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Research Article

Characterization and construction of total perfect codes in Semi-Cayley graphs

Masoumeh Koohestani[1,†], Doost Ali Mojdeh[1,*], Mohsen Ghasemi[2]

[1]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[†]m.koohestani@umail.umz.ac.ir
[*]damojdeh@umz.ac.ir

[2]Department of Mathematics, Urmia University, P. O. Box 575615-1818, Urmia, Iran
m.ghasemi@urmia.ac.ir

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Abstract: In this paper, we investigate the existence and construction of total perfect codes in semi-Cayley graphs. We first establish two equivalent characterizations for their existence: one in terms of a suitable vertex partition and the other through the defining subsets of the graph. These characterizations provide structural criteria for analyzing and constructing such codes. Furthermore, for a vertex subset $T = M_0 \cup N_1$, we consider its adjugation, defined by $T^* = M_1 \cup N_0$, and prove a necessary and sufficient condition for this operation to preserve the total perfect code property. As an application, we identify a distinguished family of total perfect codes generated from a vertex subset T that is itself a total perfect code. Finally, we unify and extend several previously known results and characterize the relationship between the existence of total perfect codes in semi-Cayley graphs and the property that the graph is a pseudocover of a complete graph.

Keywords: graph partition, pseudocover, semi-Cayley graph, total perfect code.

AMS Subject Classification: 05C25, 05C69, 94B99

1. Introduction

Throughout this paper, all graphs are assumed to be finite, connected, and undirected, and all groups are finite. Let G be a finite group with identity element 1_G. We define two disjoint copies of G as follows:

$$G_0 = \{(g)_0 \mid g \in G\},\ G_1 = \{(g)_1 \mid g \in G\}.$$

* *Corresponding Author*

Let R, L, and S be subsets of G such that $R = R^{-1}$ and $L = L^{-1}$, and $1_G \notin R \cup L$. The *semi-Cayley* graph $\mathrm{SC}(G; R, L, S)$ is defined to be the graph with vertex set $G_0 \cup G_1$, where two vertices $(g)_i$ and $(h)_j$, where $i, j \in \{0, 1\}$, are adjacent if and only if one of the following conditions holds:

(1) $i = j = 0$ and $hg^{-1} \in R$;

(2) $i = j = 1$ and $hg^{-1} \in L$;

(3) $i = 0, j = 1$ and $hg^{-1} \in S$.

In the literature, the term *bi-Cayley* graph is sometimes used in place of semi-Cayley graph.

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A *perfect code* in Γ is a subset C of $V(\Gamma)$ such that no two vertices in C are adjacent and every vertex in $V(\Gamma) \setminus C$ is adjacent to exactly one vertex in C. Also, C is called a *total perfect code* in Γ, if every vertex in $V(\Gamma)$ has exactly one neighbor in C. In graph theory, perfect codes are also referred to as *efficient dominating sets* [6, 7] or *independent perfect dominating sets* [9].

The systematic study of perfect codes began with Norman Biggs in 1973 [3], when he investigated their existence in distance-transitive graphs. Since then, extensive research has been devoted to perfect codes and total perfect codes in broader classes of graphs, including grid graphs and product graphs [1, 5]. Characterizing perfect codes and total perfect codes in various graph classes has become an important area of research because of their applications in the efficient design of communication networks.

More recently, two important classes of graphs, namely Cayley graphs and Cayley sum graphs, have attracted considerable attention due to their rich algebraic structures [2, 4, 8, 10, 13, 15]. However, despite the structural similarities between semi-Cayley graphs and Cayley graphs, relatively little work has been devoted to perfect codes in semi-Cayley graphs [11, 12]. This gap in the literature motivates our investigation of total perfect codes in this class of graphs.

In graph theory, a graph is called *regular* if all vertices have the same degree. Consequently, $\mathrm{SC}(G; R, L, S)$ is regular if and only if $|R| = |L|$, in which case its degree is $|R| + |S|$. In this paper, we focus on total perfect codes in regular semi-Cayley graphs.

The structure of the paper is as follows. In Section 2, we review definitions, notations and basic results about semi-Cayley graphs which would be applied in next section. Section 3 presents our main results on total perfect codes in semi-Cayley graphs. In particular, we establish a necessary and sufficient condition for their existence via a suitable vertex partition, identify a distinguished family of codes generated by such a partition, and characterize their connection with semi-Cayley graphs that are pseudocovers of complete graphs. Section 4 is devoted to summarizing the results of this paper and outlining future research directions on perfect codes and total perfect codes in semi-Cayley graphs.

