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Research Article

Local multiset dimension of triangular snake graph with an application to truss structures

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Abstract: A truss is a framework made up of beams and joints, creating a rigid structure, usually arranged in a triangular formation, that supports a load and is the building block of many structures. Truss structures, are essential in engineering for spanning long distances efficiently and can be monitored with high accuracy, using the concept of local multiset dimension (LMD) in graph theory. LMD determines the minimum reference points needed to uniquely locate adjacent vertices in a graph. Applying this concept to truss structures, increases the efficiency of structural monitoring by deducing the optimal placement for sensors, and real-time monitoring, improving the safety, reliability, and performance of truss systems across diverse engineering applications. The local multiset basis of triangular snake graphs is discussed in this paper, and their local multiset dimension is determined. As an application, LMD is obtained for some common truss structures.

Keywords: local multiset dimension, metric dimension, multiset dimension.

AMS Subject Classification: 05C12

1. Introduction

Consider $G(V, E)$ to be a connected, simple and undirected graph having order $o(G) = |V|$ and size $s(G) = |E|$. Distance between any two vertices $u, v \in V(G)$ is denoted by $d(u, v)$ and diameter is denoted by $diam(G)$. For the terms that are not defined here, we refer to [7, 8].

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Truss structures, in the field of civil engineering, are renowned for their strength and stability. They are often based on the triangular snake graph structure - a derived graph of a path in graph theory. This structure employs a network of interconnected triangular units, resembling a snake's body, with each triangle forming a stable unit. The triangular snake graph structure offers several advantages, including inherent load distribution capabilities and resistance to deformation under stress. By leveraging the inherent stability of triangles and their ability to efficiently transfer loads along their edges, truss structures built on this foundation can span long distances with minimal material usage. Additionally, the modular nature of the triangular units allows for scalability and adaptability, enabling the construction of trusses tailored to specific architectural or engineering requirements.

Trusses, as fundamental structural elements, play a critical role in various engineering applications, including bridges, buildings, aerospace structures, and mechanical systems. The ability to monitor and assess the structural health of trusses is essential for ensuring safety, reliability, and longevity. Traditional methods of structural monitoring often rely on global measurements or sensor placements, which may not fully capture localized damage or structural vulnerabilities. In recent years, there has been growing interest in leveraging advanced mathematical tools, such as graph theory and the concept of local multiset dimension, to develop novel approaches for truss monitoring and assessment.

Graph theory provides a framework for representing and analyzing the connectivity of structural systems, including trusses. Trusses can be modeled as graphs, where vertices represent joints or connection points, and edges represent structural members connecting the joints. By converting a truss into a graph, one can utilize algorithms and mathematical tools from graph theory to solve engineering problems related to the truss design, such as quantifying the minimum number of local sensors required to uniquely deduce the position of every joint on the truss within a specified region or neighborhood.

The use of local multiset dimension offers several advantages in monitoring trusses. Since traditional global monitoring techniques may struggle to detect localized damage or defects in truss structures, local multiset dimension enables the identification of critical regions or neighborhoods within the truss where damage or degradation has occurred. By focusing on these localized areas, maintenance efforts can be targeted more effectively, leading to timely repairs and improved structural integrity. Local multiset dimension provides insights into the redundancy and robustness of truss structures at the local level. By analyzing the local connectivity patterns and measuring the local multiset dimension, engineers can evaluate the structural performance of individual components or sections within the truss. This information is valuable for assessing the overall resilience of the truss to various loading conditions and environmental factors. Optimal sensor placement is crucial for effective structural monitoring and health assessment. Local multiset dimension can guide the placement of sensors by identifying key locations within the truss where measurements are most informative. By strategically positioning sensors based on local multiset dimension analysis, engineers can maximize the coverage of critical areas and minimize the number of sensors required, thereby reducing monitoring costs and complexity. Local multiset dimension analysis can be integrated into real-time monitoring systems for continuous assessment of truss structures. By monitoring changes in local connectivity patterns and multiset dimension values over time, engineers can detect anomalies, such as sudden load redistributions or progressive damage accumulation, and take proactive measures to prevent catastrophic failures. In summary, the use of local multiset dimension in monitoring trusses offers significant advantages in terms of localized damage detection, robustness assessment, sensor placement optimization, and real-time monitoring capabilities. By leveraging this advanced mathematical tool, engineers can enhance the safety, reliability, and performance of truss structures in diverse engineering applications.

In 2017, R. Simanjuntak et al. [10] outlined a new concept called multiset dimension, inspired from metric dimension [9, 11–15]. Here, the multiset of the distances was studied, instead of examining the ordered k-tuple. They established the multiset dimension bounds for arbitrary graphs.

