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2-distance injective coloring of graphs

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Abstract: For a given graph $G = (V(G), E(G))$, a 2-distance coloring of G is a proper vertex coloring of the graph G such that any pair of vertices at a distance of at most 2 receive different colors. A vertex coloring of a graph G is called an injective coloring if any two vertices u and v with a common neighbor, receive different colors.

A 2-distance injective coloring of a graph G is a vertex coloring in which any two vertices that share a common neighbor are assigned different colors, and any two vertices that share a common 2-distance neighbor are also assigned different colors.

For any vertex $u \in V(G)$, the 2-distance neighbors of u are denoted by $N_G^2(u) = \{v \in V(G) : d_G(u,v) = 2\}$, and the 2-distance degree of u is given by $deg_G^2(u) = |N_G^2(u)|$. Furthermore, let $\Delta(G)$ and $\Delta^{(2)}(G)$ represent the maximum degree and maximum 2-distance degree of G, respectively.

In this paper, we initiate the study of the 2-distance injective coloring of a graph. We demonstrate that, for any graph G,

$$\Delta^{(2)} + 1 \leq \chi_{2i}(G) \leq \Delta(\Delta - 1)[1 + (\Delta - 1)^2] + 1$$

with sharp bounds, where $\chi_{2i}(G)$ represents the 2-distance injective chromatic number of G. In particular, the trees T and the grid graphs $P_n \square P_m$ achieve the lower bound.

Additionally, we explore the relationship between 2-distance injective coloring and 2-distance coloring of a graph, and determine $\chi_{2i}(G)$ for several well-known graphs G.

Keywords: graph coloring, 2-distance coloring, injective coloring, 2-distance injective coloring.

AMS Subject Classification: 05C15

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1. Introduction and preliminaries

Graph coloring is one of the most popular areas of research. Many applications can be found in the literature, such as resource allocation for various businesses, scheduling in libraries, computer processes, daily life tasks, manufacturing gadgets, printed circuit analysis, and timetabling or work scheduling, satellite area scheduling, and frequency transfer. These are just some of the many applications that exist today, with many more anticipated in the future.

For standard terminology and notation in graph theory, we use [4, 17] as references. Let $G = (V, E)$, where the set of vertices is $V = V(G)$ and the set of edges is $E = E(G)$, be a simple, finite, undirected graph. The set $N(v) = \{u \mid uv \in E\}$ is the open neighborhood of a vertex $v \in V$, and the set $N[v] = N(v) \cup \{v\}$ is its closed neighborhood. The degree of v, denoted by $\deg(v)$, is the cardinality of $|N(v)|$. $\Delta(G)$, $\delta(G)$ and $\omega(G)$ denote the maximum degree, the minimum degree, and clique number of G, respectively. Any vertex of degree 1 and its adjacent vertex are referred to as a pendant (or leaf) and a support vertex, respectively. An isolated vertex and a full vertex are vertices of degree 0 and degree $n-1$, respectively, where $n = |V(G)|$. For two vertices u and v, the distance $d_G(u, v)$ is the length of a shortest path between u and v.

A proper vertex coloring of a graph G is a labeling $f : V \to \{1, 2, 3, \ldots, k\}$ (where the labels are colors) such that adjacent vertices receive different labels. Graph coloring problems have attracted significant attention among researchers since their introduction in the 19th century. The chromatic number of a given graph G, denoted by $\chi(G)$, is the minimum number of colors required for a proper coloring of G.

Let $f : V \to \{1, 2, 3, \ldots, k\}$ be a function such that for any pair of vertices $u, v \in V(G)$ with $d(u, v) \leq 2$, we have $f(u) \neq f(v)$. Then f is called a 2-*distance* k-*coloring* of a graph G. This concept was introduced for the first time in 1969 by Kramer and Kramer [8], either as a 2-distance coloring of a graph G or as the usual proper coloring of G^2 (see also their survey on general distance coloring in [9]). Some applications of this concept can be found in [7].

The smallest integer $k \geq 1$ for which G can be 2-distance colored with k colors is called the 2-distance chromatic number of G, denoted by $\chi_2(G) = \chi(G^2)$, where G^2 is the square of the graph G.

For a graph G, assume that $f : V(G) \to \{1, 2, 3, \ldots, k\}$ is a function such that for any two vertices u, v with a common neighbor, we have $f(u) \neq f(v)$. Then f is called an injective coloring. In other words, if we observe a path $P_3 = xyz$ in the graph, then $f(x) \neq f(z)$. The smallest integer $k \geq 1$ for which G admits an injective k-coloring is called the injective chromatic number $\chi_i(G)$ of G.

The injective coloring of graphs originated in Complexity Theory on Random Access Machines and has been applied to the theory of error codes and the design of computer networks [1]. One of the important applications of injective coloring of hypercubes lies in the theory of error correction codes. It should be noted that this coloring is not necessarily a proper coloring.

This concept was first introduced in 2002 by Hahn et al. [6] and has been further

studied in [2, 3, 5, 11, 12, 14, 15]. Injective coloring of a graph G is related to the usual coloring of the square G^2. It is evident that if G has a 2-distance coloring, then G also admits an injective coloring. Thus, the inequality $\chi_i(G) \leq \chi_2(G)$ holds. Since, in the injective coloring of a graph G, all vertices with a common neighbor receive different colors, it follows that $\Delta(G) \leq \chi_i(G) \leq |V(G)|$.
For any vertex $u \in V(G)$, the 2-*distance neighbors of* u, denoted by $N_G^2(u)$, are defined as $N_G^2(u) = \{v \in V(G) : d_G(u, v) = 2\}$ and the 2-distance degree of u is given by $deg_G^2(u) = |N_G^2(u)|$. The maximum and minimum 2-distance degrees of G are denoted by $\Delta^{(2)}(G)$ and $\delta^{(2)}(G)$, respectively.

Remark 1. It should be noted that, if necessary, the number 2 in all the symbols above can be generalized to $k \geq 2$.

For the *Cartesian product* of two simple graphs G and H, denoted by $G\Box H$, the vertex set is $V(G) \times V(H)$, and the edge set is $E(G\Box H) = \{(g_1, h_1)(g_2, h_2) : g_1g_2 \in E(G) \textit{ and } h_1 = h_2, \textit{ or } g_1 = g_2 \textit{ and } h_1h_2 \in E(H)\}$.
The Cartesian product produces many important graphs, such as ladder graphs, grid graphs, toroidal graphs, and cylindrical graphs.
Now, we generalize the notion of injective coloring and 2-distance coloring as 2-*distance injective coloring* of a graph. The motivation behind this generalization is to study how it behaves in relation to injective graph coloring, usual graph coloring, and 2-distance graph coloring. In particular, we aim to examine the following conjectures posed on injective coloring [10] and 2-distance coloring [16] from the perspective of 2-distance injective colorings. Although we do not delve into these conjectures here, we believe that the concept of 2-distance injective coloring could contribute to discussions surrounding these conjectures.

Conjecture 1. [10] Let G be a planar graph with maximum degree Δ.
Then $\chi_i(G)$ satisfies the following bounds:

(a) $\chi_i(G) \leq 5$, if $\Delta = 3$, (b) $\chi_i(G) \leq \Delta + 5$, if $4 \leq \Delta \leq 7$, (c) $\chi_i(G) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$, otherwise.

Conjecture 2. [16] Let G be a planar graph with maximum degree Δ.
Then $\chi_2(G)$ satisfies the following bounds:

(a) $\chi_2(G) \leq 7$, if $\Delta = 3$, (b) $\chi_2(G) \leq \Delta + 5$, if $4 \leq \Delta \leq 7$, (c) $\chi_2(G) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$, otherwise.