2. Preliminaries

In this section, we review some notation and basic results that will be used in the subsequent sections. In addition, we recall several definitions related to semi-Cayley graphs, which can be found in [12].
Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We write $u \sim v$, where $u, v \in V(\Gamma)$, to indicate that u and v are adjacent. For a subset $X \subseteq V(\Gamma)$, let $\Gamma[X]$ denote the subgraph of Γ induced by X. We also use $\Gamma(u)$ for the neighbourhood of a vertex $u \in V(\Gamma)$, that is, the set of vertices adjacent to u in Γ. Let K_n denote the complete graph on n vertices.

Definition 1. Let Γ and $\tilde{\Gamma}$ be graphs, and let

$$p : V(\Gamma) \to V(\tilde{\Gamma})$$

be a surjective map. We say that Γ is a *cover* of $\tilde{\Gamma}$ (with *covering projection* p) if for every $u \in V(\Gamma)$, the map p induces a one-to-one correspondence between the neighbors of u and the neighbors of $p(u)$. In other words,

$$p : \Gamma(u) \to \tilde{\Gamma}((p(u))$$

is a bijection for every $u \in V(\Gamma)$. For a vertex $v \in V(\tilde{\Gamma})$, the set

$$p^{-1}(v) = \{u \in V(\Gamma) \mid p(u) = v\}$$

is called the *fibre* over v. If every fibre has the same size k, Γ is called a *k-fold* cover of $\tilde{\Gamma}$.

Definition 2. Let Γ and $\tilde{\Gamma}$ be graphs. We say that Γ is a *pseudocover* of $\tilde{\Gamma}$ if there exists a surjective map

$$p : V(\Gamma) \to V(\tilde{\Gamma})$$

such that the following hold:

(1) For every vertex $v \in V(\tilde{\Gamma})$, the subgraph induced by the fibre $p^{-1}(v)$, i.e., $\Gamma[p^{-1}(v)]$ is a matching.

(2) Let Γ^* be the graph obtained from Γ by deleting all edges contained within the fibres. Then $p : \Gamma^* \to \tilde{\Gamma}$ is a covering projection.

The map p is called a *pseudocovering*, and the set $p^{-1}(v)$ is called the *fibre* over v. If every fiber has exactly k vertices, then p is called a *k-fold* pseudocovering.

Suppose that G is a finite group. For any subsets $H, K \subseteq G$, define

$$HK = \{hk \mid h \in H,\ k \in K\}$$

whenever both H and K are nonempty, and $HK = \emptyset$ when either H or K is empty. In particular, if $H = \{h\}$ or $K = \{k\}$, then we write hK or Hk, respectively. When $H = K$, we denote HK by H^2. Evidently, $HK \subseteq G$.
Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a regular semi-Cayley graph with a bijection $\alpha : G \to G$, where $(1_G)^\alpha = 1_G$ and $R^\alpha = L$. Every subset $T \subseteq V(\Gamma)$ can be written as a union $T = M_0 \cup N_1$, where $M_0 \subseteq G_0$ and $N_1 \subseteq G_1$. The *adjugate* of T is defined by

$$T^* = M_1 \cup N_0.$$

Moreover, for each $x \in G$, the subsets $x \circ_\alpha T$ and $x \blacksquare T^*$ associated with T and T^*, respectively, are defined as follows:

$$x \circ_\alpha T = (xM)_0 \cup (x^\alpha N)_1, \; x \blacksquare T^* = \begin{cases} (xM)_1 \cup (x^{-1}N)_0, & \text{if } \; S \neq S^{-1}, \\ (xM)_1 \cup (xN)_0, & \text{if } \; S = S^{-1}. \end{cases}$$

Furthermore, for a subset $H \subseteq G$, we use the notation

$$H \circ_\alpha T = \cup_{h \in H} h \circ_\alpha T, \; H \blacksquare T^* = \cup_{h \in H} h \blacksquare T^*.$$

We remark that each element $x \in G$ induces an automorphism of $\mathrm{SC}(G; R, L, S)$ defined by

$$\breve{x} : V(\mathrm{SC}(G; R, L, S)) \to V(\mathrm{SC}(G; R, L, S)), \quad (u)_i \mapsto (ux)_i.$$

Consequently, we have

$$T^{\breve{x}} = (Mx)_0 \cup (Nx)_1, \quad (T^*)^{\breve{x}} = (Mx)_1 \cup (Nx)_0.$$

We now state the following proposition, which follows directly from the definition of total perfect codes.

Proposition 1. *Let Γ be a graph. Then, the following items hold:*

(1) If C is a total perfect code of Γ, then $\{\Gamma(c) \mid c \in C\}$ is a partition of $V(\Gamma)$.

