

Augmented Sombor index of graph operations

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Abstract: The augmented Sombor index (ASO) of a finite simple graph G with no component isomorphic to K_2 is defined by $ASO(G) = \sum_{uv \in E(G)} \sqrt{(d_u^2 + d_v^2)/(d_u + d_v - 2)}$. In this paper, we derive explicit formulas for the ASO index of graphs resulting from various graph operations, including the Cartesian product, join, corona product, and wreath product. We establish rigorous bounds for the ASO index under these operations and provide Nordhaus-Gaddum type results. Our work extends the mathematical theory of this recently introduced index and provides further theoretical results that may support its potential applications in chemical graph theory.

Keywords: Augmented Sombor index, topological index, graph operation, bounds, Cartesian product, join, corona product, wreath product.

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1. Introduction

Chemical graph theory provides a fundamental bridge between mathematics and chemistry, where molecular structures are modeled as graphs with atoms represented by vertices and chemical bonds by edges. Within this framework, *topological indices* play a crucial role as numerical descriptors derived from a graph's structure, designed to correlate with physicochemical properties of corresponding molecules [6].

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The predictive power of these indices in Quantitative Structure-Property Relationship (QSPR) and Quantitative Structure-Activity Relationship (QSAR) studies has established their importance in chemical compound design and analysis [3].

Among various topological indices, degree-based variants have attracted significant attention due to their simplicity and strong correlative ability. A notable recent addition is the *Sombor index*, introduced by Gutman [7] and defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Extensive research has led to investigations of its extremal values across various graph classes [4, 8, 10] and the introduction of several variants including the elliptic Sombor index [1], Euler Sombor index [2], and diminished Sombor index [11].

Das, Gutman, and Ali [5] introduced the *augmented Sombor index* (ASO), defined for a graph G containing no component isomorphic to P_2 by

$$ASO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u)^2 + d_G(v)^2}{d_G(u) + d_G(v) - 2}}.$$

This condition ensures that every denominator in the sum is positive. Throughout this paper, we consider finite simple graphs that are not necessarily connected, provided that no component is isomorphic to K_2 . Isolated vertices make no contribution to the index. In their original work, the authors investigated the chemical applicability of the ASO index to physicochemical properties of octane isomers and established extremal results for trees, quasi-trees, connected graphs, and unicyclic graphs.

Despite these developments, further mathematical investigation of the ASO index remains of interest. Graph operations provide standard constructions for building larger graphs from smaller ones, and background on product graphs and related operations can be found in [9]. Studying the behavior of a degree-based topological index under these operations requires analyzing the resulting vertex degrees and partitioning the edge set according to the operation.

The main contributions of this paper are as follows:

- We derive exact edge-sum formulas for the ASO index of graphs obtained through Cartesian products, joins, corona products, and wreath products.
- We establish upper and lower bounds for these operations under explicit hypotheses that ensure the index is well-defined and the relevant degree conditions are satisfied.
- We obtain sum and product Nordhaus–Gaddum-type inequalities for the ASO index of a graph and its complement, under conditions ensuring that both indices are well-defined.

2. Preliminaries

Throughout this paper, graphs are finite and simple, but are not necessarily connected. For a graph $G = (V(G), E(G))$, we write $n_G = |V(G)|$ and $m_G = |E(G)|$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, and the minimum and maximum degrees are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Whenever necessary, the graphs used in a graph operation are understood to be disjoint copies.

We recall several fundamental graph operations that provide mechanisms for constructing complex graphs from simpler ones.

Definition 1 ([9]). The *Cartesian product* of two graphs G and H , denoted by $G \square H$, has vertex set $V(G \square H) = V(G) \times V(H)$, where two vertices (u, x) and (v, y) are adjacent if and only if either $u = v$ and $xy \in E(H)$, or $x = y$ and $uv \in E(G)$. In $G \square H$, each vertex (u, v) has degree $d_G(u) + d_H(v)$.

Definition 2 ([9]). The *join* of two graphs G and H , denoted by $G + H$, is obtained from their disjoint union by adding every edge joining a vertex of G to a vertex of H . Thus, $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. In $G + H$, the degrees of vertices in G and H are, respectively, $d_{G+H}(u) = d_G(u) + n_H$ and $d_{G+H}(v) = d_H(v) + n_G$.

Definition 3 ([9]). The *corona product* of two graphs G and H , denoted by $G \circ H$, is obtained by taking one copy of G and n_G disjoint copies of H , and joining each vertex of G to every vertex in its corresponding copy of H . In $G \circ H$, the degrees of vertices in G and in a copy of H are, respectively, $d_{G \circ H}(u) = d_G(u) + n_H$ and $d_{G \circ H}(v) = d_H(v) + 1$.