Definition 1. [10] In a connected graph G, let $\mathscr{B} = \{b_i | 1 \leq i \leq k\}$ be a subset of $V(G)$. The multiset representation of a vertex v in $V(G)$ with reference to $\mathscr{B}, r_m(v|\mathscr{B}) = \{d(v, b_1), d(v, b_2), \ldots, d(v, b_k)\}$, is the set of distances between the vertices in $\mathscr{B}$ and v, together with their multiplicity.. For all $u, v \in V(G)$, if $r_m(u|\mathscr{B}) \neq r_m(v|\mathscr{B})$, then $\mathscr{B}$ is called an m-resolving set of G. Multiset basis is the m-resolving set having least cardinality and the multiset dimension (MD) of G, denoted by $md(G)$ is its cardinality. G is said to have an infinite multiset dimension, if G has no m-resolving set and is written as $md(G) = \infty$.

Novi H. Bong et al. [6] established, for some classes of graphs, the properties of multiset dimension.

In 2019, Ridho Alfarisi et al. [4], introduced a new concept called local multiset dimension, where rather than checking the uniqueness of the multiset for every vertex pair in the graph, only the vertices adjacent to one another are examined.

Definition 2. [4] In a connected graph $G(V, E)$, let $\mathscr{B} = \{b_i | 1 \leq i \leq k\} \subseteq V(G)$. With reference to $\mathscr{B}$, the multiset representation of a vertex $v \in V(G)$ is $r_m(v|\mathscr{B}) = \{d(v, b_1), d(v, b_2), \ldots, d(v, b_k)\}$ where each $d(v, b_i)$ is the distance between each $b_i \in \mathscr{B}$ and v in $V(G)$, together with their multiplicity. For every edge uv in $E(G)$, if $r_m(v|\mathscr{B}) \neq r_m(u|\mathscr{B})$, then the resolving set $\mathscr{B}$ is called a local resolving set of G. The local multiset basis (lmsb) of G is the local resolving set with minimum cardinality and local multiset dimension (LMD), denoted by $lmd(G)$ is its cardinality. If G has no $lmsb$, then $lmd(G) = \infty$.

Ridho Alfarisi et al. [4] established sharp bounds for some families of graphs and determined the exact value for path, complete, cycle, trees, caterpillar graphs and others.

An example for comparison between Metric Dimension and Local Multiset Dimension is give in Figure 1.

Theorem 1. *[4] Let $P_n (n \geq 2)$ be a path of length n, then $lmd(G) = 1$.*

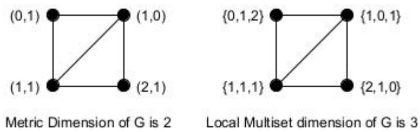


Figure 1: A comparison of metric dimension and local multiset dimension for a graph G

Theorem 2. *[4] Let $C_n(n \geq 3)$ be a cycle of length n, then*
$$lmd(G) = \begin{cases} 1, & if\ \ n \geq 4\ and\ n \equiv 0(mod\, 2) \\ 3, & if\ \ n \geq 7\ and\ n \equiv 1(mod\, 2) \\ \infty, & if\ \ n = 3, 5 \end{cases}.$$

Theorem 3. *[4] Let $K_n(n \geq 3)$ be a complete graph, then $lmd(K_n) = \infty$.*

Note 4. [4] A graph G has $lmd(G) = 1$, if and only if G is bipartite.

Theorem 5. *[3] Let $C_n^{'}(n \geq 4)$ be a pan graph constructed by adding a pendant vertex to a vertex of the cycle C_n. Then, $lmd(C_n^{'}) = \begin{cases} 1, & if\ \ n \equiv 0(mod\, 2) \\ 2, & if\ \ n \equiv 1(mod\, 2) \end{cases}$.*

Ridho Alfarisi et al. [5] and R. Adawiyah et al. [1, 2] have explored LMD for many more classes of graphs.

All Triangular snake graphs defined here are based on the path P_n with $V_1 = \{v_i | 1 \leq i \leq n\}$, the vertex set and $E_1 = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$, the edge set.

Triangular Snake $TS_n(n \geq 2)$ is a graph derived with a cycle C_3, replacing each edge of a path $P_n(n \in \mathbb{Z}^+)$. Let $V(TS_n) = V_1 \cup V_2$ where $V_2 = \{u_j | 1 \leq j \leq n-1\}$ such that edge set is $E(TS_n) = E_1 \cup E_2$ where $E_2 = \{u_i v_i, u_i v_{i+1} | 1 \leq i \leq n-1\}$ are the remaining edges, with $|V| = 2n-1$ and $|E| = 3(n-1)$.