(2) For two disjoint total perfect codes C_1 and C_2 of Γ, we have $|C_1| = |C_2|$. Furthermore, if we consider $\Gamma[C_1 \cup C_2]$ and delete all edges internal to C_1 and to C_2, then the resulting graph is a perfect matching.

(3) Let C is a total perfect code and $\sigma \in Aut(\Gamma)$. Then, C^σ is a total perfect code of Γ.

Proof. (1) Let C be a total perfect code of Γ. We first show that

$$V(\Gamma) = \bigcup_{c \in C} \Gamma(c).$$

The inclusion $\bigcup_{c \in C} \Gamma(c) \subseteq V(\Gamma)$ is immediate. Conversely, for any $v \in V(\Gamma)$, the definition of a total perfect code guarantees a unique $c \in C$ such that $v \sim c$. Hence $v \in \Gamma(c)$, proving that $V(\Gamma) \subseteq \bigcup_{c \in C} \Gamma(c)$.

Now, suppose $\Gamma(c_1) \cap \Gamma(c_2) \neq \emptyset$ for distinct $c_1, c_2 \in C$. Then some vertex v is adjacent to both c_1 and c_2, contradicting the definition of a total perfect code. Therefore, the sets $\{\Gamma(c) \mid c \in C\}$ are pairwise disjoint and form a partition of $V(\Gamma)$.

(2) Let C_1 and C_2 be total perfect codes of Γ. For each $c_1 \in C_1$, since C_2 is a total perfect code of Γ, there exists a unique $c_2 \in C_2$ such that $c_1 \sim c_2$. Conversely, since C_1 is also a total perfect code, each $c_2 \in C_2$ is adjacent to a unique vertex of C_1. Thus, the adjacency relation defines a bijection between C_1 and C_2. Consequently,

$$|C_1| = |C_2|.$$

Moreover, after deleting the edges internal to C_1 and C_2, the remaining edges of $\Gamma[C_1 \cup C_2]$ form a perfect matching between C_1 and C_2.

(3) Let C be a total perfect code of Γ and let $\sigma \in \mathrm{Aut}(\Gamma)$. For any $v \in V(\Gamma)$, write $v = \sigma(u)$ for some $u \in V(\Gamma)$. Since C is a total perfect code, there exists a unique $c \in C$ such that $u \sim c$. As σ preserves adjacency, we have $v = \sigma(u) \sim \sigma(c) \in C^\sigma$. If $v \sim \sigma(c')$ for some $c' \in C$, then applying σ^{-1} yields $u \sim c'$, contradicting the uniqueness of c. Hence, C^σ is a total perfect code of Γ.

□

The next proposition is an immediate consequence of the definition of pseudocovering together with the definition of a total perfect code.

Proposition 2. *Let Γ be a graph and let n be a natural number. Then, Γ is a pseudocover of K_n if $V(\Gamma)$ admits a partition $\{C_1, C_2, \ldots, C_n\}$ such that each C_i is a total perfect code of Γ.*

Proof. Let $\{C_1, C_2, \ldots, C_n\}$ be a partition of $V(\Gamma)$ into total perfect codes. By Proposition 1(2), all sets C_is have the same cardinality, say c, and for every $i \neq j$, the graph obtained from $\Gamma[C_i \cup C_j]$ by deleting the edges within C_i and C_j is a perfect matching between C_i and C_j. Define

$$p : V(\Gamma) = \bigcup_{i=1}^{n} C_i \to V(K_n) = \{v_1, \ldots, v_n\}$$

by $p(u) = v_i$ whenever $u \in C_i$. Then, $p^{-1}(v_i) = C_i$ for each i, so every fiber has size c. Moreover, for every $u \in C_i$, the restriction

$$p : \Gamma(u) \to K_n(p(u))$$

is a bijection, since u has exactly one neighbor in each $C_j (j \neq i)$, while v_i is adjacent to all $n-1$ remaining vertices of K_n. Hence, the conditions of Definition 2 are satisfied, and therefore Γ is a c-fold pseudocover of K_n. □

3. Total perfect codes

In this section, we present our results on total perfect codes in semi-Cayley graphs. We begin by stating the following remark concerning the neighborhoods of vertices in a semi-Cayley graph. Recall that the notation $\mathrm{SC}(G; R, L, S)$ denotes a semi-Cayley graph with $L = R^\alpha$, where α is a bijection of G such that $(1_G)^\alpha = 1_G$.