Definition 4 ([9]). The *wreath product*, also known as the *lexicographic product*, of two graphs G and H , denoted by $G \wr H$, has vertex set $V(G \wr H) = V(G) \times V(H)$. Two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either

1. $u_1 u_2 \in E(G)$, or
2. $u_1 = u_2$ and $v_1 v_2 \in E(H)$.

In $G \wr H$, each vertex (u, v) has degree $n_H d_G(u) + d_H(v)$.

The *augmented Sombor index* (ASO) of a graph G having no component isomorphic to K_2 is defined by $\text{ASO}(G) = \sum_{uv \in E(G)} \sqrt{h(d_G(u), d_G(v))}$, where $h(x, y) = \frac{x^2 + y^2}{x + y - 2}$. The condition that G has no component isomorphic to K_2 ensures that every denominator in the sum is positive. Isolated vertices do not affect the index.

We establish fundamental properties of the function h that will be used throughout the paper.

Lemma 1. For positive real numbers x and y satisfying $x + y > 2$, the function $h(x, y) = \frac{x^2 + y^2}{x + y - 2}$ has the following properties:

1. $h(x, y) = h(y, x)$;

2. if $x = y > 1$, then $h(x, x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$;
3. $h(x, y) \geq 4$, with equality if and only if $x = y = 2$.

Proof. The first assertion follows immediately from the definition. For the second assertion, substituting $y = x$ gives $h(x, x) = \frac{2x^2}{2x-2} = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$. For the third assertion, observe that $h(x, y) - 4 = \frac{x^2+y^2-4x-4y+8}{x+y-2} = \frac{(x-2)^2+(y-2)^2}{x+y-2}$. The denominator is positive by assumption, and the numerator is nonnegative. Equality holds if and only if $x = y = 2$. $\square$

The following lemma extends Lemma 1 of [5], which assumes $n > 4$, to graphs of order $n \geq 3$. The equality cases for $n = 3$ and $n = 4$ are included explicitly.

Lemma 2 ([5]). *Let G be a finite simple graph of order $n \geq 5$. For any pendent edge $uv \in E(G)$ with $d_G(u) > d_G(v) = 1$, we have*

$$5 \leq h(d_G(u), 1) \leq n + \frac{2}{n-2},$$

with left equality if and only if $d_G(u) \in \{2, 3\}$, and right equality if and only if $d_G(u) = n - 1$.

The same inequalities and the lower equality condition remain valid for $n = 3, 4$ by direct checking; for $n = 4$, equality in the upper bound also occurs when $d_G(u) = 2$.

Lemma 3 ([5]). *Let G be a finite simple graph of order $n \geq 4$. For any non-pendent edge $uv \in E(G)$ with $d_G(u), d_G(v) \geq 2$, we have*

$$4 \leq h(d_G(u), d_G(v)) \leq n + \frac{1}{n-2},$$

with left equality if and only if $d_G(u) = d_G(v) = 2$, and right equality if and only if $d_G(u) = d_G(v) = n - 1$.

For $n = 3$, every non-pendent edge satisfies $d_G(u) = d_G(v) = 2$, and $h(2, 2) = 4 = 3 + \frac{1}{3-2}$. Thus, the same bounds and equality conditions hold for $n = 3$.

3. ASO index of graph operations

3.1. Cartesian Product

Theorem 1. *Let G and H be nonempty finite simple graphs such that $G \square H$ has no component isomorphic to K_2 . Then*

$$\text{ASO}(G \square H) = \sum_{u \in V(G)} \text{ASO}_H(u) + \sum_{v \in V(H)} \text{ASO}_G(v),$$

where

$$\text{ASO}_H(u) = \sum_{vw \in E(H)} \sqrt{h(d_G(u) + d_H(v), d_G(u) + d_H(w))}$$

and

$$\text{ASO}_G(v) = \sum_{uw \in E(G)} \sqrt{h(d_G(u) + d_H(v), d_G(w) + d_H(v))}.$$

Proof. The edge set of $G \square H$ is the disjoint union of two classes: edges arising from H with a fixed vertex of G , and edges arising from G with a fixed vertex of H .