Also, since the notion of 2-distance injective coloring is a natural generalization of injective coloring, it is reasonable to predict that it has applications in various fields in the real world and would also prove useful in coding theory, much like injective coloring. At present, we formally define this concept.

Definition 1. A 2-*distance injective coloring* using k colors of a graph G is a function $f : V(G) \to \{1, 2, 3, \ldots, k\}$ such that any pair of vertices with a common neighbor receive different colors, and any two vertices with a common 2-distance neighbor also receive different colors. In other words If $N(u) \cap N(v) \neq \emptyset$ or $N_G^2(u) \cap N_G^2(v) \neq \emptyset$ then $f(u) \neq f(v)$. Or for a path $P_3 = ywz$ or a path $P_5 = xywzt$ in the graph G, $f(y) \neq f(z)$ and $f(x)$, $f(t)$, $f(w)$ are pairwise different.

For a 2-distance injective k-coloring function f, the set of color classes $\{v \in V(G) \mid f(v) = i,\ 1 \leq i \leq k\}$ forms a 2-distance injective k-coloring class of G with color i. The smallest k such that G has a 2-distance injective k-coloring (or simply a 2-distance injective coloring if k is clear from the context) is called the 2-distance injective chromatic number, $\chi_{2i}(G)$, of G. It is worth noting that this type of coloring is not necessarily a proper coloring (see Figure 1).

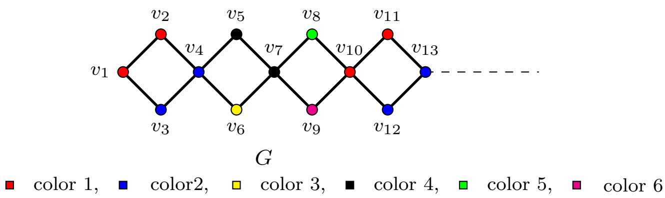

Figure 1. **An example for non proper 2-distance injective coloring of a graph G**

An alternate way of understanding the 2-distance injective chromatic number of a graph G is through the concept of the 2-distance common neighbor graph $G^{(2i)}$.

Definition 2. For a given graph G, a 2-distance common neighbor graph $G^{(2i)}$ has the vertex set $V(G^{(2i)}) = V(G)$, and two distinct vertices are adjacent in $G^{(2i)}$ if and only if they have a common neighbor or a common 2-distance neighbor in G.

Considering that a vertex subset S is independent in $G^{(2i)}$ if and only if no two vertices in S share a common neighbor or a common 2-distance neighbor in G, it follows that $\chi_{2i}(G) = \chi(G^{(2i)})$. The following example illustrates this concept.
It is easy to see that if f is a χ_{2i}-chromatic function of G, then $f(w) = 1 = f(y)$, $f(c) = 2 = f(d)$, $f(x) = 3 = f(u)$, $f(v) = 4 = f(z)$, and $f(b) = 5$. This demonstrates that $\chi(G^{(2i)}) = 5$.

Hahn et al. showed that all neighbors of a common vertex must receive distinct colors, which implies $\chi_i(G) \geq \Delta(G)$. A natural candidate for the 2-distance injective coloring of graphs is the maximum 2-distance degree of G, $\Delta^{(2)}(G)$. It is straightforward to observe that $\chi_{2i}(G) \geq \Delta^{(2)}(G)$.

This paper is organized as follows: In Section 2, we determine the sharp lower and upper bounds for the 2-distance injective chromatic number of any graph and show that any tree T achieves the lower bound. Additionally, we compare 2-distance injective coloring with 2-distance coloring in Section 3. In Section 4, we investigate the 2-distance injective coloring of well-known graphs. Notably, in Section 5, we prove

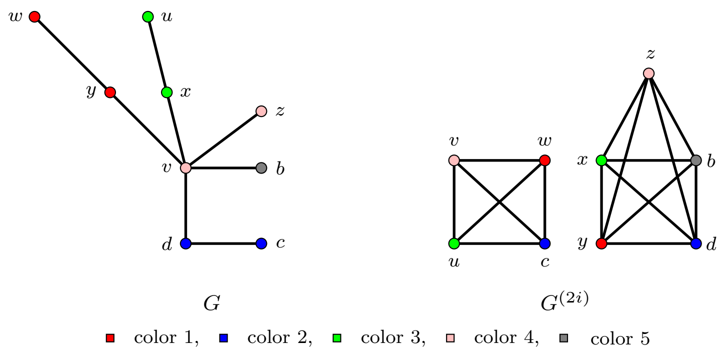

Figure 2. **An example for 2-distance common neighbor graph $G^{(2i)}$ of G**

that grid graphs $P_n \square P_m$ also achieve the lower bound. Finally, we outline future directions for this concept.

2. 2-distance injective: Basic results

In this section, we establish sharp upper and lower bounds for the 2-distance injective chromatic number of arbitrary graphs and determine its precise value for a tree T. To this end, we require Brooks' theorem, as stated below.

Proposition 1. ([17], Proposition 5.1.7, 5.1.13) *If G is a simple connected graph, then $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.*

This upper bound is optimal for cliques and odd cycles, so $\chi(G) = \Delta(G) + 1$.

Brooks' Theorem

Theorem 1. ([17], Theorem 5.1.22) *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Theorem 2. *If G is a simple graph with maximum degree Δ and maximum 2-distance degree $\Delta^{(2)}$, then:*

$$\Delta^{(2)} + 1 \leq \chi_{2i}(G) \leq \Delta(\Delta - 1)[1 + (\Delta - 1)^2] + 1.$$

These bounds are sharp.

Proof. Let G be an arbitrary graph, and let v_0 be a vertex with the maximum 2-distance degree in G. The vertex v_0, along with the vertices at 2-distance from it,

forms a clique of size $\Delta^{(2)}+1$ in $G^{(2i)}$. Therefore, $\chi_{2i}(G)=\chi(G^{(2i)})\geq\Delta^{(2)}+1$. The lower bound is sharp for trees T (Theorem 3) and grid graphs (Theorem 7).
For the upper bound, consider any vertex $v\in V(G)=V(G^{(2i)})$. There are at most $\Delta(\Delta-1)$ vertices in G that are at distance 2 from v. Each such vertex u at distance 2 from v can have at most $\Delta(\Delta-1)$ vertices at distance 2 from itself. All vertices in these two categories are adjacent to v in $G^{(2i)}$. Thus:

$$\deg_{G^{(2i)}}(v)\leq\Delta(\Delta-1)+(\Delta(\Delta-1)((\Delta-1)(\Delta-1))=\Delta(\Delta-1)(1+(\Delta-1)^2).$$

This implies that the maximum degree of any vertex in $G^{(2i)}$ is at most $\Delta(\Delta-1)(1+(\Delta-1)^2)$. By Definition 2 and Proposition 1 and Brooks' theorem:

$$\chi_{2i}(G)=\chi(G^{(2i)})\leq\Delta(\Delta-1)(1+(\Delta-1)^2)+1.$$

This bound is sharp for cycles C_5 and C_{10}, where the graph $C_5^{(2i)}$ and $C_{10}^{(2i)}$ contain the component K_5. (Proposition 4, (2)). $\square$

Theorem 3. *For any tree T of order at least three,*

$$\chi_{2i}(T)=\Delta^{(2)}+1.$$

Proof. From Theorem 2, we know $\chi_{2i}(T)\geq\Delta^{(2)}(T)+1$.
Now, we show that $\chi_{2i}(T)=\Delta^{(2)}(T)+1$. Since T is a tree, any vertex v can form a clique with the vertices at 2-distance, resulting in a clique of size $\deg^2(v)+1$. Thus, $\Delta(T^{(2i)})=\Delta^2(T)$. By Brooks' theorem: $\chi_{2i}(T)=\chi(T^{(2i)})\leq\Delta(T^{(2i)})+1=\Delta^2(T)+1$. Therefore

$$\chi_{2i}(T)=\Delta^{(2)}(T)+1.$$

$\square$

3. 2-distance injective coloring versus coloring and 2-distance coloring

In this section, we examine the relationship between the 2-distance injective coloring of graphs and the concepts of graph coloring and 2-distance coloring.