Remark 1. Let $\Gamma = \mathrm{SC}(G; R, L, S)$ and let $T = M_0 \cup N_1$ be a subset of $V(\Gamma)$. Then, the neighborhood of vertices $(m)_0, (n)_1 \in T$ can be obtained as follows:

$$\Gamma\left((m)_0\right) = \{(rm)_0, (sm)_1 \mid r \in R, s \in S\}, \quad \Gamma\left((n)_1\right) = \{(r^\alpha n)_1, (s^{-1}n)_0 \mid r \in R, s \in S\}.$$

Hence,

$$\begin{aligned}
\cup_{t\in T}\Gamma(t) &= \bigcup_{(m)_0\in M_0,\ (n)_1\in N_1} \left(\bigcup_{r\in R} ((rm)_0 \cup (r^\alpha n)_1) \cup \bigcup_{s\in S} ((sm)_1 \cup (s^{-1}n)_0) \right) \\
&= \bigcup_{r\in R,\ s\in S} \left((rM)_0 \cup (r^\alpha N)_1 \cup (sM)_1 \cup (s^{-1}N)_0\right) \\
&= \bigcup_{r\in R,\ s\in S} (r \circ_\alpha T \cup s \,\blacksquare\, T^*)
\end{aligned}$$

The following theorem investigates the relationship between total perfect codes and partitions of the vertex set in regular semi-Cayley graphs. We note that similar relationships have also been established for Cayley graphs and Cayley sum graphs [10, 15].

Theorem 1. *Assume* $\Gamma = \mathrm{SC(G; R, L, S)}$ *is a regular semi-Cayley graph and* $T = M_0 \cup N_1 \subseteq V(\Gamma)$, *where* $M, N \subseteq G$. *Then, the items (1) and (2) in the following are equivalent. Moreover, they are equivalent with (3) provided that* $\alpha \in \mathrm{Aut(G)}$ *and* $S = S^{-1}$.

(1) T *is a total perfect code.*

(2) The family of subsets $\{r \circ_\alpha T, s \,\blacksquare\, T^* \mid r \in R, s \in S\}$ *forms a partition of* $V(\Gamma)$.

(3) $|T| = \frac{2|G|}{|R|+|S|}$, $(R^2 \setminus \{1_G\}) \circ_\alpha T \cap T = (S^2 \setminus \{1_G\}) \,\blacksquare\, T^* \cap T^* = R \circ_\alpha T \cap S \,\blacksquare\, T^* = \emptyset$.

Proof. (1) $\Rightarrow$ (2) Since $T = M_0 \cup N_1$ is a total perfect code of Γ, Proposition 1 implies that $\{\Gamma(t) \mid t \in T\}$ is a partition for $V(\Gamma)$. Using Remark 1, the $|R| + |S|$ subsets $r \circ_\alpha T$ and $s \blacksquare T^*$, where $r \in R$ and $s \in S$, form a partition for G.
(2) $\Rightarrow$ (1) Suppose that $\{r \circ_\alpha T, s \blacksquare T^* \mid r \in R, s \in S\}$ is a partition of $V(\Gamma)$. We prove that every vertex of Γ is adjacent to a unique vertex in T by considering following two cases:

(i) Let $(x)_0 \in V(\Gamma)$. Then, we have $(x)_0 \in r \circ_\alpha T$ or $(x)_0 \in s \blacksquare T^*$, for a unique $r \in R$ or $s \in S$. Consequently, $(x)_0 = (rm)_0$ or $(x)_0 = \left(s^{-1}n\right)_0$, for some $m \in M$ and $n \in N$, which means that $(x)_0 \sim (m)_0$ or $(x)_0 \sim (n)_1$.

(ii) We consider $x_1 \in V(\Gamma)$. Based on the preceding argument, we conclude that $(x)_1 = \left(r^\alpha \tilde{n}\right)_1$ or $(x)_1 = (s\tilde{m})_1$, for a unique $r \in R$ or $s \in S$, which leads to the fact that $(x)_1 \sim (\tilde{n})_1$ or $(x)_1 \sim (\tilde{m})_0$.

We now assume that $\alpha \in \mathrm{Aut(G)}$ and $S = S^{-1}$. We aim to show that item (2) is equivalent to item (3).
(2) $\Rightarrow$ (3) Let $\{r \circ_\alpha T, s \blacksquare T^* \mid r \in R, s \in S\}$ be a partition of $V(\Gamma)$. So, we have $2|G| = |T|(|R| + |S|)$. To establish the other statements in item (3), we distinguish the following three cases:

(i) Assume $t \in (R^2 \setminus \{1_G\}) \circ_\alpha T \cap T$. So, there exist $r, \tilde{r} \in R$, where $r^{-1} \neq \tilde{r}$, such that $t \in r^{-1}\tilde{r} \circ_\alpha T \cap T$. Then, we have $t \in M_0 \cap (r^{-1}\tilde{r}M)_0$ or $t \in N_1 \cap ((r^{-1}\tilde{r})^\alpha N)_1$ which leads to $r \circ_\alpha T \cap \tilde{r} \circ_\alpha T \neq \emptyset$ and this is a contradiction.

