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A note on the Sombor coindex of graphs

Zhang Yun[1,†], Xuewu Zuo[1,‡], Akbar Jahanbani[2,*], Murat Cancan[3]

[1] General Education Department, Anhui Xinhua University, Hefei, 230088, China
[†] xinhuazy2022@163.com
[‡] xinhuazxw@163.com

[2] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
[*] akbar.jahanbani92@gmail.com

[3] Faculty of Education, Van Yuzuncu Yil University, Van, Turkey
mcancan@yyu.edu.tr

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Abstract: In this note, we point out several critical inaccuracies in a recent paper on the Sombor coindex of graphs [On Sombor coindex of graphs, Commun. Comb. Optim. 8 (2023), no. 3, 513–529.]. We identify fundamental errors in the main results and provide concrete counterexamples to refute them.

Keywords: Sombor coindex, graph operations, topological indices, corrections, counterexamples.

AMS Subject Classification: 05C07, 05C35

1. Introduction and preliminaries

The Sombor index and its coindex have attracted considerable attention in chemical graph theory. In [3], the authors studied various properties of the Sombor coindex, including bounds, relations to other coindices, and explicit formulas for graph operations. While the paper contains interesting results, some of the presented statements are incorrect. In this note, we identify these inaccuracies and provide corrected versions with rigorous proofs.

We follow the notation of [3]. Let G be a finite simple graph with n vertices and m edges. The degree of a vertex u is denoted by $d_G(u)$, with Δ and δ being the maximum

* *Corresponding Author*

and minimum degrees, respectively. The complement of G is $\overline{G}$, and $\overline{m} = \binom{n}{2} - m$. The Sombor coindex is defined as

$$\overline{SO}(G) = \sum_{uv \notin E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

The first Zagreb coindex and the forgotten coindex are respectively defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)], \qquad \overline{F}(G) = \sum_{uv \notin E(G)} [d_G(u)^2 + d_G(v)^2].$$

For further background on the Sombor index and its coindex, we refer the reader to recent work [2].

2. Corrections

We now correct several errors in [3]. For each, we state the original claim, provide a counterexample, and give a corrected version with proof. In some cases, our corrections improve the original bounds.

2.1. Theorem 5

In [3], Theorem 5, the authors proved that for any graph G on n vertices with minimum degree δ,

$$\overline{SO}(G) + \overline{SO}(\overline{G}) \leq 2\overline{M}_1(G) - \left(1 - \frac{1}{\sqrt{2}}\right) n(n-1)\delta. \tag{2.1}$$

This claim is incorrect. Consider the path $G = P_3$, the degree sequence is $(1, 2, 1)$, hence $n = 3$, $\delta = 1$, and $\overline{M}_1(P_3) = 2$. We compute:

$$\overline{SO}(P_3) = \sqrt{1^2 + 1^2} = \sqrt{2} \approx 1.414,$$

and $\overline{P_3}$ has degree sequence $(1, 0, 1)$, so

$$\overline{SO}(\overline{P_3}) = \sqrt{1^2 + 0^2} + \sqrt{0^2 + 1^2} = 2.$$

Thus the left-hand side of 2.1 equals 3.414. The right-hand side is

$$2 \cdot 2 - \left(1 - \frac{1}{\sqrt{2}}\right) \cdot 3 \cdot 2 \cdot 1 = 4 - 1.757 = 2.243.$$

Clearly, inequality (2.1) is false.

Theorem 1 (Corrected Theorem 5). *Let G be a graph on n vertices with m edges and minimum degree $\delta(G) = \delta$. Let $\overline{G}$ have minimum degree $\delta(\overline{G})$ (with the convention $\delta(\overline{G}) = 0$ if $\overline{G}$ has no edges). Then*

$$\overline{SO}(G) + \overline{SO}(\overline{G}) \leq 2\overline{M}_1(G) - (2 - \sqrt{2})\big(\delta\,\overline{m} + \delta(\overline{G})\,m\big). \tag{2.2}$$

Proof. Applying Theorem 3 of [3] to G and to $\overline{G}$ gives

$$\overline{SO}(G) \leq \overline{M}_1(G) - (2 - \sqrt{2})\delta\,\overline{m},$$

and $\overline{SO}(\overline{G}) \leq \overline{M}_1(\overline{G}) - (2 - \sqrt{2})\delta(\overline{G})\,m$. It is known that $\overline{M}_1(G) = \overline{M}_1(\overline{G})$ (see [1, 3]). Adding the two inequalities yields (2.2). $\square$