Lemma 1. ([6]) *Let G be a graph with diameter 2 and independence number α. Then $\alpha\leq\chi_i(G)$.*

From Lemma 1, we deduce the following corollary:

Corollary 1. *If G is a graph with diameter 2 and independence number α, then $\alpha\leq\chi_i(G)\leq\chi_{2i}(G)$.*

Lemma 2. ([6]) *For a k-regular graph G with $\chi_i(G) = k$, the value of k divides $n = |V(G)|$.*

As an immediate consequence, we have:

Corollary 2. *For any k-regular graph G with $\chi_{2i}(G) = k$, it follows that $k \mid n = |V(G)|$.*

Proof. Since G is k-regular, we know $\chi_i(G) \geq k$. On the other hand, $\chi_i(G) \leq \chi_{2i}(G) = k$. Thus, $\chi_i(G) = k$, and the result follows from Lemma 2. □

Lemma 3. ([6]) *Let G be a connected graph distinct from K_2. Then $\chi(G) \leq \chi_i(G)$.*

From Lemma 3, we derive the following corollary:

Corollary 3. $\chi(G) \leq \chi_i(G) \leq \chi_{2i}(G)$.

The inequalities in Corollary 3 may or may not hold as equalities. Consider the following examples:
For a complete graph K_n ($n \neq 2$), we have $\chi(G) = \chi_i(G) = \chi_{2i}(G) = n$.
For an odd cycle graph C_n, where $n \not\equiv 0 \pmod 3$, we have $\chi(C_n) = \chi_i(C_n) = 3 < 4 \leq \chi_{2i}(C_n)$.
If $G = K_4 - e$ (a complete graph on four vertices with one edge removed), then $\chi(G) = 3 < 4 = \chi_i(G) = \chi_{2i}(G)$.
For a spider graph G with $2n+1$ vertices, obtained by subdividing one edge of a star S_{n+1} ($n \geq 3$), we have $\chi(G) = 2 < n = \chi_i(G) < n+1 = \chi_{2i}(G)$.

Proposition 2. *Let G be a graph where any two adjacent vertices are included in either a triangle or the endpoints of a path P_5. Then $\chi(G) \leq \chi_{2i}(G)$.*
Conversely, if the endpoints of every path P_3 or P_5 in G are adjacent, then $\chi_{2i}(G) \leq \chi(G)$.

Proof. Since any two adjacent vertices are contained in either a triangle or are the endpoints of a path P_5, these two vertices must be assigned different colors in any 2-distance injective coloring. Therefore, any 2-distance injective coloring can be considered as a proper coloring of G, implying $\chi(G) \leq \chi_{2i}(G)$.
Conversely, given the construction of G, any proper coloring ensures that the endpoints of every path P_3 or P_5 are assigned different colors. Thus, the usual proper coloring of G serves as a 2-distance injective coloring, implying $\chi_{2i}(G) \leq \chi(G)$. □

Proposition 3. *Let G be a graph where any three vertices on a path P_3 form a triangle, or any two of them are the endpoints of a path P_5. Then $\chi_2(G) \leq \chi_{2i}(G)$.*
Conversely, if the endpoints of every path P_5 in G are adjacent or share a common neighbor, then $\chi_{2i}(G) \leq \chi_2(G)$.

Proof. If any three vertices of P_3 form a triangle or any two of them are endpoints of a path P_5, then these three vertices must be assigned distinct colors in any 2-distance injective coloring. Thus, any 2-distance injective coloring serves as a 2-distance coloring, implying $\chi_2(G) \leq \chi_{2i}(G)$.
Conversely, given the construction of G, any 2-distance coloring ensures that adjacent vertices and any three vertices on a path P_3 receive distinct colors. By assumption, this 2-distance coloring also serves as a 2-distance injective coloring, implying $\chi_{2i}(G) \leq \chi_2(G)$. □

If G is a graph of order n and diameter at most 2, then $G^2 = K_n$. Thus, $\chi_2(G) = n$, and $\chi_2(G) \geq \chi_{2i}(G)$. This yields the following corollary:

Corollary 4. *For graphs with diameter at most* 2*, we have:* $\chi(G^2) = \chi_2(G) \geq \chi_{2i}(G) = \chi(G^{2i})$.

However, the inequality in Corollary 4 may be strict. For example, $\chi_{2i}(P_3) = 2 < 3 = \chi(G^2)$. Another example is a graph G formed from the cycle $C_4 = xyzu$ by adding a vertex v adjacent to both x and z. Here, $\chi_{2i}(G) = 3 < 5 = \chi(G^2)$.
Finally, we state a result establishing conditions under which $\chi_2(G)$ and $\chi_{2i}(G)$ are equal.

Theorem 4. *Suppose G is a graph of order n and diameter* 2*. Then $\chi_2(G) = \chi_{2i}(G)$ if and only if every edge of G lies on a cycle C_3 or C_5.*

Proof. From Corollary 4, $\chi_2(G) = n$.
If $n = \chi_2(G) = \chi_{2i}(G)$, then from Definition 2, $G^{(2i)} = K_n$. Hence, any two vertices of G are either adjacent or share a common neighbor, ensuring that each edge of G lies on a C_3 or C_5.
Conversely, if each edge of G lies on a C_3 or C_5, then any two vertices of G are either adjacent or share a common neighbor. Consequently, x and y in G are adjacent in $G^{(2i)}$, establishing $\chi_2(G) = \chi_{2i}(G)$. □

For graphs G with diameter at least three, we cannot directly compare $\chi_{2i}(G)$ and $\chi_2(G)$. In other words, we may find families of graphs G with diameter at least 3 for which $\chi_2(G) > \chi_{2i}(G)$, families of graphs for which $\chi_{2i}(G) > \chi_2(G)$ and families of graphs for which $\chi_{2i}(G) = \chi_2(G)$. Therefore, these two concepts ($\chi_{2i}(G)$ and $\chi_2(G)$) are generally not comparable. In line with the incomparability of these two concepts, the following results can also be considered: $X_{2i}(T)$ versus $X_2(T)$.

Lemma 4. ([13] Proposition 10) *For any tree T, $\chi_2(T) = \Delta(T) + 1$.*

Example 1. *Consider the trees in Figure 3, which have a diameter of 4. It is clear that, for the family of trees in Figure 3, $\chi_{2i}(T) = \Delta^{(2)}(T) + 1$. Now, from Lemma 4, we have:*

For the family of trees T in (I), $\Delta(T) > \Delta^{(2)}(T)$, so we have $\chi_2(T) > \chi_{2i}(T)$.

For the family of trees T in (II), $\Delta(T) = \Delta^{(2)}(T)$, so we have $\chi_2(T) = \chi_{2i}(T)$.

For the family of trees T in (III), $\Delta(T) < \Delta^{(2)}(T)$, so we have $\chi_2(T) < \chi_{2i}(T)$.