The Double Triangular Snake $DTS_n(n \geq 2)$ is derived from two triangular snakes TS_n sharing a common path P_n. Let $V(DTS_n) = V_1 \cup V_2 \cup V_3$ be the vertex set of DTS_n, where $V_2 = \{u_i | 1 \leq i \leq n-1\}$ and $V_3 = \{w_i | 1 \leq i \leq n-1\}$ are the remaining vertices such that the edge set is $E(DTS_n) = E_1 \cup E_2 \cup E_3$ where $E_2 = \{u_i v_i, u_i v_{i+1} | 1 \leq i \leq n-1\}$ and $E_3 = \{w_i v_i, w_i v_{i+1} | 1 \leq i \leq n-1\}$ with $|V| = 3n-2$ and $|E| = 5(n-1)$.

The Alternate Triangular Snake $ATS_n(n \geq 3)$ is derived with a cycle C_3, replacing every alternate edge of a path $P_n(n \in \mathbb{Z}^+)$. Let $V(ATS_n) = V_1 \cup V_2$ and $E(ATS_n) = E_1 \cup E_2$ be the vertex set and edge set of ATS_n, where the sets differ and are given in two cases, depending on the value of n.

1. $n \equiv 0 \pmod 2$. There are two types of snake graphs in this case.

   (a) A_1TS_n
   The vertex set $V(A_1TS_n) = V_1 \cup V_2$ where $V_2 = \left\{u_i | 1 \leq i \leq \frac{n}{2}\right\}$ are the

remaining vertices such that the edge set of A_1TS_n is $E(A_1TS_n) = E_1 \cup E_2$ where $E_2 = \{u_iv_{2i-1}, u_iv_{2i} | 1 \le i \le \frac{n}{2}\}$ with $|V| = \frac{3n}{2}$.

(b) A_2TS_n
The vertex set $V(A_2TS_n) = V_1 \cup V_2$ where $V_2 = \{u_i | 1 \le i \le \frac{n-2}{2}\}$ are the remaining vertices such that the edge set is $E(A_2TS_n) = E_1 \cup E_2$ where $E_2 = \{u_iv_{2i}, u_iv_{2i+1} | 1 \le i \le \frac{n-2}{2}\}$ with $|V| = \frac{3n-2}{2}$.

2. $n \equiv 1 \pmod 2$, the graph A_oTS_n
The vertex set $V(A_oTS_n) = V_1 \cup V_2$ where $V_2 = \{u_i | 1 \le i \le \frac{n-1}{2}\}$ are the remaining vertices such that the edge set is $E(A_oTS_n) = E_1 \cup E_2$ where $E_2 = \{u_iv_{2i-1}, u_iv_{2i} | 1 \le i \le \frac{n-1}{2}\}$ with $|V| = \frac{3n-1}{2}$.

The Double Alternate Triangular Snake $DATS_n (n \ge 3)$ is derived from two of the same Alternate Triangular Snakes sharing a common path P_n. Let $V(DATS_n) = V_1 \cup V_2 \cup V_3$ be the vertex set of $DATS_n$, where the vertex set and the edge set $E(ATS_n) = E_1 \cup E_2 \cup E_3$ are given in two cases depending on the value of n.

1. $n \equiv 0 \pmod 2$. There are two types of snake graphs in this case.

(a) DA_1TS_n
The vertex set $V(DA_1TS_n) = V_1 \cup V_2 \cup V_3$ where $V_2 = \{u_i | 1 \le i \le \frac{n}{2}\}$ and $V_3 = \{w_i | 1 \le i \le \frac{n}{2}\}$ are the remaining vertices such that the edge set is $E(DA_1TS_n) = E_1 \cup E_2 \cup E_3$ where $E_2 = \{u_iv_{2i-1}, u_iv_{2i} | 1 \le i \le \frac{n}{2}\}$ and $E_3 = \{w_iv_{2i-1}, w_iv_{2i} | 1 \le i \le \frac{n}{2}\}$ with $|V| = 2n$.

(b) DA_2TS_n
The vertex set $V(DA_2TS_n) = V_1 \cup V_2 \cup V_3$ where $V_2 = \{u_i | 1 \le i \le \frac{n-2}{2}\}$ and $V_3 = \{w_i | 1 \le i \le \frac{n-2}{2}\}$ are the remaining vertices such that the edge set is $E(DA_2TS_n) = E_1 \cup E_2 \cup E_3$ where $E_2 = \{u_iv_{2i}, u_iv_{2i+1} | 1 \le i \le \frac{n-2}{2}\}$ and $E_3 = \{w_iv_{2i}, w_iv_{2i+1} | 1 \le i \le \frac{n-2}{2}\}$ with $|V| = 2n - 2$.