For each $u \in V(G)$, the edges $\{(u, v)(u, w) : vw \in E(H)\}$ contribute exactly $\text{ASO}_H(u)$, since $d_{G \square H}(u, v) = d_G(u) + d_H(v)$. Similarly, for each $v \in V(H)$, the edges $\{(u, v)(w, v) : uw \in E(G)\}$ contribute exactly $\text{ASO}_G(v)$. Summing over all vertices of G and H , respectively, accounts for every edge of $G \square H$ exactly once. The stated identity follows. $\square$

Corollary 1. *Let G and H be nonempty r -regular and s -regular graphs, respectively, and put $D = r + s$. If $D \geq 2$, then*

$$\text{ASO}(G \square H) = (n_G m_H + n_H m_G) \frac{D}{\sqrt{D-1}}.$$

If $D = 0$, then $G \square H$ is edgeless and $\text{ASO}(G \square H) = 0$. If $D = 1$, then $G \square H$ is 1-regular and has a component isomorphic to K_2 , so its ASO index is undefined.

Proof. Every vertex of $G \square H$ has degree $D = r + s$, and $|E(G \square H)| = n_G m_H + n_H m_G$. If $D \geq 2$, every edge has both endpoint degrees equal to D ; hence its contribution is $\sqrt{h(D, D)} = \sqrt{\frac{2D^2}{2D-2}} = \frac{D}{\sqrt{D-1}}$. Multiplying this contribution by the number of edges proves the formula.

If $D = 0$, both factors are edgeless and so is their Cartesian product. If $D = 1$, every vertex of the product has degree 1; therefore, each component is isomorphic to K_2 , and the ASO index is undefined. $\square$

3.2. Join

Theorem 2. *Let G and H be nonempty finite simple graphs satisfying $n_G + n_H \geq 3$. Then*

$$\text{ASO}(G + H) = \text{ASO}_G^+ + \text{ASO}_H^+ + \text{ASO}_{GH},$$

where

$$\begin{aligned} \text{ASO}_G^+ &= \sum_{uv \in E(G)} \sqrt{h(d_G(u) + n_H, d_G(v) + n_H)}, \\ \text{ASO}_H^+ &= \sum_{uv \in E(H)} \sqrt{h(d_H(u) + n_G, d_H(v) + n_G)}, \\ \text{ASO}_{GH} &= \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{h(d_G(u) + n_H, d_H(v) + n_G)}. \end{aligned}$$

Proof. The join $G + H$ consists of three disjoint classes of edges: edges within G , edges within H , and edges joining a vertex of G to a vertex of H .

In $G + H$, a vertex $u \in V(G)$ has degree $d_G(u) + n_H$, while a vertex $v \in V(H)$ has degree $d_H(v) + n_G$. The three sums in the statement give precisely the contributions of the three respective edge classes. Since $n_G + n_H \geq 3$, the join has no component isomorphic to K_2 , so its ASO index is well-defined. $\square$

For convenience, define

$$\Theta(m, z) = \begin{cases} 0, & m = 0, \\ \frac{mz}{\sqrt{z-1}}, & m > 0 \text{ and } z \geq 2, \end{cases}$$

for nonnegative integers m and real numbers z such that $m = 0$ or $z \geq 2$. Thus, an empty edge class contributes zero without requiring evaluation of a quotient with zero denominator.

Corollary 2. *Let G and H be nonempty r -regular and s -regular graphs, respectively, with $n_G + n_H \geq 3$. Then*

$$\text{ASO}(G + H) = \Theta(m_G, r + n_H) + \Theta(m_H, s + n_G) + n_G n_H \sqrt{\frac{(r + n_H)^2 + (s + n_G)^2}{r + n_H + s + n_G - 2}}.$$

Proof. Every edge within G has endpoint degrees $r + n_H$, and every edge within H has endpoint degrees $s + n_G$. If the corresponding edge class is nonempty, its contribution per edge is, respectively, $\frac{r+n_H}{\sqrt{r+n_H-1}}$ or $\frac{s+n_G}{\sqrt{s+n_G-1}}$. If $m_G > 0$, then $r \geq 1$ and $n_H \geq 1$, so $r + n_H \geq 2$. Similarly, if $m_H > 0$, then $s + n_G \geq 2$. Hence both occurrences of Θ are well-defined, including when the corresponding edge class is empty.

There are $n_G n_H$ edges between the two factors. Each such edge has endpoint degrees $r + n_H$ and $s + n_G$, so its contribution is $\sqrt{\frac{(r+n_H)^2 + (s+n_G)^2}{r+n_H+s+n_G-2}}$. The denominator is positive because $n_G + n_H \geq 3$. Summing the contributions proves the result. $\square$

3.3. Corona Product

Theorem 3. *Let G and H be nonempty finite simple graphs such that $n_H \geq 2$ or $\delta(G) \geq 1$. Then*

$$\text{ASO}(G \circ H) = \text{ASO}_G^* + n_G \text{ASO}_H^* + \text{ASO}_{GH}^*,$$

where

$$\begin{aligned} \text{ASO}_G^* &= \sum_{uv \in E(G)} \sqrt{h(d_G(u) + n_H, d_G(v) + n_H)}, \\ \text{ASO}_H^* &= \sum_{uv \in E(H)} \sqrt{h(d_H(u) + 1, d_H(v) + 1)}, \\ \text{ASO}_{GH}^* &= \sum_{u \in V(G)} \sum_{v \in V(H)} \sqrt{h(d_G(u) + n_H, d_H(v) + 1)}. \end{aligned}$$

Proof. The edge set of $G \circ H$ consists of three disjoint classes: the edges within G , the edges within its n_G attached copies of H , and the edges joining each vertex of G to the vertices of its corresponding copy of H .