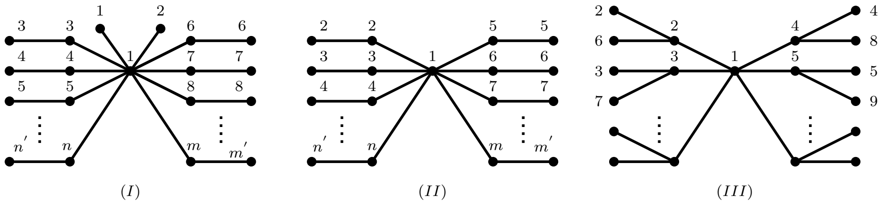

Figure 3. The 2-distance injective coloring of graphs G related to Example 1 for diameter 4

Assume that G is graph, obtained from a cycle C_{2n} $(n \geq 3)$ with additional edges that make the vertices with even indices adjacent to each other. Then the graph G is called an n-sun graph. Any sun graph is simply an n-sun graph for some $n \geq 3$. A graph with $2n$ vertices, obtained from cycle C_n by attaching one leaf to each vertex of C_n, is called an n-sunlet graph.

From the definitions of n-sun graph and n-sunlet graph, it is straightforward to see that.

Example 2. *1. For the sun graph G with diameter 3, we have $\chi_2(G) < \chi_{2i}(G)$.*
2. For the sunlet graph G with diameter at least 3, $\chi_2(G) \leq \chi_{2i}(G)$.

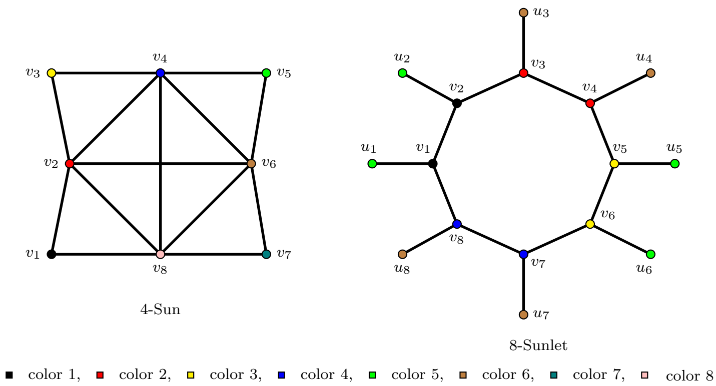

Figure 4. The 2-distance injective coloring of sun graph and sunlet graph

A double star $S_{m,n}$ is a graph with two support vertices and $m + n$ leaves, where m leaves are adjacent to one support vertex and n leaves are adjacent to the other support vertex.

Example 3. *For the double star $S_{m,n}$, we have:*
$\chi_2(G) = 2 + max\{m, n\} > 1 + max\{m, n\} = \chi_{2i}(G)$.

4. 2-distance injective coloring of specified graphs

In this section, we study the 2-distance injective coloring of some familiar graphs.

Proposition 4. *For the path P_n and the cycle C_n, we have*

$$(1)\ \chi_{2i}(P_n) = \begin{cases} 1 & n = 2 \\ 2 & n = 3, 4 \\ 3 & n \geq 5 \end{cases}.$$

And for a positive integer k,

$$(2)\ \chi_{2i}(C_n) = \begin{cases} 3 & n = 3k \\ 2 & n = 4 \\ 4 & n \in \{6k+1, 6k+2, 6k+4, 6k+5\},\ n \neq 10 \\ 5 & n \in \{5, 10\} \end{cases}.$$

Proof. To determine the 2-distance injective chromatic number of the path P_n and the cycle C_n, we proceed as follows:

(1) **For path** P_n, let $n = 2$. It is obvious that $\chi_{2i}(P_2) = 1$. Let $n = 3, 4$. We assign color 1 to the vertices v_1, v_2 and color 2 to the vertices v_3, v_4. Since every two vertices with a common neighbor receive distinct colors, this assignment is a 2-distance injective coloring of P_n for $n = 3, 4$. Let $n \geq 5$. We color the vertices of P_n following the sequence $1, 1, 2, 2, 3, 3, 1, 1, \cdots$ (see Figure 5).

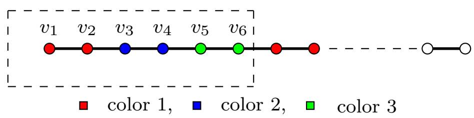

Figure 5. **The 2-distance injective coloring of P_n**

Then, it is easy to see that this coloring is a 2-distance injective coloring of the path P_n.

(2) **For a cycle** C_n, let $V(C_n) = \{v_i : 1 \leq i \leq n\}$. For the proof, there are four cases to consider for the 2-distance injective coloring of C_n.

• **For** C_{3k}: Let $k = 1$. It is obvious that $\chi_{2i}(C_3) = 3$.
Let $k \geq 2$. From Theorem 2, $\chi_{2i}(C_{3k}) \geq 3$. Now, we show a 2-distance injective coloring of C_{3k} with 3 colors. We assign colors 1, 2, and 3 to the vertices v_j, v_k, and v_l, respectively, where $j \equiv 1$, $k \equiv 2$, and $l \equiv 3 \pmod 3$.

With this coloring, every three successive vertices are assigned colors a, b, c or b, c, a or c, a, b and every five successive vertices are assigned colors a, b, c, a, b or b, c, a, b, c or c, a, b, c, a. Therefore, from Definition 1, this coloring is a 2-distance injective coloring of C_{3k}. Thus $\chi_{2i}(C_{3k}) = 3$.

• **For** C_4: The vertices v_1, v_2, v_3, v_4 can be colored with colors $1, 1, 2, 2$ respectively. This is a 2-distance injective coloring of C_4.

• **For** C_{6k+1} **and** C_{6k+2}: We prove that, for 2-distance injective coloring of C_{6k+1} and C_{6k+2}, at least 4 colors are needed.

Let f be a 2-distance injective coloring function of the cycle C_{6k+1}. For f on v_i, we consider the following subcases.

(i) Let $f(v_1) = 1$, $f(v_2) = 2$ and $f(v_3) = 3$. Then,

$$f(v_4) = 1 \text{ or } f(v_4) = 3.$$

If $f(v_4) = 1$, then the vertices v_i, v_j and v_l with $i \equiv 1$ $(i \neq 6k + 1)$, $j \equiv 2$ and $l \equiv 3$ $(mod\ 3)$ must be assigned $f(v_i) = 1$, $f(v_j) = 2$ and $f(v_l) = 3$ respectively. Since $f(v_{6k-2}) = 1 = f(v_1)$, the resulting coloring is not a 2-distance injective coloring of C_{6k+1}.

If $f(v_4) = 3$, then the vertices v_i, v_j and v_l with $i \equiv 0$ *and* 1 $(i \neq 6k+1)$, $l \equiv 3$ *and* 4 $(mod\ 6)$ and $j \equiv 2$ $(mod\ 3)$ must be assigned $f(v_i) = 1$, $f(v_l) = 3$ and $f(v_j) = 2$ respectively. Since $f(v_{6k}) = 1$ and $f(v_1) = 1$, the resulting coloring is not a 2-distance injective coloring of C_{6k+1}.

(ii) Let $f(v_1) = 1$, $f(v_2) = 2 = f(v_3)$. Then,

$$f(v_4) = 1 \text{ or } f(v_4) = 3.$$

If $f(v_4) = 1$, then the vertices v_i, v_j and v_l with $i \equiv 1$ $(mod\ 3)$, $(i \neq 6k + 1)$, $j \equiv 2$ *and* 3 and $l \equiv 5$ *and* 6 $(mod\ 6)$ should be assigned $f(v_i) = 1$, $f(v_j) = 2$ and $f(v_l) = 3$ respectively. Since $f(v_{6k-2}) = 1 = f(v_1)$, this coloring is not a 2-distance injective coloring of C_{6k+1}.