2. $n = 1 \pmod 2$, the graph DA_oTS_n
The vertex set $V(DA_oTS_n) = V_1 \cup V_2 \cup V_3$ where $V_2 = \{u_i | 1 \le i \le \frac{n-1}{2}\}$ and $V_3 = \{w_i | 1 \le i \le \frac{n-1}{2}\}$ are the remaining vertices such that the edge set is $E(DA_oTS_n) = E_1 \cup E_2 \cup E_3$ where $E_2 = \{u_iv_{2i-1}, u_iv_{2i} | 1 \le i \le \frac{n-1}{2}\}$ and $E_3 = \{w_iv_{2i-1}, w_iv_{2i} | 1 \le i \le \frac{n-1}{2}\}$ with $|V| = 2n - 1$.

In this paper, the *lmsb* of some snake graphs $TS_n, ATS_n, DTS_n, DATS_n$ are analyzed and their local multiset dimension is deduced, extending to the study of LMD for some known truss structures.

2. Lower bound for LMD of triangular snake graphs

In this section, the lower bound for LMD of Triangular Snake (TS_n), Double Triangular Snake (DTS_n), Alternate Triangular Snake (ATS_n) and Double Alternate Triangular Snake ($DATS_n$) graphs are discussed.

Lemma 1. *Let G be a graph with lmsb $\mathscr{B} = \{b_1, b_2\}$. Then, $d(b_1, b_2) \equiv 0 \pmod 2$.*

Proof. For a given $\mathscr{B} = \{b_1, b_2\}$, assume that $d(b_1, b_2) = p \equiv 1 \pmod 2$ and let there exist a b_1b_2-path: $v_0 (= b_1) - v_1 - v_2 - \ldots - v_{p-1} - v_p (= b_2)$ of length p in G. Then, for the adjacent vertices $v_{\frac{p-1}{2}}$ and $v_{\frac{p+1}{2}}$, $r_m\left(v_{\frac{p-1}{2}}|\mathscr{B}\right) = r_m\left(v_{\frac{p+1}{2}}|\mathscr{B}\right) = \left\{d\left(v_{\frac{p-1}{2}}, b_1\right), d\left(v_{\frac{p-1}{2}}, b_2\right)\right\}$, a contradiction to the fact that $\mathscr{B}$ is an lmsb of G. Hence, when $lmd(G) = 2, d(b_1, b_2) \equiv 0 \pmod 2$. □

Lemma 2. *Let G be a graph K_3 with a pendant edge. Then, $lmd(G) = 2$.*

Proof. Let G be a graph with vertex set $V(G) = \{v_i | 0 \le i \le 3\}$ and edge set $E(G) = \{v_1v_2, v_2v_3, v_3v_1, v_0v_1\}$ where $v_i, 1 \le i \le 3$ form the graph K_3 and v_0 is the pendent vertex adjacent to v_1. Since G contains a cycle C_3, it is not bipartite and thus, $lmd(G) \ge 2$.
Now, consider $\mathscr{B} = \{v_0, v_2\}$. Then, $r_m(v_0|\mathscr{B}) = \{0, 2\}, r_m(v_1|\mathscr{B}) = \{1, 1\}, r_m(v_2|\mathscr{B}) = \{0, 2\}$ and $r_m(v_3|\mathscr{B}) = \{1, 2\}$. Since $v_0 \not\sim v_2$, and all adjacent vertices carry different multisets, $\mathscr{B}$ is a minimum local multiset basis of G and $lmd(G) \le 2$. □

Lemma 3. *Let G be a graph K_4 with an added vertex v_0 adjacent to two vertices of K_4. Then, $lmd(G) = 3$.*

Proof. Let G be a graph with vertex set $V(G) = \{v_i | 0 \le i \le 4\}$ and edge set $E(G) = \{v_iv_j | i \ne j, 1 \le i \le 4\} \cup \{v_0v_1, v_0v_2\}$ where $v_i, 1 \le i \le 4$ form the graph K_4 and v_0 is the added vertex adjacent to v_1 and v_2. Since G contains a cycle C_3, it is not bipartite and thus, $lmd(G) \ge 2$. Now, if $\mathscr{B}$ contains only vertices of K_4, then G cannot be resolved since $lmd(K_4) = \infty$. Thus, $v_0 \in \mathscr{B}$. If $\mathscr{B} = \{v_0, v_1\}\, or \,\{v_0, v_2\}$, then the adjacent vertices v_3, v_4 do not get resolved and receive the multiset $\{1, 2\}$. Similarly, if $\mathscr{B} = \{v_0, v_3\}\, or \,\{v_0, v_4\}$, then the adjacent vertices v_1, v_2 do not get resolved and receive the multiset $\{1, 1\}$. Thus, $\mathscr{B}$ is not a local multiset basis and at least one more vertex is necessary to be in $\mathscr{B}$ to resolve G. Thus, $lmd(G) \ge 3$.

