{aligned}$$

Proof. For edges within G, both endpoints have degree $r+n_H$, so each edge contributes $\sqrt{h(r+n_H, r+n_H)} = \sqrt{\frac{(r+n_H)^2}{r+n_H-1}}$. There are $\frac{n_G r}{2}$ such edges. Similarly for edges within H. For edges between G and H, the endpoints have degrees $r+n_H$ and $s+n_G$, so each edge contributes $\sqrt{\frac{(r+n_H)^2+(s+n_G)^2}{r+n_H+s+n_G-2}}$. There are $n_G n_H$ such edges. □

3.3. Corona product

Theorem 3. *Let G and H be two graphs. Then the ASO index of their corona product $G \circ H$ is:*

$$ASO(G \circ H) = ASO^*_G + n_G \cdot ASO^*_H + ASO^*_{GH}$$

where

$$\begin{aligned} ASO^*_G &= \sum_{uv \in E(G)} \sqrt{h(d_G(u)+n_H, d_G(v)+n_H)} \\ ASO^*_H &= \sum_{uv \in E(H)} \sqrt{h(d_H(u)+1, d_H(v)+1)} \\ ASO^*_{GH} &= \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{h(d_G(u)+n_H, d_H(v)+1)}. \end{aligned}$$

Proof. The corona product $G \circ H$ contains three types of edges: edges within the copy of G, edges within the n_G copies of H, and edges connecting G to each copy of H. In $G \circ H$, the degree of a vertex $u \in V(G)$ becomes $d_G(u)+n_H$, and the degree of a vertex v in a copy of H becomes $d_H(v)+1$. The three terms correspond to the contributions from these three edge types. □

Corollary 3. *For $G \circ K_1$ (corona with singleton graphs), we have:*

$$ASO(G \circ K_1) = \sum_{uv \in E(G)} \sqrt{h(d_G(u)+1, d_G(v)+1)} + \sum_{u \in V(G)} \sqrt{h(d_G(u)+1, 1)}.$$

Proof. When $H = K_1$, we have $n_H = 1$, $E(H) = \emptyset$, and $d_H(v) = 0$. Substituting these values into the general formula gives the result. □

3.4. Wreath product

The wreath product (also known as the lexicographic product) is an important graph operation that finds applications in chemical graph theory and network design. In this section, we investigate the ASO index of wreath product graphs.

Theorem 4. *Let G and H be two graphs with $n_1 = |V(G)|$, $n_2 = |V(H)|$, $m_1 = |E(G)|$, and $m_2 = |E(H)|$. Then the ASO index of their wreath product $G \wr H$ is given by:*

$$\begin{aligned} ASO(G \wr H) = &\sum_{u\in V(G)} \sum_{xy\in E(H)} \sqrt{h\left(d_G(u)n_2 + d_H(x), d_G(u)n_2 + d_H(y)\right)} \\ &+ \sum_{uv\in E(G)} \sum_{x\in V(H)} \sum_{y\in V(H)} \sqrt{h\left(d_G(u)n_2 + d_H(x), d_G(v)n_2 + d_H(y)\right)}. \end{aligned}$$

Proof. The edge set of $G \wr H$ can be partitioned into two disjoint sets:

- **Internal edges**: Edges within each copy of H. For each vertex $u \in V(G)$, there is a copy of H, and for each edge $xy \in E(H)$, there is an edge between (u, x) and (u, y). The degree of (u, x) is $d_G(u)n_2 + d_H(x)$ and the degree of (u, y) is $d_G(u)n_2 + d_H(y)$. Thus, the contribution from these edges is:

$$\sum_{u\in V(G)} \sum_{xy\in E(H)} \sqrt{h\left(d_G(u)n_2 + d_H(x), d_G(u)n_2 + d_H(y)\right)}.$$

- **External edges**: Edges between different copies of H corresponding to edges in G. For each edge $uv \in E(G)$ and for each pair of vertices $x \in V(H)$ and $y \in V(H)$, there is an edge between (u, x) and (v, y). The degree of (u, x) is $d_G(u)n_2 + d_H(x)$ and the degree of (v, y) is $d_G(v)n_2 + d_H(y)$. Thus, the contribution from these edges is:

$$\sum_{uv\in E(G)} \sum_{x\in V(H)} \sum_{y\in V(H)} \sqrt{h\left(d_G(u)n_2 + d_H(x), d_G(v)n_2 + d_H(y)\right)}.$$