If $f(v_4) = 3$, then the vertices v_i, v_j and v_l with $i \equiv 1$ *and* 6, $(i \neq 6k+1)$, $j \equiv 2$ *and* 3 and $l \equiv 4$ *and* 5 $(mod\ 6)$ should be assigned $f(v_i) = 1$, $f(v_j) = 2$ and $f(v_l) = 3$ respectively. Since $f(v_{6k}) = 1 = f(v_1)$, this coloring is not a 2-distance injective coloring of C_{6k+1}.

(iii) Let $f(v_1) = 1 = f(v_2)$, $f(v_3) = 2$. Then,

$$f(v_4) = 2 \text{ or } f(v_4) = 3.$$

If $f(v_4) = 2$, then the vertices v_i, v_j and v_l with $i \equiv 1$ *and* 2 $(i \neq 6k+1)$, $j \equiv 3$ *and* 4 and $l \equiv 5$ *and* 6 $(mod\ 6)$ should be labeled by $f(v_i) = 1$, $f(v_j) = 2$ and $f(v_l) = 3$

respectively. By these assignments, the vertex v_{6k+1} cannot be assigned any color in $\{1, 2, 3\}$. Thus $f(v_{6k+1}) = 4$.

If $f(v_4) = 3$, then the vertices v_i, v_j and v_l with $i \equiv 1$ *and* 2, $(i \neq 6k + 1)$, and $l \equiv 4$ *and* 5, and $j \equiv 0$ *and* 3 $(mod\ 6)$ should be assigned $f(v_i) = 1$, $f(v_j) = 2$ and $f(v_l) = 3$ respectively. Since $f(v_{6k}) = 2 = f(v_3)$, this coloring is not a 2-distance injective coloring of C_{6k+1}.

From parts (i), (ii), and (iii), we deduce that $\chi_{2i}(C_{6k+1}) \geq 4$ and since we can find a 2-distance injective coloring for C_{6k+1} with 4 colors, we conclude that $\chi_{2i}(C_{6k+1}) = 4$.

Let f be a 2-distance injective coloring function for the cycle C_{6k+2}. In terms of f on v_i, we consider the subcases for C_{6k+1}.

Corresponding to subcase (i), if $f(v_4) = 1$, then

$$f(v_{6k-2}) = 1, f(v_{6k-1}) = 2 \text{ and } f(v_{6k}) = 3.$$

Since $f(v_1) = 1 = f(v_4)$, $f(v_2) = 2$ and $f(v_3) = 3$, so $f(v_{6k+1})$ and $f(v_{6k+2})$ cannot be assigned the colors $1, 2, 3$. Therefore, $f(v_{6k+1}) = 4 = f(v_{6k+2})$.

If $f(v_4) = 3$, then $f(v_{6k-3}) = 3 = f(v_{6k-2})$, $f(v_{6k-1}) = 2$, and $f(v_{6k}) = 1$. Since $f(v_1) = 1$, $f(v_2) = 2$, and $f(v_3) = 3 = f(v_4)$, so $f(v_{6k+1})$ and $f(v_{6k+2})$ cannot be assigned the colors $1, 2, 3$. Therefore, $f(v_{6k+1}) = 4 = f(v_{6k+2})$.

Corresponding to subcase (ii), if $f(v_4) = 1$, then

$$f(v_{6k-3}) = 2, f(v_{6k-2}) = 1, f(v_{6k-1}) = 3 \text{ and } f(v_{6k}) = 3.$$

Since $f(v_1) = 1$, $f(v_2) = 2 = f(v_3)$, and $f(v_4) = 1$, $f(v_{6k+1})$ and $f(v_{6k+2})$ cannot be assigned the colors $1, 2, 3$. Therefore, $f(v_{6k+1}) = 4 = f(v_{6k+2})$.

If $f(v_4) = 3$, then $f(v_{6k-3}) = 2$, $f(v_{6k-2}) = 3 = f(v_{6k-1})$, and $f(v_{6k}) = 1$. Since $f(v_1) = 1$, $f(v_2) = 2 = f(v_3)$ and $f(v_4) = 3$, the vertices v_{6k+1} and v_{6k+2} cannot be assigned colors $1, 2, 3$. Therefore, $f(v_{6k+1}) = 4 = f(v_{6k+2})$.

Corresponding to subcase (iii), if $f(v_4) = 2$, then

$$f(v_{6k-3}) = 2 = f(v_{6k-2}) \text{ and } f(v_{6k-1}) = 3 = f(v_{6k}).$$

Since $f(v_1) = 1 = f(v_2)$, $f(v_3) = 2 = f(v_4)$, $f(v_{6k+1})$ and $f(v_{6k+2})$ cannot be assigned colors $1, 2, 3$. Therefore, $f(v_{6k+1}) = 4 = f(v_{6k+2})$.

If $f(v_4) = 3$, then $f(v_{6k-3}) = 2$, $f(v_{6k-2}) = 3 = f(v_{6k-1})$, and $f(v_{6k}) = 2$. Since $f(v_1) = 1 = f(v_2)$ and $f(v_3) = 2$, $f(v_4) = 3$, the vertices $f(v_{6k+1})$ and $f(v_{6k+2})$ cannot be assigned colors $1, 2, 3$. Therefore, $f(v_{6k+1}) = 4 = f(v_{6k+2})$.

These analysis show that $\chi_{2i}(C_{6k+2}) = 4$.

• **For** C_{6k+4}. Let f be a 2-distance injective coloring function for the cycle C_{6k+4} with $k \geq 2$. In terms of f on v_i, we consider the subcases for C_{6k+1}.
From these suppositions, we deduce that $\chi_{2i}(C_{6k+4}) \geq 4$. Now, we give a 2-distance injective coloring of the cycle C_{6k+4} with 4 colors. Consider the following assignments:

$$f(v_i) = 1 \text{ for } i \equiv 1 \,(\text{mod } 3), \ (i \leq 6k-5 \text{ or } i = 6k),$$

$$f(v_j) = 2 \text{ for } j \equiv 2 \,(\text{mod } 3), \ (j \leq 6k-4 \text{ or } j = 6k+1),$$

$$f(v_l) = 3 \text{ for } l \equiv 3 \,(\text{mod } 3), \ (l \leq 6k-6 \text{ or } l = 6k-1, 6k+2),$$

and assign $f(v_{6k-3}) = f(v_{6k-2}) = f(v_{6k+3}) = f(v_{6k+4}) = 4$. Therefore, $\chi_{2i}(C_{6k+4}) = 4$ for $k \geq 2$.

• **For** C_{6k+5}. Let f be a 2-distance injective coloring function for the cycle C_{6k+5}. As in the case of the cycle C_{6k+4}, we have $\chi_{2i}(C_{6k+5}) \geq 4$. Now, we give a 2-distance injective coloring of the cycle C_{6k+5} with 4 colors. Consider the following assignments:

$$f(v_i) = 1 \text{ for } i \equiv 1 \,(\text{mod } 3), \ (i \leq 6k+1),$$

$$f(v_j) = 2 \text{ for } j \equiv 2 \,(\text{mod } 3), \ (j \leq 6k+2),$$

$$f(v_l) = 3 \text{ for } l \equiv 3 \,(\text{mod } 3), \ (l \leq 6k+3),$$

and assign $f(v_{6k+4}) = f(v_{6k+5}) = 4$. Therefore, $\chi_{2i}(C_{6k+5}) = 4$.