Now, consider $\mathscr{B} = \{v_0, v_2, v_3\}$. Then, $r_m(v_0|\mathscr{B}) = \{0, 2, 1\}$; $r_m(v_1|\mathscr{B}) = \{1, 1, 1\}$; $r_m(v_2|\mathscr{B}) = \{1, 1, 0\}$; $r_m(v_3|\mathscr{B}) = \{2, 0, 1\}$ and $r_m(v_4|\mathscr{B}) = \{2, 1, 1\}$. From this, all the adjacent vertices carry different multisets, $\mathscr{B}$ is a minimum local multiset basis of G and $lmd(G) \le 3$. □

Note 6. A graph G' with a subgraph G constructed by K_4 with an added vertex v_0 adjacent to two vertices of K_4, will also have $lmd(G) \ge 3$.

Lemma 4. *Let G be a graph such that $G \in \{TS_n, DTS_n, ATS_n, DATS_n\}, n \ge 3$. Then, $lmd(G) \ge 2$.*

Proof. Since G contains a subgraph C_3, G is not a bipartite graph (from [8], a graph G is not bipartite if it contains a cycle of odd length), and in consequence, G is not 2-chromatic. From Note 4, $|\mathscr{B}| \geq 2$. Thus, $lmd(G) \geq 2$ for all $G \in \{TS_n, DTS_n, ATS_n, DATS_n\}$. □

Lemma 5. *Let G be a graph such that $G \in \{TS_n, DTS_n, ATS_n, DATS_n\}$, whenever $n \geq 4$ and $n \equiv 0 \pmod 2$. Then, $lmd(G) \geq 3$.*

Proof. Let $\mathscr{B}$ be a multiset basis of G. From Lemma 4, $|\mathscr{B}| \geq 2$. The proof is discussed by contradiction. Assume that $lmd(G) < 3$, say $|\mathscr{B}| = 2$ and $\mathscr{B} = \{b_1, b_2\}$. Depending on the possible vertex pairs in G, there exists at least one of the following:

- A shortest path of odd length between b_1 and b_2 in G.
- At least one edge $xy \in E(G)$ such that $d(x, b_i) = d(y, b_i)$, for all $b_i \in \mathscr{B}$.

Thus, depending on the vertices in $\mathscr{B}$, the following points arise.

Case 1: $b_1 \in V_1, b_2 \in V_2$ (or V_3 in the case of DTS_n and $DATS_n$),
Let $b_1 = v_i$ and $b_2 = u_j($ or $w_j), (1 \leq i \leq j \leq n-1)$. In this case, there exists at least one pair of adjacent vertices that receive the same multiset as given below, depending on the structure of G.

(i) $G = TS_n$ or DTS_n,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_t|\mathscr{B}) = \{i-t, j-t+1\}$, when $1 \leq t < i$ and $i \neq 1$.
(b) $r_m(u_{t-1}|\mathscr{B}) = r_m(v_t|\mathscr{B}) = \{t-i, t-j\}$, when $j+1 < t \leq n$ and $j \neq n-1$.
(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{t-i, j-t+1\}$, for $t = \frac{i+j}{2}$, when both i and j are even (or odd).

(ii) $G = A_1TS_n$ or DA_1TS_n,

(a) $r_m(v_{2t-1}|\mathscr{B}) = r_m(u_k|\mathscr{B}) = \{i-2t+1, 2j-2t+1\}$, when $1 \leq t < \left\lceil \frac{i-1}{2} \right\rceil$, and $i \neq 1$.
(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = \{2t-i, 2t-2j+1\}$, when $j < t \leq \frac{n}{2}$ and $j \neq \frac{n}{2}$.
(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{t-i, 2j-t+1\}$, for $t = \frac{i-1+2j}{2}$, when both i and j are odd or when i is odd and j is even.

(iii) $G = A_2TS_n$ or DA_2TS_n,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = \{i-2t, 2j-2t+1\}$, when $1 \leq t < \left\lfloor \frac{i-1}{2} \right\rfloor$ and $i \neq 1, 2$.
(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t+1}|\mathscr{B}) = \{2t-i+1, 2t-2j+1\}$, when $j < t \leq \frac{n-2}{2}$ and $j \neq \frac{n-2}{2}$.
(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{2t-i+1, 2j-2t+1\}$, for $t = \frac{i+2j+1}{2}$, when both i and j are even or when i is even and j is odd.