The degree of a vertex $u \in V(G)$ in the corona is $d_G(u) + n_H$, whereas a vertex v in an attached copy of H has degree $d_H(v) + 1$. The contributions from the three edge classes are therefore the three sums displayed in the statement, with the contribution from the copies of H multiplied by n_G .

It remains to verify that the index is well-defined. If $n_H \geq 2$, each vertex of G has degree at least 2 in the corona, so no attachment edge can form a K_2 component. Moreover, each endpoint of an internal edge of a copy of H has degree at least 2. If $n_H = 1$, then $H \cong K_1$, and the condition $\delta(G) \geq 1$ ensures that no vertex of G is isolated. Hence no K_2 component is created, and the index is well-defined. The stated identity follows. $\square$

Corollary 3. *Let G be a nonempty finite simple graph with $\delta(G) \geq 1$. Then*

$$\text{ASO}(G \circ K_1) = \sum_{uv \in E(G)} \sqrt{h(d_G(u) + 1, d_G(v) + 1)} + \sum_{u \in V(G)} \sqrt{h(d_G(u) + 1, 1)}.$$

Proof. For $H = K_1$, we have $n_H = 1$, $E(H) = \emptyset$, and $d_H(v) = 0$. Substituting these values into Theorem 3 gives the formula. The condition $\delta(G) \geq 1$ ensures that $G \circ K_1$ has no component isomorphic to K_2 , so the index is well-defined. $\square$

3.4. Wreath Product

The wreath product, also known as the lexicographic product, is an important graph operation. We now give an exact formula for the ASO index of this product.

Theorem 4. *Let G and H be nonempty finite simple graphs such that $G \wr H$ has no component isomorphic to K_2 . For $u \in V(G)$ and $x \in V(H)$, put $\alpha_{u,x} = n_H d_G(u) + d_H(x)$. Then*

$$\text{ASO}(G \wr H) = \sum_{u \in V(G)} \sum_{xy \in E(H)} \sqrt{h(\alpha_{u,x}, \alpha_{u,y})} + \sum_{uv \in E(G)} \sum_{x,y \in V(H)} \sqrt{h(\alpha_{u,x}, \alpha_{v,y})}.$$

Proof. The edge set of $G \wr H$ is the disjoint union of two classes. By the degree formula in the preliminaries, $\alpha_{u,x} = d_{G \wr H}(u, x)$.

First, for each $u \in V(G)$, the edges within the copy of H indexed by u are the edges $(u, x)(u, y)$ with $xy \in E(H)$. The degrees of their endpoints are $n_H d_G(u) + d_H(x)$ and $n_H d_G(u) + d_H(y)$, respectively. Their total contribution is therefore the first sum.

Second, for each edge $uv \in E(G)$, every vertex (u, x) is adjacent to every vertex (v, y) , where $x, y \in V(H)$. The degrees of these endpoints are $n_H d_G(u) + d_H(x)$ and $n_H d_G(v) + d_H(y)$, respectively. Summing over all such edges and all ordered choices x, y gives the second sum. Each edge of $G \wr H$ is counted exactly once in these two classes. The stated formula follows. $\square$

Corollary 4. *Let G and H be nonempty r -regular and s -regular graphs, respectively, and put $D = n_H r + s$. If $D \geq 2$, then $\text{ASO}(G \wr H) = (n_G m_H + m_G n_H^2) \frac{D}{\sqrt{D-1}}$. If $D = 0$, then $G \wr H$ is edgeless and $\text{ASO}(G \wr H) = 0$. If $D = 1$, then $G \wr H$ is 1-regular and has a component isomorphic to K_2 , so its ASO index is undefined.*

Proof. Every vertex of $G \wr H$ has degree $D = n_H r + s$, and $|E(G \wr H)| = n_G m_H + m_G n_H^2$. If $D \geq 2$, every edge contributes $\sqrt{h(D, D)} = \sqrt{\frac{2D^2}{2D-2}} = \frac{D}{\sqrt{D-1}}$. Multiplying by the number of edges proves the formula.