• **For** C_5 **and** C_{10}. For C_5, since no two vertices can be assigned the same color, we have $\chi_{2i}(C_5) = 5$.
For C_{10}, since with 4 colors, at least 2 vertices at distance 4 must take the same color. On the other hand, by assigning color 1 to v_1, v_4, color 2 to v_2, v_5, color 3 to v_3, v_6, color 4 to v_7, v_8, and color 5 to v_9, v_{10}, we show that $\chi_{2i}(C_{10}) = 5$.
Thus, the result holds. □

The following results are straightforward.

Observation 5. Let n be an integer. Then we have:

- For the complete graph K_n where $n \geq 3$, $\chi_{2i}(K_n) = n$.
- For the fan graph F_n where $n \geq 3$, $\chi_{2i}(F_n) = n+1$.
- For the wheel graph W_n where $n \geq 4$, $\chi_{2i}(W_n) = n+1$.
- For the star graph S_n where $n \geq 3$, $\chi_{2i}(S_n) = n$.

In a complete bipartite graph, any two vertices of a partite set have a common neighbor, and any two vertices of two different partite sets have no common neighbor or no common 2-distance neighbor. Thus, the 2-distance injective coloring is also an injective coloring. Therefore, we can state the following.

Proposition 5. *Let $K_{n,m}$ be the complete bipartite graph with m or $n \geq 2$. Then* $\chi_{2i}(K_{n,m}) = \chi_i(K_{n,m}) = \max\{n, m\}$.

Proof. Without loss of generality, let $n \geq 2$ and $m \geq 1$. It is observed that all vertices of one partite set must be assigned distinct colors. On the other hand, each vertex of the smaller partite set can be assigned one of the colors from the larger partite set such that all vertices of the smaller partite set are assigned different colors. Therefore, $\chi_{2i}(K_{n,m}) = \max\{n, m\} = \chi_i(K_{n,m})$. □

If G is a connected r-regular bipartite graph, then any two vertices of one partite set have even distance, and any two vertices of two different partite sets have odd distance. Using Proposition 5, we have the following result.

Corollary 5. *Let G be a k-regular bipartite graph ($k \geq 2$) with diameter at most 4, and further assume that each partite set is of order n. Then $\chi_{2i}(G) = n$.*

Proof. Since two vertices of two different partite sets have odd distance, they do not share a common neighbor or a common 2-distance neighbor. On the other hand, since the diameter of G is at most 4, any two vertices of one partite set share either a common neighbor or a common 2-distance neighbor. Using the same reasoning as in the proof of Proposition 5, we conclude that $\chi_{2i}(G) = n$. □

If G is a complete r-partite graph, then any two vertices share a common neighbor. Thus, we have the following observation.

Observation 6. For any complete ($r \geq 3$) partite graph of order n, $K_{n_1,\dots,n_r}$, we have $\chi_{2i}(K_{n_1,\dots,n_r}) = n = \chi_i(K_{n_1,\dots,n_r})$.

5. Grid graphs

We know very well that graph operations are important methods for constructing new graphs and play a key role in network design and analysis.

In this section, we discuss the 2-distance injective coloring of a family of Cartesian products of two paths (ladder and grid graphs) and determine their 2-distance injective chromatic number.
A ladder graph L_n is a connected special grid graph $P_2 \square P_n$ with $2n$ vertices and $3n-2$ edges, where $V(L_n) = \{v_{ij} : i = 1, 2 \textit{ and } 1 \leq j \leq n\}$.

Remark 2. If $n = 2$, then $L_n = C_4$. Let $n \geq 3$. For $n = 3$, the vertices v_{12}, v_{21}, v_{23} induce a clique K_3 in $L_3^{(2i)}$, since any two of them share a common neighbor. Therefore, $\chi(L_3^{(2i)}) \geq 3 = \Delta^{(2)}(L_3) + 1$. If $n = 4$, The vertices $v_{12}, v_{21}, v_{23}, v_{14}$ induce a clique K_4 in $L_4^{(2i)}$, hence $\chi_{2i}(L_4) = \chi(L_4^{(2i)}) \geq 4 = \Delta^{(2)}(L_4)+1$. If $n \geq 5$, the vertices $v_{12}, v_{21}, v_{23}, v_{14}, v_{25}$ induce a clique K_5 in $L_n^{(2i)}$, and each vertex has at most 4 vertices as 2-distance common neighbors. Hence, $\chi_{2i}(L_n) = \chi(L_n^{(2i)}) \geq 5 = \Delta^{(2)}(L_4) + 1$ for $n \geq 5$.

From Remark 2, we can state and prove the following.

Proposition 6. *Let $G \cong L_n$ be a ladder graph. Then*

$$\chi_{2i}(L_n) = \begin{cases} 3 = \Delta^{(2)}(L_3) + 1 & \text{if } n = 3 \\ 4 = \Delta^{(2)}(L_4) + 1 & \text{if } n = 4 \\ 5 = \Delta^{(2)}(L_n) + 1 & \text{if } n \geq 5 \end{cases}.$$

Proof. We show that the given graph L_n can be 2-distance injectively colored with $\Delta^{(2)}(L_n) + 1$ colors.

1. For $n = 3$: From Remark 2, $\chi_{2i}(L_3) \geq 3$. We assign color 1 to v_{i1}, color 2 to v_{i2}, and color 3 to v_{i3}. This assignment is a valid 2-distance injective coloring with 3 colors of L_3. Thus, $\chi_{2i}(L_3) = 3$.

2. For $n = 4$: From Remark 2, $\chi_{2i}(L_4) \geq 4$. We assign:

- color 1 to v_{11} and v_{21}, color 2 to v_{12} and v_{22},
- color 3 to v_{13} and v_{23}, color 4 to v_{14} and v_{24}.

This assignment is a valid 2-distance injective coloring of L_4. Therefore, $\chi_{2i}(L_4) = 4$.

3. For $n \geq 5$: From Remark 2, $\chi_{2i}(L_n) \geq 5$. We assign color i $(1 \leq i \leq 5)$ to the vertices v_{1m} and v_{2m} where $m \equiv i \pmod 5$. This assignment is a valid 2-distance injective coloring of L_n for $n \geq 5$. Therefore, $\chi_{2i}(L_n) = 5$.
In Figure 6, the 2-distance injective coloring of L_n is illustrated.

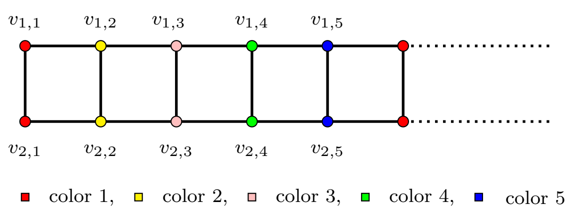

Figure 6. **2-distance injective coloring of L_n $(n \geq 5)$ with 5 colors**

$\square$

Our aim is to present the 2-distance injective chromatic number for grid graphs. Let $G_{m,n}$ represent the grid graph $P_m \Box P_n$ $(m,n > 2)$, with mn vertices and $2mn-m-n$ edges.
Let $V(P_k) = \{v_1, v_2, \ldots, v_k\}$ and $V(P_m \Box P_n) = V(G_{m,n}) = \{v_{ij} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. Here, v_{ij} denotes an arbitrary vertex of $G_{m,n}$, located in the i-th row and j-th column.

Remark 3. In the following, we aim to determine the maximum 2-distance injective chromatic number for $G_{m,n}$ with $m,n \geq 2$.