Summing these two contributions gives the desired formula. □

Corollary 4. *If G is r-regular and H is s-regular, then:*

$$ASO(G \wr H) = (n_1m_2 + m_1n_2^2) \cdot \frac{rn_2 + s}{\sqrt{rn_2 + s - 1}}.$$

Proof. Since G is r-regular, all vertices in G have degree r. Since H is s-regular, all vertices in H have degree s. Therefore, in $G \wr H$, every vertex has degree $rn_2 + s$. For any edge in $G \wr H$, both endpoints have degree $rn_2 + s$, so:

$$h(rn_2 + s, rn_2 + s) = \frac{2(rn_2 + s)^2}{2(rn_2 + s) - 2} = \frac{(rn_2 + s)^2}{rn_2 + s - 1}.$$

The number of internal edges is $n_1 m_2$, and the number of external edges is $m_1 n_2^2$. Thus, the total number of edges is $n_1 m_2 + m_1 n_2^2$, and each edge contributes $\sqrt{h(rn_2+s, rn_2+s)} = \frac{rn_2+s}{\sqrt{rn_2+s-1}}$. Multiplying gives the result. □

Example 1. Consider the wreath product $C_3 \wr C_3$, where C_3 is the cycle graph on 3 vertices. We have $n_1 = 3$, $n_2 = 3$, $m_1 = 3$, and $m_2 = 3$. Since C_3 is 2-regular, $r = 2$ and $s = 2$. Then:

$$ASO(C_3 \wr C_3) = (3 \cdot 3 + 3 \cdot 3^2) \cdot \frac{2 \cdot 3 + 2}{\sqrt{2 \cdot 3 + 2 - 1}} = (9 + 27) \cdot \frac{8}{\sqrt{7}} = 36 \cdot \frac{8}{\sqrt{7}} = \frac{288}{\sqrt{7}}.$$

4. Bounds for ASO index of graph operations

In this section, we establish various bounds for the augmented Sombor index under graph operations. These bounds provide efficient ways to estimate the ASO index without explicit computation.

4.1. Bounds for the function $h(x, y)$

We begin by establishing fundamental bounds for the function $h(x, y) = \frac{x^2+y^2}{x+y-2}$ that will be used throughout this section.

Lemma 4. *For any real numbers $x, y \geq 2$, the function $h(x, y)$ satisfies:*

1. $\frac{1}{2}(x + y) + 1 < h(x, y) \leq x + y$
2. $2 \leq \sqrt{h(x, y)} \leq \sqrt{x + y}$

with equality in the upper bound of (1) and lower bound of (2) when $x = y = 2$, and equality in the upper bound of (2) when $x = y = 2$.

Proof. (1) For the lower bound, consider:

$$h(x, y) - \left(\frac{1}{2}(x + y) + 1\right) = \frac{x^2 + y^2}{x + y - 2} - \frac{x + y}{2} - 1.$$

Using a common denominator $2(x + y - 2)$, we have:

$$= \frac{2(x^2 + y^2) - (x + y)(x + y - 2) - 2(x + y - 2)}{2(x + y - 2)}.$$

Simplifying the numerator:

$$\begin{aligned} &2x^2 + 2y^2 - (x^2 + 2xy + y^2 - 2x - 2y) - 2x - 2y + 4 \\ &= x^2 + y^2 - 2xy + 4 = (x - y)^2 + 4. \end{aligned}$$

Since $(x-y)^2+4 \geq 4 > 0$ and $2(x+y-2) > 0$ for $x, y \geq 2$, the expression is positive. Thus, $h(x,y) > \frac{1}{2}(x+y)+1$.
For the upper bound, we show that $h(x,y) \leq x+y$. Consider:

$$(x+y) - h(x,y) = x+y - \frac{x^2+y^2}{x+y-2} = \frac{(x+y)(x+y-2)-(x^2+y^2)}{x+y-2}.$$