2.2. On the proof of Theorem 6 in [3]

In the proof of Theorem 6 in [3], the authors set $a_k = \sqrt{d_i^2 + d_j^2}$ and $b_k = \delta$, and choose $a = \delta$, $A = \Delta$, $b = \delta$ and $B = \Delta$.
With these values, their algebraic computation

$$\frac{(ab + AB)^2}{4abAB} = \frac{1}{4}\left(\frac{\delta}{\Delta} + \frac{\Delta}{\delta}\right)^2 \tag{1}$$

is correct. However, this choice violates the hypotheses of the Pólya–Szegö inequality, since $a_k = \sqrt{d_i^2 + d_j^2} \leq \sqrt{2}\Delta$, which requires $A \geq \sqrt{2}\Delta$, not $A = \Delta$. Therefore, the lemma cannot be applied with these parameters. The proof is thus invalid.

Theorem 2 (Corrected Theorem 6). *Let G be a graph with m edges and $\delta > 0$. Then*

$$\sqrt{\overline{m}\,\overline{F}(G)} \leq \frac{\Delta + \delta}{2\sqrt{\Delta\delta}}\,\overline{SO}(G). \tag{2.3}$$

Proof. If $\overline{m} = 0$, then $\overline{SO}(G) = 0$ and $\overline{F}(G) = 0$, so the inequality holds. Hence, assume $\overline{m} > 0$. Let $V(G) = \{v_1, \ldots, v_n\}$ and let $d_i = d_G(v_i)$. For each edge $v_iv_j \in E(\overline{G})$, define

$$a_{ij} = \sqrt{d_i^2 + d_j^2}, \qquad b_{ij} = \delta.$$

Then $0 < \delta \leq a_{ij} \leq \sqrt{2}\Delta$ and $b_{ij} = \delta$ for all $ij \in E(\overline{G})$. Applying the Pólya–Szegö inequality (Lemma 1 of [3]) with

$$a = \sqrt{2}\delta, \quad A = \sqrt{2}\Delta, \quad b = \delta, \quad B = \delta,$$

we obtain

$$\frac{\sum_{ij\in E(\overline{G})} a_{ij}^2 \sum_{ij\in E(\overline{G})} b_{ij}^2}{\left(\sum_{ij\in E(\overline{G})} a_{ij}b_{ij}\right)^2} \leq \frac{(ab+AB)^2}{4abAB}.$$

Now,

$$\sum_{ij\in E(\overline{G})} a_{ij}^2 = \sum_{ij\in E(\overline{G})} (d_i^2+d_j^2) = \overline{F}(G),$$

$$\sum_{ij\in E(\overline{G})} b_{ij}^2 = \overline{m}\,\delta^2,$$

and

$$\sum_{ij\in E(\overline{G})} a_{ij}b_{ij} = \delta \sum_{ij\in E(\overline{G})} \sqrt{d_i^2+d_j^2} = \delta\,\overline{SO}(G).$$

Using the correct computation,

$$\frac{(ab+AB)^2}{4abAB} = \frac{(\Delta+\delta)^2}{4\Delta\delta}.$$

Therefore,

$$\frac{\overline{F}(G)\cdot\overline{m}\,\delta^2}{\delta^2\,\overline{SO}(G)^2} \leq \frac{(\Delta+\delta)^2}{4\Delta\delta},$$

which simplifies to

$$\frac{\overline{m}\,\overline{F}(G)}{\overline{SO}(G)^2} \leq \frac{(\Delta+\delta)^2}{4\Delta\delta}.$$

Taking square roots yields

$$\sqrt{\overline{m}\,\overline{F}(G)} \leq \frac{\Delta+\delta}{2\sqrt{\Delta\delta}}\,\overline{SO}(G).$$

This proves (2.3). □

2.2.1. Remark on edge cases for Theorem 6

Inequality (2.3) is proved for $\delta > 0$. If $\overline{m} = 0$ (i.e., G is complete), then both sides are zero and the inequality holds. For the case $\delta = 0$, the inequality can be obtained as a limiting case ($\delta \to 0^+$), provided $\overline{SO}(G) > 0$; if $\overline{SO}(G) = 0$, then $\overline{F}(G) = 0$ and the result is trivial. Thus the theorem remains valid in these boundary cases.