Case 2: $b_1, b_2 \in V_1$,
In this case, there exists at least one pair of adjacent vertices that receive the same multiset as given below, depending on the structure of G.

(i) $G = TS_n$ or DTS_n,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_t|\mathscr{B}) = \{i-t, j-t\}$, when $1 \leq t < i$ and $i \neq 1$.

(b) $r_m(u_{t-1}|\mathscr{B}) = r_m(v_t|\mathscr{B}) = \{t-i, t-j\}$, when $j < t \leq n$ and $j \neq n$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{t-i, j-t\}$, for $t = \frac{i+j-1}{2}$, when i is even and j is odd (or i is odd and j is even).

(ii) $G = A_1TS_n$ or DA_1TS_n,

(a) $r_m(v_{2t-1}|\mathscr{B}) = r_m(u_t|\mathscr{B}) = \{i-2t+1, j-2t+2\}$, when $1 \leq t < \left\lceil \frac{i}{2} \right\rceil$ and $i \neq 1$.

(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = \{2t-i, 2t-2j+1\}$, when $\left\lceil \frac{j+1}{2} \right\rceil < t \leq \frac{n}{2}$ and $j \neq n$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{t-i, j-t\}$, for $t = \frac{i-1+j}{2}$, when i is even and j is odd (or i is odd and j is even).

(iii) $G = A_2TS_n$ or DA_2TS_n,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = \{i-2t, j-2t\}$, when $1 \leq t < \left\lfloor \frac{i-1}{2} \right\rfloor$ and $i \neq 1, 2$.

(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t+1}|\mathscr{B}) = \{2t-i+1, 2t-j+1\}$, when $\left\lfloor \frac{j+1}{2} \right\rfloor < t \leq \frac{n-2}{2}$ and $j \neq n-1, n$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{2t-i+1, j-2t\}$, for $t = \frac{i+j-1}{2}$, when i is even and j is odd (or i is odd and j is even).

Case 3: Both $b_1, b_2 \in V_2$ (or V_3 in the case of DTS_n and $DATS_n$).
Let $b_1 = u_i$ and $b_2 = u_j$ (or $b_1 = w_i$ and $b_2 = w_j$) $, (1 \leq i \leq j \leq n-1)$. In this case, there exists at least one pair of adjacent vertices that receive the same multiset as given below, depending on the structure of G.

(i) $G = TS_n$ or DTS_n,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_t|\mathscr{B}) = \{i-t+1, j-t+1\}$, when $1 \leq t < i$ and $i \neq 1$.

(b) $r_m(u_{t-1}|\mathscr{B}) = r_m(v_t|\mathscr{B}) = \{t-i, t-j\}$, when $j+1 < t \leq n$ and $j \neq n-1$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{t-i, j-t+1\}$, for $t = \frac{i+j}{2}$, when both i and j are odd (or even).

(ii) $G = A_1TS_n$ or DA_1TS_n,

(a) $r_m(v_{2t-1}|\mathscr{B}) = r_m(u_t|\mathscr{B}) = \{2i-2t+1, 2j-2t+1\}$, when $1 \leq t < i$ and $i \neq 1$.

(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = \{2t-2i+1, 2t-2j+1\}$, when $j < t \leq \frac{n}{2}$ and $j \neq \frac{n}{2}$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{2t-2i+1, 2j-2t+1\}$, for $t = \frac{2i+2j-2}{2}$.

(iii) $G = A_2TS_n$ or DA_2TS_n,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = \{2i-2t+1, 2j-2t+1\}$, $1 \le t < i$ and $i \ne 1$.

(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t+1}|\mathscr{B}) = \{2t-2i+1, 2t-2j+1\}$, when $j < t \le \frac{n-2}{2}$ and $j \ne \frac{n-2}{2}$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{2t-2i+1, 2j-2t+1\}$, for $t = \frac{2i+2j}{2}$.

Case 4: $b_1 \in V_2$ and $b_2 \in V_3$ (in the case of DTS_n and $DATS_n$),
Let $b_1 = u_i$ and $b_2 = w_j (1 \le i \le j \le n-1)$. In this case, there exists at least one pair of adjacent vertices that receive the same multiset as given below, depending on the structure of G.

(i) $G = DTS_n$,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_t|\mathscr{B}) = r_m(w_t|\mathscr{B}) = \{i-t+1, j-t+1\}$, when $1 \le t < i$ and $i \ne 1$.