(1) $\mathbf{m,n \geq 5.}$ The set of vertices

$$N^2(v_{ij}) = \{v_{(i+2)j}, v_{(i-2)j}, v_{i(j+2)}, v_{i(j-2)}, v_{(i+1)(j+1)}, v_{(i+1)(j-1)}, v_{(i-1)(j+1)}, v_{(i-1)(j-1)}\}$$

with $3 \leq i \leq m-3$ and $3 \leq j \leq n-3$ is one of the largest sets in the 2-distance neighborhood of the vertex v_{ij}. Any pair of vertices in this set has a common neighbor or a common 2-distance neighbor, and the size of this set is $|N^2(v_{ij})| = \Delta^{(2)}(G_{m,n}) = 8$.
For $m,n \geq 5$, such a set exists. For instance, consider $N^2_{G_{m,n}}(v_{33})$. This set, together with v_{ij}, forms a maximum clique of size 9, i.e., a K_9 in the graph $G^{(2i)}_{m,n}$ for $m,n \geq 5$. Thus, $\chi_{2i}(G_{m,n}) \geq 9 = \Delta^{(2)}(G_{m,n}) + 1$.

(2) **Let $\mathbf{m = 4}$ *and* $\mathbf{n \geq 5}$**. Then, the set of vertices

$$N^2(v_{ij}) = \{v_{(i-2)j}, v_{i(j+2)}, v_{i(j-2)}, v_{(i+1)(j+1)}, v_{(i+1)(j-1)}, v_{(i-1)(j+1)}, v_{(i-1)(j-1)}\}$$

with $2 \leq i \leq 3$ and $3 \leq j \leq n-3$, is one of the largest sets in the 2-distance neighborhood of the vertex v_{ij}, where any pair of vertices in this set has either a common neighbor or a common 2-distance neighbor. The size of this set is $|N^2(v_{ij})| = \Delta^{(2)}(G_{4,n}) = 7$.
This set, together with v_{ij}, forms a maximum clique of size 8, i.e., a K_8 in the graph $G^{(2i)}_{4,n}$. Thus, $\chi_{2i}(G_{4,n}) \geq 8 = \Delta^{(2)}(G_{4,n}) + 1$.

(3) **Let $\mathbf{m = 3}$ *and* $\mathbf{n \geq 5}$**. Then, the set

$$N^2(v_{2j}) = \{v_{2(j-2)}, v_{2(j+2)}, v_{1(j-1)}, v_{1(j+1)}, v_{3(j-1)}, v_{3(j+1)}\}$$

with $3 \leq j \leq n-3$, is one of the largest sets in the 2-distance neighborhood of the vertex v_{2j}. Another vertex has at most 6 vertices in its 2-distance neighborhood, and $|N^2(v_{2j})| = \Delta^{(2)}(G_{m,n}) = 6$.
This set, together with v_{2j}, forms a maximum clique of size 7, i.e., a K_7 in the graph $G^{(2i)}_{3,n}$. Thus, $\chi_{2i}(G_{3,n}) \geq 7 = \Delta^{(2)}(G_{3,n}) + 1$.

It is also not hard to see that

(4) **Let $\mathbf{m = 4 = n}$**. Then, one can find the set $N^2(v_{33}) = \{v_{31}, v_{22}, v_{42}, v_{13}, v_{24}, v_{44}\}$, which, along with v_{33} forms a maximum clique of size 7, K_7 in $G^{(2i)}_{4,4}$. Thus, $\chi_{2i}(G_{4,4}) \geq 7 = \Delta^{(2)}(G_{4,4}) + 1$.

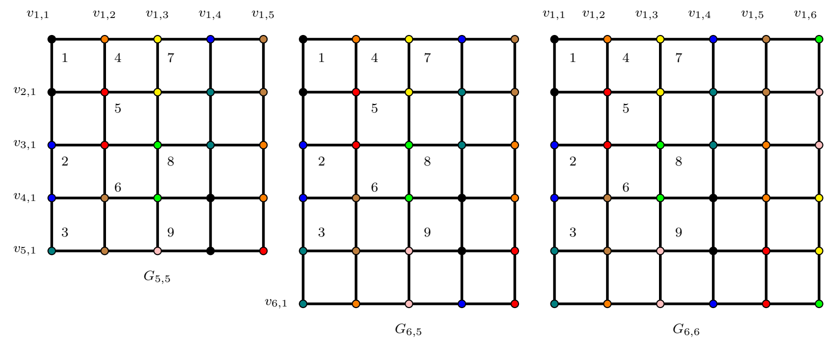

Figure 7. **2-distance injective coloring of $G_{5,5}, G_{6,5}$ *and* $G_{6,6}$**

(5) **Let m $= 3$ *and* n $= 4$.** Then, one can find the set $N^2(v_{23}) = \{v_{21}, v_{12}, v_{32}, v_{14}, v_{34}\}$, which, along with v_{23} forms a maximum clique of size 6, K_6 in $G_{3,4}^{(2i)}$. Thus $\chi_{2i}(G_{3,4}) \geq 6 = \Delta^{(2)}(G_{3,4}) + 1$.

(6) **Let m $= 3 =$ n.** Then, one can find the set $N^2(v_{22}) = \{v_{11}, v_{31}, v_{13}, v_{33}\}$, which, together with v_{22} forms a maximum clique of size 5, K_5 in $G_{3,3}^{(2i)}$. Thus $\chi_{2i}(G_{3,3}) \geq 5 = \Delta^{(2)}(G_{3,3})+1$.

Now, according to the results we have seen in Remark 3, the following theorem can be stated and proven.

Theorem 7. *Let $G_{m,n}$ be a grid graph with $m, n \geq 2$. Then*

$$\chi_{2i}(G_{m,n}) = \begin{cases} 9 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m, n \geq 5 \\ 8 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 4, n \geq 5 \\ 7 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 3, n \geq 5 \\ 7 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 4, n = 4 \\ 6 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 3, n = 4 \\ 5 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 3, n = 3 \end{cases}.$$

Proof. We show that the given graph $G_{m,n}$ can be 2-distance injectively colored with $\Delta^{(2)}(G_{m,n}) + 1$ colors.
1. For $m = 5 = n$, $m = 6, n = 5$, and $m = 6, n = 6$, the grid graphs $G_{5,5}$, $G_{6,5}$ and $G_{6,6}$, denoted in Figure 8 is 2-distance injectively colored using 9 colors, where $9 = \Delta^{(2)}(G_{5,5}) + 1 = \Delta^{(2)}(G_{6,5}) + 1 = \Delta^{(2)}(G_{6,6}) + 1$. Since $G_{6,5}$ is isomorphic to $G_{5,6}$, we conclude that $\chi_{2i}(G_{m,n}) = 9$ for $5 \leq m, n \leq 6$.
Let $m, n \geq 7$. Let the i-th row denote the vertices v_{ij} with $1 \leq j \leq n$ and the j-th column denote the vertices v_{ij} with $1 \leq i \leq m$. For i-th row ($i \geq 7$) and j-th column ($j \geq 7$), if $i \equiv k \ (mod\, 6)$ and $j \equiv l \ (mod\, 6)$ with $1 \leq k, l \leq 6$, then we assign the colors of k-th row to the i-th row and we assign the colors of the l-th column to the j-th column. Since we use 9 colors, and $\Delta^{(2)}(G_{m,n}) + 1 = 9$, we then have $\chi_{2i}(G_{m,n}) = 9$ for $m, n \geq 7$. Therefore $\chi_{2i}(G_{m,n}) = 9$ for $m, n \geq 5$.