(ii) Let $t \in (S^2 \setminus \{1_G\}) \blacksquare T^* \cap T^*$. Therefore, there exists $s, \tilde{s} \in S$, with $s^{-1} \neq \tilde{s}$, for which $t \in s^{-1}\tilde{s} \blacksquare T^* \cap T^*$. We obtain $t \in M_1 \cap (s^{-1}\tilde{s}M)_1$ or $t \in N_0 \cap (s^{-1}\tilde{s}N)_0$. Hence, $s \blacksquare T^* \cap \tilde{s} \blacksquare T^* \neq \emptyset$ and this contradicts the assumption in item (2).

(iii) We now consider $R \circ_\alpha T \cap S \blacksquare T^* \neq \emptyset$. It follows that there exists $r \in R$ and $s \in S$ such that $r \circ_\alpha T \cap s \blacksquare T^* \neq \emptyset$ and this is a contradiction.

(3) $\Rightarrow$ (2) We prove that, for every $r \in R$ and $s \in S$, all sets $r \circ_\alpha T$ and $s \blacksquare T^*$ are pairwise disjoint. We proceed by considering the following three cases:

(i) Suppose that $t \in r \circ_\alpha T \cap \tilde{r} \circ_\alpha T$, where $r, \tilde{r} \in R$ are distinct. Then either $t \in (rM)_0 \cap (\tilde{r}M)_0$ or $t \in (r^\alpha N)_1 \cap (\tilde{r}^\alpha N)_1$. Consequently, $r^{-1}\tilde{r}M \cap M \neq \emptyset$ or $(r^{-1}\tilde{r})^\alpha N \cap N \neq \emptyset$, which contradicts the assumption that $(R^2 \setminus \{1_G\}) \circ_\alpha T \cap T = \emptyset$.

(ii) Let $t \in s \blacksquare T^* \cap \tilde{s} \blacksquare T^*$, where $s, \tilde{s} \in S$ are distinct. Then we obtain that $s^{-1}\tilde{s} \blacksquare T^* \cap T^* \neq \emptyset$ which is a contradiction.

(iii) If $t \in r \circ_\alpha T \cap s \blacksquare T^*$, then this contradicts the assumption $R \circ_\alpha T \cap S \blacksquare T^* = \emptyset$.

Moreover, it is obvious that $|\{r \circ_\alpha T, s \blacksquare T^* \mid r \in R, s \in S\}| = |T|(|R| + |S|)$, and from the statement in item (3), this is equal to $2|G| = |V(\Gamma)|$. Thus, item (2) is proved. $\square$

In Theorem 1, let α be the identity map and $T \subseteq V(\Gamma)$ be a total perfect code of Γ. Then $\{rT, sT \mid r \in R, s \in S\}$ forms a partition of $V(\Gamma)$.
The next result, which follows from Theorem 1, concerns the computation of the cardinalities of the subsets M and N in a total perfect code $T = M_0 \cup N_1$.

Corollary 1. *Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be regular, and $T = M_0 \cup N_1$ be a vertex subset of Γ. If $T = M_0 \cup N_1$ is a total perfect code of Γ, then one of the following items holds:*

(1) $|M| = |N| = \frac{|G|}{|R|+|S|}$, when both M and N are nonempty sets.

(2) We have $|R| = |S|$. Moreover, if $M = \emptyset$, then $|N| = \frac{|G|}{|S|}$; and if $N = \emptyset$, then $|M| = \frac{|G|}{|S|}$.

Proof. Suppose that α a bijection of G with $(1_G)^\alpha = 1_G$ and $R^\alpha = L$. From Theorem 1, $\{r \circ_\alpha T, s \bullet T^* \mid r \in R, s \in S\}$ is a partition of G. Note that each set $r \circ_\alpha T$ contains exactly $|M|$ elements of G_0 and $|N|$ elements of G_1. Also, every set $s \bullet T^*$ includes exactly $|M|$ elements of G_1 and $|N|$ elements of G_0. So, we have $|M||R| + |N||S| = |N||R| + |M||S|$. This implies the following two cases.