(b) $r_m(u_{t-1}|\mathscr{B}) = r_m(v_t|\mathscr{B}) = r_m(u_{t-1}|\mathscr{B}) = \{t-i+1, t-j+1\}$, when $j+1 < t \le n$ and $j \ne n-1$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{t-i, j-t+1\}$, for $t = \frac{i+j}{2}$, when both i and j are odd (or even).

(ii) $G = DA_1TS_n$,

(a) $r_m(v_{2t-1}|\mathscr{B}) = r_m(u_t|\mathscr{B}) = r_m(w_t|\mathscr{B}) = \{2i-2t+1, 2j-2t+1\}$, when $1 \le t < i$ and $i \ne 1$.

(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = r_m(u_t|\mathscr{B}) = \{2t-2i+1, 2t-2j+1\}$, when $j < t \le \frac{n}{2}$ and $j \ne \frac{n}{2}$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{2t-2i+1, 2j-2t+1\}$, for $t = \frac{2i+2j-2}{2}$.

(iii) $G = DA_2TS_n$,

(a) $r_m(u_t|\mathscr{B}) = r_m(v_{2t}|\mathscr{B}) = r_m(w_t|\mathscr{B}) = \{2i-2t+1, 2j-2t+1\}$, $1 \le t < i$ and $i \ne 1$.

(b) $r_m(u_t|\mathscr{B}) = r_m(v_{2t+1}|\mathscr{B}) = r_m(w_t|\mathscr{B}) = \{2t-2i+1, 2t-2j+1\}$, when $j < t \le \frac{n-2}{2}$ and $j \ne \frac{n-2}{2}$.

(c) $r_m(v_t|\mathscr{B}) = r_m(v_{t+1}|\mathscr{B}) = \{2t-2i+1, 2j-2t+1\}$, for $t = \frac{2i+2j}{2}$.

From the above cases, it can be seen that, $\mathscr{B}$ is not a local multiset basis of G and two vertices are not sufficient to form a local multiset basis of G. Thus, $|\mathscr{B}| > 2 \Rightarrow lmd(G) \ge 3$. □

3. Upper bound of LMD for triangular snake graphs

In this section, the upper bound for LMD of various triangular snake graphs are discussed.

Lemma 6. *Let $G \in \{TS_n, DTS_n\}, n \geq 3$ be a triangular snake graph. Then,* $lmd(G) \leq \begin{cases} 2, & \text{if } n \equiv 1 \pmod 2 \\ 3, & \text{if } n \equiv 0 \pmod 2 \end{cases}.$

Proof. To prove the upper bound, consider the following cases.

Case 1: $n \equiv 1 \pmod 2$,
Suppose $\mathscr{B} = \{v_1, v_n\}$, then the resolving multiset generated for each vertex is given by $r_m(v_i|\mathscr{B}) = \{i-1, n-i\}$ and $r_m(u_j|\mathscr{B}) (= r_m(w_j|\mathscr{B})) = \{j, n-j\}$.

Case 2: $n \equiv 0 \pmod 2$
Suppose $\mathscr{B} = \{v_1, v_{n-1}, v_n\}$, then the resolving multiset generated for each vertex is given by

$r_m(v_i|\mathscr{B}) = \begin{cases} \{i-1, n-i, n-1-i\}, & \text{if } 1 \leq i \leq n-1 \\ \{i-1, n-i, n-i+1\}, & \text{if } i = n \end{cases}$. Also,

$r_m(u_i|\mathscr{B}) = \begin{cases} \{i, n-i, n-1-i\}, & \text{if } 1 \leq i \leq n-2 \\ \{i, n-i, n-i\}, & \text{if } i = n-1 \end{cases}$.

From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$; $r_m(u_i|\mathscr{B}) \neq r_m(v_i|\mathscr{B})$; and $r_m(u_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B}), (r_m(w_i|\mathscr{B}) \neq r_m(v_i|\mathscr{B})$; and $r_m(w_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B}))$. Thus, adjacent vertices in $TS_n($ or $DTS_n)$ carry different multisets. Hence, $\mathscr{B}$ is a minimum local multiset basis of $TS_n($ and $DTS_n)$. Thus, $lmd(G) \leq 2$, for $n \equiv 1 \pmod 2$ and $lmd(G) \leq 3$, for $n \equiv 0 \pmod 2$. □

Lemma 7. *Let $G \in \{ATS_n, DATS_n\}, n \geq 3$ be an alternate triangular snake graph. Then,* $lmd(G) \leq \begin{cases} 2, & \text{if } n \equiv 1 \pmod 2 \\ 3, & \text{if } n \equiv 0 \pmod 2 \end{cases}.$

Proof. To prove the upper bound, consider the following cases.