If $D = 0$, then $r = s = 0$, so both factors and their wreath product are edgeless. If $D = 1$, every vertex of the product has degree 1; hence each component is isomorphic to K_2 , and the ASO index is undefined. $\square$

Example 1. Consider the wreath product $C_3 \wr C_3$, where C_3 is the cycle graph on three vertices. Here $n_G = n_H = 3$, $m_G = m_H = 3$, $r = s = 2$. Thus $D = 3 \cdot 2 + 2 = 8$, and Corollary 4 yields $\text{ASO}(C_3 \wr C_3) = (3 \cdot 3 + 3 \cdot 3^2) \frac{8}{\sqrt{7}} = 36 \frac{8}{\sqrt{7}} = \frac{288}{\sqrt{7}}$.

4. Bounds for ASO index of graph operations

In this section, we establish upper and lower bounds for the augmented Sombor index under several graph operations. Throughout, the ASO index is considered only for graphs having no component isomorphic to K_2 . Recall that $h(x, y) = \frac{x^2 + y^2}{x + y - 2}$. For positive real numbers x, y with $x + y > 2$, we have

$$h(x, y) - 4 = \frac{(x - 2)^2 + (y - 2)^2}{x + y - 2} \geq 0. \quad (4.1)$$

Consequently, $h(x, y) \geq 4$, with equality if and only if $x = y = 2$. Moreover, if uv is an edge of a graph having no component isomorphic to K_2 , then $d(u) + d(v) > 2$. Indeed, both endpoint degrees are positive, and their sum can equal 2 only when both endpoints have degree 1, in which case uv forms a K_2 -component. Thus, every edge of a graph in the domain of the ASO index has a positive denominator in the definition of h .

4.1. Bounds for the Function $h(x, y)$

We record several auxiliary bounds for the function h , which may also be useful in further investigations of the ASO index.

Lemma 4. *For real numbers $x, y \geq 2$, the function $h(x, y)$ satisfies*

1.

$$\frac{x + y}{2} + 1 < h(x, y) \leq x + y;$$

2.

$$2 \leq \sqrt{h(x, y)} \leq \sqrt{x + y}.$$

Equality in the upper bound of (1), the lower bound of (2), and the upper bound of (2) holds if and only if $x = y = 2$.

Proof. For the lower bound in (1), direct calculation gives

$$h(x, y) - \left(\frac{x+y}{2} + 1 \right) = \frac{(x-y)^2 + 4}{2(x+y-2)} > 0.$$

For the upper bound, we obtain $(x+y) - h(x, y) = \frac{2(xy-x-y)}{x+y-2}$. Since $x, y \geq 2$, $xy - x - y = (x-1)(y-1) - 1 \geq 0$. Hence $h(x, y) \leq x+y$. Equality holds if and only if $(x-1)(y-1) = 1$, which, under $x, y \geq 2$, is equivalent to $x = y = 2$.

By (4.1), $h(x, y) \geq 4$, with equality if and only if $x = y = 2$. Taking square roots in these inequalities proves (2) and its stated equality conditions. $\square$

Lemma 5. For real numbers $x, y \geq 2$,

$$h(x, y) \leq x + y - 2 + \frac{4}{x + y - 2}.$$

Equality holds if and only if $x = 2$ or $y = 2$.

Proof. Put $t = x + y \geq 4$. Since $y = t - x$, we have

$$h(x, t-x) = \frac{x^2 + (t-x)^2}{t-2} = \frac{2x^2 - 2tx + t^2}{t-2},$$

where $2 \leq x \leq t-2$.

For fixed t , define $f_t(x) = \frac{2x^2 - 2tx + t^2}{t-2}$. Then $f_t''(x) = \frac{4}{t-2} > 0$, so f_t is strictly convex on $[2, t-2]$. Therefore, its maximum on this interval is attained at an endpoint.

By symmetry, $f_t(2) = f_t(t-2) = \frac{4+(t-2)^2}{t-2} = t-2 + \frac{4}{t-2}$. If $t > 4$, strict convexity implies that equality with this maximum occurs only at the endpoints. If $t = 4$, the interval consists of the single point $x = 2$, and equality holds there as well. Thus, $h(x, y) \leq x + y - 2 + \frac{4}{x+y-2}$, with equality if and only if $x = 2$ or $y = 2$. $\square$

Lemma 6. Let $D \geq 2$ and let $2 \leq x, y \leq D$. Then $h(x, y) \leq h(D, D) = \frac{D^2}{D-1}$.