Simplifying the numerator:

$$(x+y)(x+y-2)-(x^2+y^2) = x^2+2xy+y^2-2x-2y-x^2-y^2 = 2xy-2x-2y.$$

Thus,

$$(x+y) - h(x,y) = \frac{2xy-2x-2y}{x+y-2} = \frac{2(xy-x-y)}{x+y-2}.$$

Since $x, y \geq 2$, we have $xy - x - y = x(y-1) - y \geq 2(1) - 2 = 0$, so $xy - x - y \geq 0$. Also, $x+y-2 > 0$, so $(x+y) - h(x,y) \geq 0$. Therefore, $h(x,y) \leq x+y$. Equality holds when $xy - x - y = 0$, which occurs when $x = y = 2$.
(2) The lower bound $\sqrt{h(x,y)} \geq 2$ follows from Lemma 1 since $h(x,y) \geq 4$. Equality holds when $x = y = 2$. The upper bound $\sqrt{h(x,y)} \leq \sqrt{x+y}$ follows from part (1) since $h(x,y) \leq x+y$ and the square root is increasing. Equality holds when $x = y = 2$. □

Lemma 5. *For any real numbers $x, y \geq 2$, the function $h(x,y)$ satisfies:*

$$h(x,y) \leq x+y-2+\frac{4}{x+y-2}.$$

Equality holds when $\{x,y\} = \{2, t-2\}$ for some $t = x+y \geq 4$, i.e., when one of the variables is 2.

Proof. Let $t = x+y \geq 4$. We consider $h(x,y)$ as a function of x for fixed t, with the constraint that $2 \leq x \leq t-2$. Write $y = t-x$, then:

$$h(x, t-x) = \frac{x^2+(t-x)^2}{t-2} = \frac{2x^2-2tx+t^2}{t-2}.$$

Consider the function $f(x) = h(x, t-x)$ on the interval $[2, t-2]$. To find its maximum, we compute the derivative:

$$f'(x) = \frac{4x-2t}{t-2}.$$

Setting $f'(x) = 0$ gives $4x - 2t = 0$, or $x = t/2$. The second derivative is:

$$f''(x) = \frac{4}{t-2} > 0,$$

so f is convex on $[2, t-2]$. For a convex function on a closed interval, the maximum occurs at one of the endpoints. Therefore, we compare the values at the endpoints:
At $x = 2$:

$$f(2) = h(2, t-2) = \frac{4 + (t-2)^2}{t-2} = \frac{t^2 - 4t + 8}{t-2} = (t-2) + \frac{4}{t-2}.$$

At $x = t - 2$:

$$f(t-2) = h(t-2, 2) = f(2) \quad \text{(by symmetry)}.$$

At $x = t/2$ (if $t/2 \geq 2$, i.e., $t \geq 4$):

$$f(t/2) = h(t/2, t/2) = \frac{2(t/2)^2}{t-2} = \frac{t^2/2}{t-2} = \frac{t^2}{2(t-2)}.$$

Now we show that $f(2) \geq f(t/2)$ for all $t \geq 4$:

$$f(2) - f(t/2) = \left((t-2) + \frac{4}{t-2}\right) - \frac{t^2}{2(t-2)} = \frac{2(t-2)^2 + 8 - t^2}{2(t-2)}.$$

The numerator simplifies to:

$$2(t^2 - 4t + 4) + 8 - t^2 = 2t^2 - 8t + 8 + 8 - t^2 = t^2 - 8t + 16 = (t-4)^2 \geq 0.$$

Equality holds only when $t = 4$, which corresponds to $x = y = 2$.
Thus, for any fixed $t \geq 4$, the maximum of $h(x, y)$ subject to $x + y = t$ and $x, y \geq 2$ occurs at the endpoints where one variable is 2 and the other is $t - 2$. The maximum value is $(t-2) + \frac{4}{t-2} = x + y - 2 + \frac{4}{x+y-2}$. This completes the proof. □

4.2. Bounds for Cartesian product

Theorem 5. *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then the ASO index of their Cartesian product satisfies:*

$$2(n_1 m_2 + n_2 m_1) \leq ASO(G \square H) \leq (n_1 m_2 + n_2 m_1) \cdot \frac{\Delta_G + \Delta_H}{\sqrt{\Delta_G + \Delta_H - 1}},$$

where Δ_G and Δ_H are the maximum degrees of G and H, respectively.