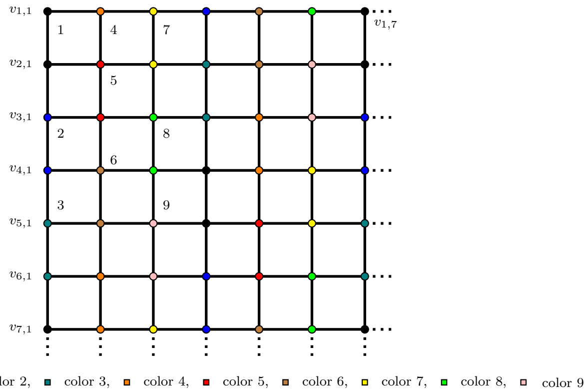

Figure 8. **2-distance injective coloring of $G_{m,n}$**

2. Let $m = 4, n \geq 5$. From Remark 3, we have $\chi_{2i}(G_{4,n}) \geq 8$. Now we will demonstrate a 2-distance injective coloring with 8 colors for $G_{4,n}$.
For $m = 4$ and $1 \leq n \leq 8$, the grid graphs $G_{4,8}$ as shown in Figure 9 ($G_{4,8}$), is 2-distance injectively colored with 8 colors, where $8 = \Delta^{(2)}(G_{4,n}) + 1$ for $5 \leq n \leq 8$. Since $G_{4,k}$ is isomorphic to $G_{k,4}$, we conclude that $\chi_{2i}(G_{4,n}) = 8$ for $1 \leq n \leq 8$.

Let $m = 4$ and $n \geq 9$. For j-th column ($j \geq 9$), if $j \equiv l$ (mod 8) with $1 \leq l \leq 8$, then we assign the colors of the l-th column to the j-th column. Since we use 8 colors for these coloring and $\Delta^{(2)}(G_{m,n}) + 1 = 8$, we then have $\chi_{2i}(G_{m,n}) = 8$ for $m = 4, n \geq 8$. Therefore $\chi_{2i}(G_{4,n}) = 8$ for $n \geq 5$.

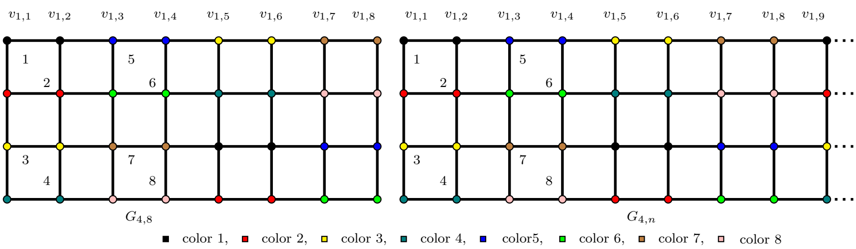

Figure 9. **2-distance injective coloring of $G_{4,n}$**

3. Let $m = 3, n \geq 5$. From Remark 3, we have $\chi_{2i}(G_{3,n}) \geq 7$. Now we demonstrate a 2-distance injective coloring with 7 colors for $G_{3,n}$.
Let f be a function with the property,
$f(v_{1i}) = 1$ for $i \equiv 1, 2$, $f(v_{3i}) = 1$ for $i \equiv 5, 6$ (mod 8),

$f(v_{1i}) = 3$ for $i \equiv 5, 6$, $f(v_{3i}) = 3$ for $i \equiv 1, 2$ (mod 8),

$f(v_{1i}) = 4$ for $i \equiv 3, 4$, $f(v_{3i}) = 4$ for $i \equiv 7, 8$ (mod 8),

$f(v_{1i}) = 6$ for $i \equiv 7, 8$, $f(v_{3i}) = 6$ for $i \equiv 3, 4$ (mod 8) and

$f(v_{2i}) = 2$ for $i \equiv 1, 2$, $f(v_{2i}) = 5$ for $i \equiv 3, 4$, $f(v_{2i}) = 7$ for $i \equiv 5, 6$ (mod 6).

It is easy to see that f is a 2-distance injective coloring function with 7 colors (Figure 10). Therefore $\chi_{2i}(G_{3,n}) = 7$.

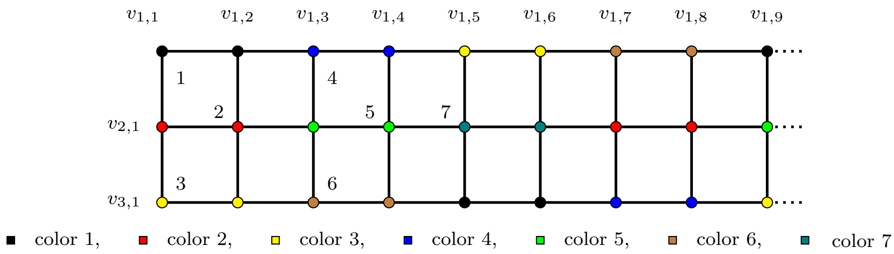

Figure 10. **2-distance injective coloring of $G_{3,n}$**

4. Let $m = 4, n = 4$. From Remark 3, we have $\chi_{2i}(G_{4,4}) \geq 7$. On the other hand, the colors assigned to the vertices of Figure 11 show that $\chi_{2i}(G_{4,4}) = 7$.

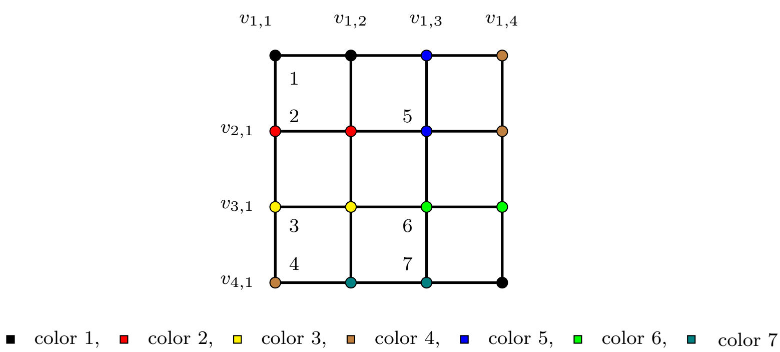

Figure 11. **2-distance injective coloring of $G_{4,4}$**

5. For $m = 3, n = 4$, and $m = 3 = n$, Remark 3 shows that $\chi_{2i}(G_{3,4}) \geq 6$ and $\chi_{2i}(G_{3,3}) \geq 5$. On the other hand, a simple investigation shows that the equalities hold. $\square$

6. Concluding remarks and future perspectives

Characterize graph G such that the 2-distance injective chromatic number satisfies

$$\chi_{2i}(G) = \Delta(\Delta - 1)[1 + (\Delta - 1)^2] + 1.$$

In Theorem 3 and Theorem 7, we showed that for trees T and grid graphs $G_{m,n}$,

$$\chi_{2i}(T) = \Delta^{(2)}(T) + 1, \qquad \chi_{2i}(G_{m,n}) = \Delta^{(2)}(G_{m,n}) + 1.$$

This indicates that these types of graphs have relatively simple coloring properties, as their 2-distance injective chromatic number is tightly determined by their 2-distance degree and neighborhood structure.

Characterize the graphs G such that $\chi_{2i}(G) = \Delta^{(2)}(G) + 1$.

For planar graphs G, discuss the 2-distance injective chromatic number of G. Characterize planar graphs G such that $\chi_{2i}(G) = \Delta^{(2)}(G) + 1$.

It is desirable to determine $\chi_{2i}(G)$ for tori graphs $G = P_n \Box C_m$ and cylinder graphs $G = C_n \Box C_m$.

As we remarked in the text, $\chi_{2i}(G) \geq \chi_i(G)$. Can Conjecture 1 be verified from the point of view of 2-distance injective coloring?

We showed that the 2-distance injective coloring and 2-distance coloring are not comparable. Verify the validity of Conjecture 2 from the perspective of 2-distance injective coloring.

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