(i) If both $|M|$ and $|N|$ are nonzero values, then we must have $|M| = |N|$. Therefore, from Theorem 1, we obtain

$$|M| = |N| = \frac{|G|}{|R| + |S|}.$$

(ii) If $|M|$ or $|N| = 0$, it follows that $|R| = |S|$. Using Theorem 1, either $|M|$ or $|N|$, whichever is nonzero, is equal to $|G|/|S|$.

□

One intriguing feature of semi-Cayley graphs is the connection between a vertex subset T and its adjugate T^*. As illustrated below, T may form a total perfect code without T^* having the same property.

Example 1. Let $\Gamma = \mathrm{SC}(\mathbb{Z}_6, \{\pm 1\}, \{\pm 1\}), \{1\})$ be a semi-Cayley graph and $\alpha \in \mathrm{Aut}(\mathbb{Z}_6)$ be the identity map. Suppose that $M = \{0, 3\}$ and $N = \{1, 4\}$. Then, $T = M_0 \cup N_1$ is a total perfect code, while T^* is not. See Figure 1.

The following theorem specifies the conditions for which the property of being a total perfect code is preserved under adjugation.

Theorem 2. *Let $\Gamma = \mathrm{SC}(G; R, R^\alpha, S)$ be a semi-Cayley graph, where $\alpha \in \mathrm{Aut(G)}$, and $T = M_0 \cup N_1 \subseteq V(\Gamma)$, where $M, N \subseteq G$. If $S = S^{-1}$, $M^\alpha = M$, and $N^\alpha = N$, then T is a total perfect code if and only if T^* is a total perfect code.*

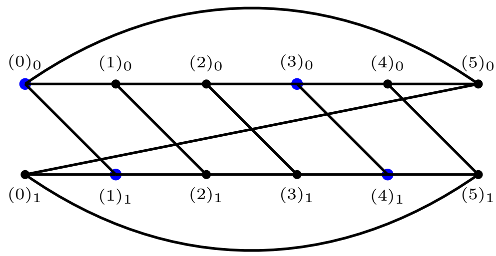

Figure 1. $\mathrm{SC}\,(\mathbb{Z}_6, \{\pm 1\}, \{\pm 1\}, \{1\})$

Proof. Suppose that T is a total perfect code of Γ. We prove that each vertex in $V(\Gamma) = G_0 \cup G_1$ is adjacent to a unique vertex in T^*. Let $(g)_1 \in G_1$ be an arbitrary vertex of Γ. Since T is a total perfect code of Γ, there exists exactly one element, either $(m)_0 \in M_0$ or $(n)_1 \in N_1$, that is adjacent to $(g)_0 \in G_0$. Hence,

$$(g)_0 = (rm)_0 \text{ or } (g)_0 = (sn)_0,$$

for $r \in R$ or $s \in S$. Consequently, we obtain $(g^\alpha)_1 = (r^\alpha m^\alpha)_1$ or $(g)_1 = (sn)_1$ which implies that $(g^\alpha)_1 \sim (m^\alpha)_1 \in M_1$ or $(g)_1 \sim (n)_0 \in N_0$.
We now consider $(g)_0 \in G_0$. Since T is a total perfect code of Γ, $(g)_1$ is adjacent to a unique vertex in T, say $(g)_1 \sim (\tilde{m})_0$ or $(\tilde{n})_1$, where $(\tilde{m})_0 \in M_0$ or $(\tilde{n})_1 \in N_1$. Consequently,

$$(g)_1 = (\tilde{s}\tilde{m})_1 \text{ or } (g)_1 = (\tilde{r}^\alpha \tilde{n})_1,$$

for some $r \in R$ and $s \in S$. Hence,

$$(g^{\alpha^{-1}})_0 = (\tilde{r}\tilde{n}^{\alpha^{-1}})_0 \text{ or } (g)_0 = (\tilde{s}\tilde{m})_0.$$

It follows that $(g^{\alpha^{-1}})_0 \sim (\tilde{n}^{\alpha^{-1}})_0 \in N_0$ or $(g)_0 \sim (\tilde{m})_1 \in M_1$. Therefore, under the assumptions $M^\alpha = M$, $N^\alpha = N$ and $S = S^{-1}$, each vertex in Γ has a unique neighbor in T^*.
Conversely, suppose that T^* is a total perfect code. By the same argument as above, it follows that every vertex in Γ is adjacent to a unique vertex in T. □

Remark 2. In Theorem 2, if we consider the simpler case where α is the identity map, then the theorem states that in the semi-Cayley graph $\Gamma = \mathrm{SC}(G; R, R, S)$ with $S = S^{-1}$, for any subset $\{t_1, t_2, \ldots, t_k\} \subseteq G$, the subsets

$$\{(t_1)_{i_1}, \ldots, (t_k)_{i_k}\} \text{ and } \{(t_1)_{1-i_1}, \ldots, (t_k)_{1-i_k}\}$$

of $V(\Gamma)$, where each $i_j \in \{0, 1\}$, are either both total perfect codes or neither is. A similar result for perfect codes in semi-Cayley graphs was proved in [11].