Proof. For the lower bound, from Lemma 4 we have $\sqrt{h(x, y)} \geq 2$ for any $x, y \geq 2$. Since $G \square H$ has $n_1 m_2 + n_2 m_1$ edges and all vertices have degree at least 2, we get the lower bound.
For the upper bound, note that the maximum degree in $G \square H$ is $\Delta_G + \Delta_H$. For any edge in $G \square H$ with endpoint degrees d_1, d_2, we have:

$$\sqrt{h(d_1, d_2)} \leq \sqrt{h(\Delta_G + \Delta_H, \Delta_G + \Delta_H)} = \frac{\Delta_G + \Delta_H}{\sqrt{\Delta_G + \Delta_H - 1}}.$$

Multiplying by the number of edges gives the upper bound. □

Corollary 5. *If G and H are regular graphs with degrees r and s respectively, then:*

$$(n_1m_2 + n_2m_1)\sqrt{\frac{r+s}{2}+1} \leq ASO(G\Box H) \leq (n_1m_2 + n_2m_1) \cdot \frac{r+s}{\sqrt{r+s-1}}.$$

Proof. For regular graphs, all vertices in $G\Box H$ have degree $r+s$. From Lemma 4, we have:

$$\sqrt{\frac{1}{2}(r+s)+1} < \sqrt{h(r+s, r+s)} \leq \frac{r+s}{\sqrt{r+s-1}}.$$

Multiplying by the number of edges gives the result. □

4.3. Bounds for join of two graphs

Theorem 6. *Let G and H be two graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then the ASO index of their join satisfies:*

$$ASO(G + H) \geq 2(m_1 + m_2 + n_1n_2)$$

and

$$\begin{aligned} ASO(G + H) \leq & m_1 \cdot \frac{\Delta_G + n_2}{\sqrt{\Delta_G + n_2 - 1}} + m_2 \cdot \frac{\Delta_H + n_1}{\sqrt{\Delta_H + n_1 - 1}} \\ & + n_1n_2 \cdot \sqrt{\frac{(\Delta_G + n_2)^2 + (\Delta_H + n_1)^2}{\Delta_G + n_2 + \Delta_H + n_1 - 2}}. \end{aligned}$$

Proof. The lower bound follows from Lemma 4 since $\sqrt{h(x, y)} \geq 2$ for all edges. The number of edges in $G + H$ is $m_1 + m_2 + n_1n_2$.
For the upper bound, note that in $G + H$, the maximum degree for vertices in G is $\Delta_G + n_2$ and in H is $\Delta_H + n_1$. The three terms correspond to the maximum possible contributions from edges within G, within H, and between G and H, respectively. □

4.4. Bounds for corona product

Theorem 7. *Let G and H be two graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then the ASO index of their corona product satisfies:*

$$ASO(G \circ H) \geq 2(m_1 + n_1m_2 + n_1n_2)$$

and

$$\begin{aligned} ASO(G \circ H) \leq & m_1 \cdot \frac{\Delta_G + n_2}{\sqrt{\Delta_G + n_2 - 1}} + n_1m_2 \cdot \frac{\Delta_H + 1}{\sqrt{\Delta_H}} \\ & + n_1n_2 \cdot \sqrt{\frac{(\Delta_G + n_2)^2 + (\Delta_H + 1)^2}{\Delta_G + n_2 + \Delta_H - 1}}. \end{aligned}$$

Proof. The lower bound follows from Lemma 4 since $\sqrt{h(x, y)} \geq 2$ for all edges. The number of edges in $G \circ H$ is $m_1 + n_1m_2 + n_1n_2$.
For the upper bound, note that in $G \circ H$, the maximum degree for vertices in G is $\Delta_G + n_2$, and for vertices in H copies is $\Delta_H + 1$. The three terms correspond to the maximum possible contributions from edges within G, within copies of H, and between G and H, respectively. □

4.5. Bounds for wreath product

Theorem 8. *Let G and H be two graphs. Then the ASO index of their wreath product satisfies:*

$$2(n_1m_2 + m_1n_2^2) \leq ASO(G \wr H) \leq (n_1m_2 + m_1n_2^2) \cdot \frac{\Delta_G n_2 + \Delta_H}{\sqrt{\Delta_G n_2 + \Delta_H - 1}},$$

where Δ_G and Δ_H are the maximum degrees of G and H, respectively.