2.2.2. Remark

We now show that the constant in our corrected inequality (2.3) is indeed smaller than the constant in the original inequality of [3], which means our result is stronger.

Let $\Delta \geq \delta > 0$. We claim that

$$\frac{\Delta+\delta}{2\sqrt{\Delta\delta}} \leq \frac{1}{2}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right). \tag{3}$$

To prove this, set $t = \sqrt{\frac{\Delta}{\delta}} \geq 1$. Then $\Delta = t^2\delta$. Substituting into the left-hand side of (3):

$$\frac{\Delta+\delta}{2\sqrt{\Delta\delta}} = \frac{t^2\delta+\delta}{2\sqrt{t^2\delta^2}} = \frac{\delta(t^2+1)}{2t\delta} = \frac{t^2+1}{2t}.$$

Similarly, the right-hand side of (3) becomes

$$\frac{1}{2}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right) = \frac{1}{2}\left(\frac{1}{t^2}+t^2\right) = \frac{t^4+1}{2t^2}.$$

Thus, it suffices to prove

$$\frac{t^2+1}{2t} \leq \frac{t^4+1}{2t^2}.$$

Multiplying both sides by $2t^2 > 0$, we obtain

$$t(t^2+1) \leq t^4+1,$$

or equivalently,

$$t^4 - t^3 - t + 1 \geq 0.$$

But

$$t^4 - t^3 - t + 1 = (t-1)^2(t^2+t+1).$$

Since $(t-1)^2 \geq 0$ and $t^2+t+1 > 0$ for all $t \geq 1$, the inequality holds for all $\Delta \geq \delta > 0$. Equality occurs if and only if $t = 1$, i.e., $\Delta = \delta$ (when G is regular).
Therefore,

$$\frac{\Delta+\delta}{2\sqrt{\Delta\delta}} \leq \frac{1}{2}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right),$$

with equality if and only if G is regular. Consequently, the corrected inequality (2.3) is sharper than the original claim in [3].

2.3. Improved bounds for Theorem 7

In this subsection, we improve the bounds given in Theorem 7 of [3]. The original theorem states that for any graph G,

$$\frac{2\delta\sqrt{\overline{m}\,\Delta\,\overline{M}_1(G)}}{\Delta+\delta} \leq \overline{SO}(G) \leq \sqrt{\overline{m}\,\Delta\,\overline{M}_1(G)}. \tag{2.4}$$

We show that both bounds can be sharpened. In particular, we replace the upper bound with a stronger one involving the forgotten coindex $\overline{F}(G)$.

Theorem 3. *For any graph G,*

$$\frac{\overline{M}_1(G)}{\sqrt{2}} \leq \overline{SO}(G) \leq \sqrt{\overline{m}\,\overline{F}(G)}. \tag{2.5}$$

Proof. For the upper bound, applying the Cauchy–Schwarz inequality directly gives

$$\overline{SO}(G)^2 = \left(\sum_{ij \in E(\overline{G})} 1 \cdot \sqrt{d_i^2 + d_j^2} \right)^2 \leq \overline{m} \sum_{ij \in E(\overline{G})} (d_i^2 + d_j^2) = \overline{m}\,\overline{F}(G).$$

Thus,

$$\overline{SO}(G) \leq \sqrt{\overline{m}\,\overline{F}(G)}.$$

For the lower bound, using $\sqrt{x^2+y^2} \geq \frac{x+y}{\sqrt{2}}$ for all $x, y \geq 0$, we obtain

$$\overline{SO}(G) = \sum_{ij \notin E} \sqrt{d_i^2 + d_j^2} \geq \frac{1}{\sqrt{2}} \sum_{ij \notin E} (d_i + d_j) = \frac{\overline{M}_1(G)}{\sqrt{2}}.$$

This completes the proof.
Equality in the lower bound $\overline{SO}(G) \geq \overline{M}_1(G)/\sqrt{2}$ occurs if and only if every non-edge ij has $d_i = d_j$. Equality in the upper bound occurs if and only if all non-edges have the same value of $\sqrt{d_i^2 + d_j^2}$, i.e., $\overline{SO}(G)^2 = \overline{m}\,\overline{F}(G)$. □