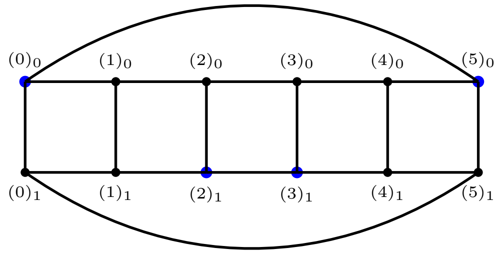

Figure 2. $\mathrm{SC}(\mathbb{Z}_6, \{\pm 1\}, \{\pm 1\}, \{0\})$

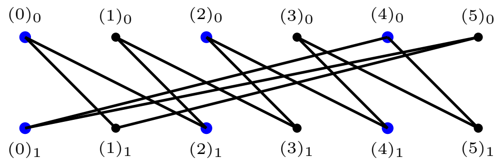

Figure 3. $\mathrm{SC}(\mathbb{Z}_6, \emptyset, \emptyset, \{1, 2\})$

Example 2. Let $\Gamma = \mathrm{SC}(\mathbb{Z}_6, \{\pm 1\}, \{\pm 1\}, \{0\})$ be a semi-Cayley graph. Suppose that $\alpha \in \mathrm{Aut}(\mathbb{Z}_6)$ is the identity map, $M = \{0, 5\}$ and $N = \{2, 3\}$. We can see that M and N satisfy the conditions in Theorem 2. Therefore, either both $T = M_0 \cup N_1$ and $T^* = N_0 \cup M_1$ are total perfect codes, or neither of them is a total perfect code. From Figure 2, both T and T^* are total perfect codes in Γ.

Note that the conditions $M^\alpha = M$, $N^\alpha = N$, and $S^{-1} = S$ in Theorem 2 are sufficient but not necessary. In the following example, both of T and T^* are total perfect codes, even though Γ does not satisfy all conditions.

Example 3. Suppose that $\Gamma = \mathrm{SC}(\mathbb{Z}_6, \emptyset, \emptyset, \{1, 2\})$ in which $\alpha \in \mathrm{Aut}(\mathbb{Z}_6)$ is the identity map. Suppose that $M = N = \{0, 2, 4\}$ and $T = M_0 \cup N_1$. It is straightforward to check that T and T^* are total perfect codes in Γ, see Figure 3. However, Γ does not satisfy all conditions in Theorem 2 because $S \neq S^{-1}$. Therefore, the conditions of this theorem are sufficient but not necessary.

Corollary 2. *Let $\Gamma = \mathrm{SC}(G; R, R^\alpha, S)$ be a semi-Cayley graph and $T = M_0 \cup N_1 \subseteq V(\Gamma)$, where $\alpha \in \mathrm{Aut(G)}$ and $M, N \subseteq G$. Suppose that $S = S^{-1}, M^\alpha = M, N^\alpha = N$. If T is a total perfect code of Γ, then $T^{\check{x}}$ and $(T^*)^{\check{x}}$ are also total perfect codes.*

Proof. Since T is a total perfect code of Γ, it follows from Proposition 1(3) that $T^{\check{x}}$ is a total perfect code because it is obtained by the right multiplication from T. Therefore, from Theorem 2, $(T^*)^{\check{x}}$ is also total perfect codes of Γ. $\square$

Note that by [14], we know that in connected semi-Cayley graphs $\mathrm{SC}(G; R, L, S)$ the union $R \cup L \cup S$ generates G. Therefore, if we consider the identity map in Corollary 2, then for a total perfect code T in the semi-Cayley graph $\mathrm{SC}(G; R, R, S)$, where $S^{-1} = S$, each of the sets Tr and Ts (for every $r \in R$ and $s \in S$) is also a total perfect code. Analogous results for Cayley graphs and Cayley sum graphs were established in [10, 15], respectively.