Proof. For the lower bound, from Lemma 4, we have $\sqrt{h(x,y)} \geq 2$ for any $x, y \geq 2$. Since $G \wr H$ has $n_1m_2 + m_1n_2^2$ edges and all vertices have degree at least 2, we obtain the lower bound.
For the upper bound, note that the maximum degree in $G \wr H$ is $\Delta_G n_2 + \Delta_H$. For any edge in $G \wr H$, the endpoint degrees are at most $\Delta_G n_2 + \Delta_H$. Thus:

$$\sqrt{h(d_1, d_2)} \leq \sqrt{h(\Delta_G n_2 + \Delta_H, \Delta_G n_2 + \Delta_H)} = \frac{\Delta_G n_2 + \Delta_H}{\sqrt{\Delta_G n_2 + \Delta_H - 1}}.$$

Multiplying by the number of edges gives the upper bound. □

4.6. Nordhaus-Gaddum type results

Theorem 9. *Let G be a graph of order $n \geq 3$. Then for the augmented Sombor index, we have:*

$$ASO(G) + ASO(\overline{G}) \leq \binom{n}{2} \cdot \sqrt{\frac{n^2 - 2n + 2}{n - 2}}.$$

Proof. Consider the complete graph K_n on the same vertex set as G. Its edge set $E(K_n)$ consists of exactly $\binom{n}{2}$ edges. This edge set is partitioned into two disjoint subsets: the edges of G and the edges of its complement $\overline{G}$. Consequently, every edge $uv \in E(K_n)$ appears in exactly one of the two graphs G or $\overline{G}$.
The sum $ASO(G) + ASO(\overline{G})$ can therefore be expressed as a single summation over all edges of the complete graph:

$$ASO(G) + ASO(\overline{G}) = \sum_{uv \in E(K_n)} \sqrt{h\big(d_{G^*}(u), d_{G^*}(v)\big)},$$

where for a given edge uv, the symbol G^* denotes the graph (either G or $\overline{G}$) that contains the edge uv, and $d_{G^*}(u)$ is the degree of vertex u in that graph.
For any edge $uv \in E(K_n)$, the degrees of its endpoints in the containing graph satisfy $1 \leq d_{G^*}(u), d_{G^*}(v) \leq n - 1$. To bound the term $\sqrt{h(d_{G^*}(u), d_{G^*}(v))}$, we distinguish two cases based on whether the edge is pendant or non-pendant in G^*.

Case 1. (pendant edge) If one of the endpoints has degree 1 in G^*, then by Lemma 2 we have

$$h\big(d_{G^*}(u), d_{G^*}(v)\big) \leq n + \frac{2}{n-2} = \frac{n^2-2n+2}{n-2}.$$

Case 2. (non-pendant edge) If both endpoints have degree at least 2 in G^*, then by Lemma 3 we have

$$h\big(d_{G^*}(u), d_{G^*}(v)\big) \leq n + \frac{1}{n-2}.$$

For $n \geq 3$, it holds that $n + \frac{2}{n-2} > n + \frac{1}{n-2}$. Hence, the maximum possible value of $h(d_{G^*}(u), d_{G^*}(v))$ is $\frac{n^2-2n+2}{n-2}$, attained when one endpoint has degree 1 and the other has degree $n-1$. Therefore, for every edge $uv \in E(K_n)$,

$$\sqrt{h\big(d_{G^*}(u), d_{G^*}(v)\big)} \leq \sqrt{\frac{n^2-2n+2}{n-2}}.$$

Since the sum $ASO(G) + ASO(\overline{G})$ consists of exactly $\binom{n}{2}$ such terms (one for each edge of K_n), we obtain

$$ASO(G) + ASO(\overline{G}) = \sum_{uv \in E(K_n)} \sqrt{h\big(d_{G^*}(u), d_{G^*}(v)\big)} \leq \binom{n}{2} \cdot \sqrt{\frac{n^2-2n+2}{n-2}},$$

which completes the proof. □

Theorem 10. *Let G be a graph of order $n \geq 3$. Then:*

$$ASO(G) \cdot ASO(\overline{G}) \leq \frac{1}{4}\binom{n}{2}^2 \cdot \frac{n^2-2n+2}{n-2}.$$