2.3.1. Remark

Since $d_i, d_j \leq \Delta$ for all vertices, we have $d_i(\Delta - d_i) \geq 0$ and $d_j(\Delta - d_j) \geq 0$, which implies

$$d_i^2 \leq \Delta d_i, \qquad d_j^2 \leq \Delta d_j.$$

Adding and summing over all $ij \in E(\overline{G})$ yields

$$\overline{F}(G) = \sum_{ij \in E(\overline{G})} (d_i^2 + d_j^2) \leq \Delta \sum_{ij \in E(\overline{G})} (d_i + d_j) = \Delta\,\overline{M}_1(G).$$

Therefore, $\sqrt{\overline{m}\,\overline{F}(G)} \leq \sqrt{\overline{m}\,\Delta\,\overline{M}_1(G)}$, which shows that the new upper bound is sharper than the original one.
We now show that the new lower bound in (2.5) is stronger than the original lower

bound in (2.4). That is, we establish that for every graph G,

$$\frac{\overline{M}_1(G)}{\sqrt{2}} \geq \frac{2\delta\sqrt{\overline{m}\,\Delta\,\overline{M}_1(G)}}{\Delta+\delta}. \tag{8}$$

If $\overline{m}=0$ (i.e., G is complete) or $\delta=0$, both sides of (8) are zero, so the inequality holds trivially. Assume now $\overline{m}>0$ and $\delta>0$. The above inequality is equivalent to

$$\frac{\overline{M}_1(G)^2}{2} \geq \frac{4\delta^2\,\overline{m}\,\Delta\,\overline{M}_1(G)}{(\Delta+\delta)^2},$$

that is, $\overline{M}_1(G) \geq \frac{8\delta^2\,\overline{m}\,\Delta}{(\Delta+\delta)^2}$. Now, using the known bound $\overline{M}_1(G) \geq 2\overline{m}\,\delta$, it is enough to prove that

$$2\overline{m}\,\delta \geq \frac{8\delta^2\,\overline{m}\,\Delta}{(\Delta+\delta)^2}.$$

After dividing both sides by $2\overline{m}\delta>0$, this reduces to $(\Delta+\delta)^2 \geq 4\Delta\delta$. This is an immediate consequence of the elementary identity

$$(\Delta+\delta)^2-4\Delta\delta=(\Delta-\delta)^2 \geq 0,$$

which holds for all real numbers Δ and δ. Therefore, (8) holds for every graph G.

2.3.2. Equality cases for the improved bounds

Equality in the lower bound $\overline{SO}(G) \geq \overline{M}_1(G)/\sqrt{2}$ occurs if and only if every non-edge ij has $d_i=d_j$. Equality in the upper bound occurs if and only if all non-edges have the same value of $\sqrt{d_i^2+d_j^2}$, i.e., $\overline{SO}(G)^2=\overline{m}\,\overline{F}(G)$.

2.4. Corollary 5

In [3], Corollary 5, the authors claimed that for the closed fence $C_n[K_2]$,

$$\overline{SO}(C_n[K_2]) = 5\big(2n(n-3)+4\big)\sqrt{2}. \tag{2.6}$$

This claim is incorrect. Take $n=3$. Since $C_3=K_3$, the composition $C_3[K_2]=K_3[K_2]$ is the complete graph K_6 (each vertex of K_3 is replaced by a K_2 and all edges between different copies are present). Hence $\overline{SO}(K_6)=0$. But formula (2.6) gives

$$5\big(2\cdot 3(0)+4\big)\sqrt{2}=20\sqrt{2}\neq 0.$$

So (2.6) is false.

Theorem 4 (Corrected Corollary 5). *For $n \geq 3$,*

$$\overline{SO}(C_n[K_2]) = 10n(n-3)\sqrt{2}. \tag{2.7}$$

Proof. The graph $C_n[K_2]$ is 5-regular on $2n$ vertices. Indeed, for any vertex (u, v) with $u \in V(C_n)$, $v \in V(K_2)$, its degree is $2 \cdot d_{C_n}(u) + d_{K_2}(v) = 2 \cdot 2 + 1 = 5$. By Corollary 1 of [3], for an r-regular graph on N vertices,

$$\overline{SO}(G) = \frac{Nr(N-1-r)}{\sqrt{2}}.$$

Thus, with $N = 2n$, $r = 5$,

$$\overline{SO}(C_n[K_2]) = \frac{2n \cdot 5 \cdot (2n-1-5)}{\sqrt{2}} = \frac{10n(2n-6)}{\sqrt{2}} = 10n(n-3)\sqrt{2}.$$

For $n = 3$, this gives 0, as expected. □

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