Proof. Consider $\phi(t) = \frac{t^2}{t-1}$, $t \geq 2$. Its derivative is $\phi'(t) = \frac{t(t-2)}{(t-1)^2} \geq 0$. Hence ϕ is nondecreasing on $[2, \infty)$, and $\frac{x^2}{x-1} \leq \frac{D^2}{D-1}$, $\frac{y^2}{y-1} \leq \frac{D^2}{D-1}$. Equivalently, $x^2 \leq \frac{D^2}{D-1}(x-1)$, $y^2 \leq \frac{D^2}{D-1}(y-1)$. Adding these inequalities yields $x^2 + y^2 \leq \frac{D^2}{D-1}(x+y-2)$. Since $x+y-2 > 0$, division by $x+y-2$ proves the result. $\square$

Lemma 7. *Let $I = [p, q]$ and $J = [r, s]$, where $0 < p \leq q$, $0 < r \leq s$, and $p + r > 2$. Then*

$$\max_{(x,y) \in I \times J} h(x, y) = \max\{h(p, r), h(p, s), h(q, r), h(q, s)\}.$$

Proof. Since $p + r > 2$, the denominator of $h(x, y)$ is positive throughout $I \times J$. For fixed $y \in J$, direct differentiation yields $\frac{\partial^2 h}{\partial x^2}(x, y) = \frac{4((y-1)^2+1)}{(x+y-2)^3} > 0$. Hence, $h(\cdot, y)$ is strictly convex on I , and its maximum over this interval is attained at p or q . Consequently, $h(x, y) \leq \max\{h(p, y), h(q, y)\}$ $((x, y) \in I \times J)$. The symmetry $h(x, y) = h(y, x)$ gives the analogous endpoint property in the second variable. Thus, $h(p, y) \leq \max\{h(p, r), h(p, s)\}$, and $h(q, y) \leq \max\{h(q, r), h(q, s)\}$. Combining these inequalities shows that no point in $I \times J$ can yield a value larger than the maximum at its four corners. Since all four corners belong to the rectangle, the asserted equality follows. $\square$

4.2. Bounds for the Cartesian Product

Theorem 5. *Let G and H be nonempty finite simple graphs such that $G \square H$ has no component isomorphic to K_2 . Then $\text{ASO}(G \square H) \geq 2(n_G m_H + n_H m_G)$. Moreover, if $\delta(G \square H) \geq 2$, then we have the following upper bound*

$$\text{ASO}(G \square H) \leq (n_G m_H + n_H m_G) \frac{\Delta(G) + \Delta(H)}{\sqrt{\Delta(G) + \Delta(H) - 1}}.$$

Proof. The number of edges in the Cartesian product is $|E(G \square H)| = n_G m_H + n_H m_G$. For every edge uv of $G \square H$, the assumption that the product has no K_2 -component gives $d_{G \square H}(u) + d_{G \square H}(v) > 2$. By (4.1), each edge contributes at least 2 to the ASO index. Summing over all edges proves the lower bound. This argument also covers the case in which the product has no edges, when both sides of the lower bound are zero.

For the upper bound, put $D = \Delta(G) + \Delta(H)$. The degree formulas for the Cartesian product give $\delta(G \square H) = \delta(G) + \delta(H) \geq 2$ and $\Delta(G \square H) = D$. In particular, $D \geq 2$, and every vertex degree in the product belongs to $[2, D]$. By Lemma 6, every edge contributes at most $\sqrt{h(D, D)} = \frac{D}{\sqrt{D-1}}$. Multiplying this bound by the number of edges proves the upper bound. $\square$

4.3. Bounds for Join

Theorem 6. *Let G and H be nonempty finite simple graphs with $n_G + n_H \geq 3$. Put*

$$\begin{aligned} a_G &= n_H + \delta(G), & b_G &= n_H + \Delta(G), \\ a_H &= n_G + \delta(H), & b_H &= n_G + \Delta(H), \end{aligned}$$

and define

$$M_J = \max\{h(a_G, a_H), h(a_G, b_H), h(b_G, a_H), h(b_G, b_H)\}.$$

Then

$$\text{ASO}(G + H) \geq 2(m_G + m_H + n_G n_H),$$

and

$$\text{ASO}(G + H) \leq \Theta(m_G, b_G) + \Theta(m_H, b_H) + n_G n_H \sqrt{M_J},$$

where Θ is as defined above.

Proof. The join has $|E(G + H)| = m_G + m_H + n_G n_H$ edges. Since $n_G + n_H \geq 3$, the join has no component isomorphic to K_2 . Therefore, every edge contributes at least 2, by (4.1), and the lower bound follows.