Proof. By the AM-GM inequality and Theorem 9:

$$\begin{aligned} ASO(G) \cdot ASO(\overline{G}) &\leq \left(\frac{ASO(G) + ASO(\overline{G})}{2}\right)^2 \\ &\leq \left(\frac{\binom{n}{2} \cdot \sqrt{\frac{n^2-2n+2}{n-2}}}{2}\right)^2 \\ &= \frac{1}{4}\binom{n}{2}^2 \cdot \frac{n^2-2n+2}{n-2}. \end{aligned}$$

□

5. Conclusion

This research has established the augmented Sombor index as a computationally tractable and theoretically rich molecular descriptor within the framework of graph operations. The explicit formulas derived for Cartesian products, joins, corona products, and wreath products provide powerful tools for analyzing complex molecular structures while maintaining mathematical elegance.

Our work advances chemical graph theory by bridging abstract graph operations with practical computational chemistry through applicable formulas. The developed methodologies enable efficient computation of topological indices for large molecular structures and provide descriptors for machine learning models predicting material properties.

The bounds established in this paper offer practical tools for estimating the ASO index without exhaustive computation, which is particularly valuable for large graphs and complex molecular structures. The Nordhaus-Gaddum type results provide fundamental constraints on the behavior of the ASO index in complementary graph structures.

The mathematical framework developed in this paper provides a solid foundation for further theoretical investigations and practical applications of the augmented Sombor index in chemical graph theory and beyond.

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References

[1] S. Ahmad, K.C. Das, and R. Farooq, *On elliptic Sombor index with applications*, Bull. Malays. Math. Sci. Soc. **48** (2025), Artice number: 108. https://doi.org/10.1007/s40840-025-01894-6.

[2] A.M. Albalahi, A.M. Alanazi, A.M. Alotaibi, A.E. Hamza, and A. Ali, *Optimizing the Euler Sombor index of (molecular) tricyclic graphs*, MATCH Commun. Math. Comput. Chem. **94** (2025), no. 2, 549–560. https://doi.org/10.46793/match.94-2.549A.

[3] A. Ali, A.A. Bhatti, and Z. Raza, *Further inequalities between vertex-degree-based topological indices*, Int. J. Appl. Comput. Math. **3** (2017), 1921–1930. https://doi.org/10.1007/s40819-016-0213-4.

[4] M. Chen and Y. Zhu, *Extremal unicyclic graphs of Sombor index*, Appl. Math. Comput. **463** (2024), 128374. https://doi.org/10.1016/j.amc.2023.128374.

[5] K.C. Das, I. Gutman, and A. Ali, *Augmented Sombor index*, MATCH Commun. Math. Comput. Chem. **95** (2026), no. 2, 523–547.
https://doi.org/10.46793/match.95-2.20125.

[6] I. Gutman, *Degree based topological indices*, Croat. Chem. Acta **86** (2013), no. 4, 351–361.
http://dx.doi.org/10.5562/cca2294.

[7] ______, *Geometric approach to degree-based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem. **86** (2021), no. 1, 11–16.

[8] I. Gutman and K.C. Das, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. **50** (2004), 83–92.

[9] R. Hammack, W. Imrich, and S. Klavžar, *Handbook of Product Graphs*, CRC Press, 2011.

[10] H. Liu, I. Gutman, L. You, and Y. Huang, *Sombor index: Review of extremal results and bounds*, J. Math. Chem. **60** (2022), 771–798.
https://doi.org/10.1007/s10910-022-01333-y.

[11] F. Movahedi, I. Gutman, I. Redžepović, and B. Furtula, *Diminished Sombor index*, MATCH Commun. Math. Comput. Chem. **95** (2026), no. 1, 141–162.
https://doi.org/10.46793/match.95-1.14125.