Remark 3. In Corollary 2, suppose that T satisfies the additional property

$$T^{\check{r}} = r \circ_\alpha T, \qquad (T^*)^{\check{s}} = s \mathbin{\scriptstyle\blacksquare} T^*, \tag{3.1}$$

for all $r \in R$ and $s \in S$. Then, by Theorem 1, Γ admits a partition whose parts are all total perfect codes. Moreover, any vertex subset T satisfying property (3.1) is called a *quasi-normal subset of* Γ.

Corollary 3. *Let* $\Gamma = \mathrm{SC}(G; R, R^\alpha, S)$ *be a semi-Cayley graph with a quasi-normal vertex subse* $T = M_0 \cup N_1$*, where* $\alpha \in \mathrm{Aut(G)}$ *and* $M, N \subseteq G$*. Suppose that* $S = S^{-1}, M^\alpha = M, N^\alpha = N$*. If* T *is a total perfect code of* Γ*, then* Γ *is a* $|T|$*-fold pseudocover of* $K_{|R|+|S|}$*.*

Proof. Suppose that $R = \{r_1, r_2, \ldots, r_n\}$ and $S = \{s_1, s_2, \ldots, s_m\}$. By Remark 3, the set

$$\{T^{\check{r_1}}, \ldots, T^{\check{r_n}}, (T^*)^{\check{s_1}}, \ldots, (T^*)^{\check{s_m}}\}$$

forms a partition of $V(\Gamma)$ in which every part is a total perfect code. Then, by Proposition 2, it follows that Γ is a $|T|$-fold pseudocover of $K_{|R|+|S|}$. □

Theorem 3. *Let* $\Gamma = \mathrm{SC}(G; R, L, S)$ *be regular and* $T = M_0 \cup N_1$*, where* $M_0 \subseteq G_0$ *and* $N_1 \subseteq G_1$*. If there exists a* $|T|$*-fold pseudocovering* $f : \Gamma \to \mathrm{K}_{|\mathrm{R}|+|\mathrm{S}|}$ *such that* T *and* T^* *are two fibers of* f*, then both* T *and* T^* *are total perfect codes of* Γ*.*

Proof. Suppose that α is a bijection of G with $(1_G)^\alpha = 1_G$ and $R^\alpha = L$. Let $f : \Gamma \to \mathrm{K}_{|\mathrm{R}|+|\mathrm{S}|}$ be a $|T|$-fold pseudocovering and let $u, v \in V(\mathrm{K}_{|\mathrm{R}|+|\mathrm{S}|})$ be two separate elements with $f^{-1}(u) = T$ and $f^{-1}(v) = T^*$. It is obvious that $T \cap T^* = \emptyset$. We now prove that each vertex in Γ is adjacent to a unique vertex of T and a unique vertex of T^* as described below:

- Let $x \in T \cup T^*$. By the definition of a pseudocovering, the induced subgraphs $\Gamma[T]$ and $\Gamma[T^*]$ are perfect matchings. Hence, there exist unique vertices $t_1 \in T$ and $t_2 \in T^*$ such that $x \sim t_1$ and $x \sim t_2$.
- Let $x \in V(\Gamma) \setminus T \cup T^*$. Since $\mathrm{K}_{|\mathrm{R}|+|\mathrm{S}|}$ is complete, we have $f(x) \sim u$ and $f(x) \sim v$. As the restriction of f to $\Gamma^*(x)$ is a bijection onto $\mathrm{K(f(x))}$, there exist vertices unique vertices $x_1, x_2 \in \Gamma^*(x)$ such that $f(x_1) = u$ and $f(x_2) = v$. Because $f^{-1}(u) = T$ and $f^{-1}(v) = T^*$, it follows that $x_1 \in T$ and $x_2 \in T^*$.

□

4. Concluding remarks

In this paper, we established two necessary and sufficient conditions for a semi-Cayley graph to admit a total perfect code. One condition is based on the existence of an appropriate vertex partition, while the other is characterized by the structural properties of the graph. We also identified conditions under which $T = M_0 \cup N_1$ is a total perfect code if and only if $T^* = M_1 \cup N_0$ is a total perfect code. As an application of this result, together with the correspondence between vertex partitions and total perfect codes, we constructed a family of total perfect codes derived from a given total perfect code. Furthermore, by exploiting this family, we established several relationships between the existence of total perfect codes in $\mathrm{SC}(G; R, L, S)$ and the pseudocover structure of the graph over $K_{|R|+|S|}$.
Several directions remain open for future research. In particular, the properties of perfect codes and total perfect codes that are also subgroups are relatively unexplored in the context of semi-Cayley graphs. Moreover, studying semi-Cayley graphs over special classes of groups, such as abelian and cyclic groups, may yield further theoretical insights and lead to new applications.

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