For the upper bound, first consider the edges within G . If $m_G > 0$, every endpoint of such an edge has degree at least $n_H + 1 \geq 2$ in the join and at most b_G . By Lemma 6, each such edge contributes at most $\frac{b_G}{\sqrt{b_G - 1}}$. Thus, the total contribution from the edges within G is at most $\Theta(m_G, b_G)$. This remains valid when $m_G = 0$, because the contribution is then zero by the definition of Θ . The same argument shows that the contribution from the edges within H is at most $\Theta(m_H, b_H)$.

Now consider an edge joining $u \in V(G)$ to $v \in V(H)$. Its endpoint degrees satisfy

$$a_G \leq d_{G+H}(u) \leq b_G, \quad a_H \leq d_{G+H}(v) \leq b_H.$$

Furthermore, $a_G + a_H = n_G + n_H + \delta(G) + \delta(H) \geq 3$. Thus, the hypotheses of Lemma 7 are satisfied, and the value of h for every joining edge is at most M_J . Since there are $n_G n_H$ joining edges, their total contribution is at most $n_G n_H \sqrt{M_J}$. Adding the bounds for the three edge classes proves the theorem. $\square$

4.4. Bounds for the Corona Product

Theorem 7. *Let G and H be nonempty finite simple graphs such that $n_H \geq 2$ or $\delta(G) \geq 1$. Put*

$$\begin{aligned} a_G &= n_H + \delta(G), & b_G &= n_H + \Delta(G), \\ a_H &= 1 + \delta(H), & b_H &= 1 + \Delta(H), \end{aligned}$$

and define $M_C = \max\{h(a_G, a_H), h(a_G, b_H), h(b_G, a_H), h(b_G, b_H)\}$. Then

$$\text{ASO}(G \circ H) \geq 2(m_G + n_G m_H + n_G n_H),$$

and

$$\text{ASO}(G \circ H) \leq \Theta(m_G, b_G) + \Theta(n_G m_H, b_H) + n_G n_H \sqrt{M_C},$$

where Θ is as defined above.

Proof. The corona has $|E(G \circ H)| = m_G + n_G m_H + n_G n_H$ edges. The stated hypothesis ensures that $G \circ H$ has no component isomorphic to K_2 . Indeed, if $n_H \geq 2$, every vertex of G is joined to at least two vertices in its attached copy of H ; if $n_H = 1$, the condition $\delta(G) \geq 1$ ensures that no base vertex is isolated in G . Consequently, each edge contributes at least 2, by (4.1), proving the lower bound.

For the upper bound, first consider the edges within G . If $m_G > 0$, each endpoint of such an edge has degree at least $n_H + 1 \geq 2$ in the corona and at most b_G . By Lemma 6, the contribution from these edges is at most $\Theta(m_G, b_G)$. If $m_G = 0$, this contribution is zero.

Each of the n_G copies of H contributes m_H internal edges. The degree in the corona of a vertex corresponding to $v \in V(H)$ is $d_H(v) + 1$. If $m_H > 0$, then $\Delta(H) \geq 1$, so every endpoint of an internal edge has degree at least 2 and at most $b_H = \Delta(H) + 1$. By Lemma 6, the total contribution from these internal edges is at most $\Theta(n_G m_H, b_H)$. If $m_H = 0$, their contribution is zero.

Finally, for every edge joining a vertex of G to a vertex in one of its attached copies of H , the endpoint degrees belong to $[a_G, b_G]$ and $[a_H, b_H]$, respectively. The hypotheses imply $a_G + a_H = n_H + \delta(G) + 1 + \delta(H) \geq 3$. Lemma 7 therefore shows that the value of h for each joining edge is at most M_C . There are $n_G n_H$ such edges, so their total contribution is at most $n_G n_H \sqrt{M_C}$. Adding the bounds for all three edge classes proves the theorem. $\square$

4.5. Bounds for the Wreath Product

Theorem 8. *Let G and H be nonempty finite simple graphs such that $G \wr H$ has no component isomorphic to K_2 . Then $\text{ASO}(G \wr H) \geq 2(n_G m_H + m_G n_H^2)$. If, in addition, $n_H \delta(G) + \delta(H) \geq 2$, then*

$$\text{ASO}(G \wr H) \leq (n_G m_H + m_G n_H^2) \frac{n_H \Delta(G) + \Delta(H)}{\sqrt{n_H \Delta(G) + \Delta(H) - 1}}.$$

Proof. The number of edges in the wreath product is $|E(G \wr H)| = n_G m_H + m_G n_H^2$. Since the product has no component isomorphic to K_2 , the sum of the endpoint degrees of every edge is greater than 2. Thus, by (4.1), every edge contributes at least 2, proving the lower bound. If the product has no edges, both sides of the lower bound are zero.

For the upper bound, put $D = n_H \Delta(G) + \Delta(H)$. The degree formulas for the wreath product give $\delta(G \wr H) = n_H \delta(G) + \delta(H) \geq 2$ and $\Delta(G \wr H) = D$. Hence $D \geq 2$, and every vertex degree belongs to $[2, D]$. By Lemma 6, every edge contributes at most $\sqrt{h(D, D)} = \frac{D}{\sqrt{D-1}}$. Multiplying by the number of edges proves the upper bound. $\square$

4.6. Nordhaus-Gaddum type results

Theorem 9. *Let G be a finite simple graph of order $n \geq 3$ such that neither G nor its complement $\overline{G}$ has a component isomorphic to K_2 . Then*

$$\text{ASO}(G) + \text{ASO}(\overline{G}) \leq \binom{n}{2} \sqrt{\frac{n^2 - 2n + 2}{n - 2}}.$$

Proof. Since neither G nor $\overline{G}$ has a component isomorphic to K_2 , both ASO indices are well-defined.

The edge sets $E(G)$ and $E(\overline{G})$ partition the edge set of the complete graph K_n . For each edge $uv \in E(K_n)$, let F_{uv} denote the unique graph among G and $\overline{G}$ that contains uv . Hence

$$\text{ASO}(G) + \text{ASO}(\overline{G}) = \sum_{uv \in E(K_n)} \sqrt{h(d_{F_{uv}}(u), d_{F_{uv}}(v))}.$$

Fix an edge $uv \in E(K_n)$, and write $x = d_{F_{uv}}(u)$, $y = d_{F_{uv}}(v)$. Since uv is an edge of F_{uv} , we have $1 \leq x, y \leq n - 1$. Furthermore, $x + y > 2$, because F_{uv} has no component isomorphic to K_2 .

We distinguish two cases.

Case 1: At least one endpoint has degree 1.

Without loss of generality, suppose $x = 1$. Then $y \geq 2$. By the symmetry of h , Lemma 2 gives $h(x, y) = h(y, 1) \leq n + \frac{2}{n-2}$.

Case 2: Both endpoint degrees are at least 2.

In this case, $2 \leq x, y \leq n - 1$. By Lemma 3, $h(x, y) \leq n + \frac{1}{n-2} < n + \frac{2}{n-2}$.

Thus, in both cases,

$$\sqrt{h(d_{F_{uv}}(u), d_{F_{uv}}(v))} \leq \sqrt{n + \frac{2}{n-2}} = \sqrt{\frac{n^2 - 2n + 2}{n-2}}.$$

Since K_n has exactly $\binom{n}{2}$ edges, summing this inequality over all its edges yields

$$\text{ASO}(G) + \text{ASO}(\overline{G}) \leq \binom{n}{2} \sqrt{\frac{n^2 - 2n + 2}{n-2}},$$

as required. □

Theorem 10. *Let G be a finite simple graph of order $n \geq 3$ such that neither G nor its complement $\overline{G}$ has a component isomorphic to K_2 . Then*

$$\text{ASO}(G) \text{ASO}(\overline{G}) \leq \frac{1}{4} \binom{n}{2}^2 \frac{n^2 - 2n + 2}{n-2}.$$

Proof. Both ASO indices are nonnegative. By the arithmetic–geometric mean inequality,

$$\text{ASO}(G) \text{ASO}(\overline{G}) \leq \left(\frac{\text{ASO}(G) + \text{ASO}(\overline{G})}{2} \right)^2.$$

Applying Theorem 9, we obtain

$$\text{ASO}(G) \text{ASO}(\overline{G}) \leq \left(\frac{\binom{n}{2}}{2} \sqrt{\frac{n^2 - 2n + 2}{n-2}} \right)^2 = \frac{1}{4} \binom{n}{2}^2 \frac{n^2 - 2n + 2}{n-2}.$$

This completes the proof. □

5. Conclusion

In this paper, we investigated the augmented Sombor index under four graph operations: the Cartesian product, join, corona product, and wreath product. By partitioning the edge sets of the resulting graphs into the corresponding edge classes and using the associated vertex-degree formulas, we obtained exact edge-sum expressions for the ASO index. We also established lower bounds in terms of the number of edges, upper bounds under explicit minimum-degree conditions, and sum and product Nordhaus–Gaddum-type inequalities for a graph and its complement, whenever both ASO indices are well-defined.

The results provide a common framework for studying how these graph operations affect the ASO index. However, the upper bounds obtained here are not asserted to be sharp in all cases. Further work could seek sharper bounds, characterize the equality cases, and determine extremal graphs for specific classes of factors. Another direction is to investigate the chemical applicability of the resulting indices on appropriate molecular graph classes, subject to the structural constraints of those classes and supported by separate empirical analysis.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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