Case 1: $n \equiv 1 \pmod 2$ and the graph is A_oTS_n or DA_oTS_n,
Suppose $\mathscr{B} = \{v_1, v_n\}$, then the resolving multiset generated for each vertex is given by $r_m(v_i|\mathscr{B}) = \{i-1, n-i\}$ and $r_m(u_j|\mathscr{B}) (= r_m(w_j|\mathscr{B})) = \{2j-1, n-2j+1\}$.
From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$; $r_m(u_i|\mathscr{B}) \neq r_m(v_{2i-1}|\mathscr{B})$ and $r_m(u_i|\mathscr{B}) \neq r_m(v_{2i}|\mathscr{B}), (r_m(w_i|\mathscr{B}) \neq r_m(v_{2i-1}|\mathscr{B})$ and $r_m(w_i|\mathscr{B}) \neq r_m(v_{2i}|\mathscr{B})$. Thus, in this case adjacent vertices in A_oTS_n or DA_oTS_n carry different multisets. Hence, $\mathscr{B}$ is a minimum local multiset basis of A_oTS_n and DA_oTS_n. Thus, $lmd(G) \leq 2$, for $n \equiv 1 \pmod 2$.

Case 2: $n \equiv 0 \pmod 2$ and the graph is A_1TS_n or DA_1TS_n,
When $n \geq 4$, suppose $\mathscr{B} = \{v_1, v_{n-1}, v_n\}$, then the resolving multiset generated for each vertex is given by

$r_m(v_i|\mathscr{B}) = \begin{cases} \{i-1, n-i, n-i-1\}, & \text{if } 1 \le i \le n-1 \\ \{i-1, n-i, n-i+1\}, & \text{if } i = n \end{cases}$. Also,
$r_m(u_j|\mathscr{B}) \,(= r_m(w_j|\mathscr{B})) = \begin{cases} \{2j-1, n-2j+1, n-2j\}, & \text{if } 1 \le j \le \frac{n}{2}-1 \\ \{2j-1, n-2j+1, n-2j+1\}, & \text{if } j = \frac{n}{2} \end{cases}$.
From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$; $r_m(u_i|\mathscr{B}) \neq r_m(v_{2i-1}|\mathscr{B})$; and $r_m(u_i|\mathscr{B}) \neq r_m(v_{2i}|\mathscr{B})$ $(r_m(w_i|\mathscr{B}) \neq r_m(v_{2i-1}|\mathscr{B})$; and $r_m(w_i|\mathscr{B}) \neq r_m(v_{2i}|\mathscr{B}))$. Thus, in this case adjacent vertices in A_1TS_n or DA_1TS_n carry different multisets. Hence, $\mathscr{B}$ is a minimum local multiset basis of A_1TS_n and DA_1TS_n. Thus, $lmd(G) \le 3$ for $n \equiv 0 \pmod 2$.

Case 3: $n \equiv 0 \pmod 2$ and the graph is A_2TS_n or DA_2TS_n,
When $n \ge 4$, suppose $\mathscr{B} = \{v_1, v_{n-1}, v_n\}$, then the resolving multiset generated for each vertex is given by
$r_m(v_i|\mathscr{B}) = \begin{cases} \{i-1, n-i, n-i-1\}, & \text{if } 1 \le i \le n-1 \\ \{i-1, n-i, n-i+1\}, & \text{if } i = n \end{cases}$. Also,
$r_m(u_j|\mathscr{B}) \,(= r_m(w_j|\mathscr{B})) = \{2j, n-2j, n-2j\}, 1 \le j \le \frac{n-2}{2}$.
From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$; $r_m(u_i|\mathscr{B}) \neq r_m(v_{2i}|\mathscr{B})$; and $r_m(u_i|\mathscr{B}) \neq r_m(v_{2i+1}|\mathscr{B})$ $(r_m(u_i|\mathscr{B}) \neq r_m(v_{2i}|\mathscr{B})$; and $r_m(u_i|\mathscr{B}) \neq r_m(v_{2i+1}|\mathscr{B}))$. Thus, in this case adjacent vertices in A_2TS_n or DA_2TS_n carry different multisets. Hence, $\mathscr{B}$ is a minimum local multiset basis of A_2TS_n and DA_2TS_n. Thus, $lmd(G) \le 3$ for $n \equiv 0 \pmod 2$. □

4. LMD of triangular snake graphs

In this section, the local multiset dimension of Triangular snake (TS_n), Double Triangular Snake (DTS_n), Alternate Triangular Snake (ATS_n) and Double Alternate Triangular Snake ($DATS_n$) graphs are deduced.

Note 7. When $n = 2$, TS_2 (or ATS_2) is the graph K_3. From Theorem 3, $lmd(TS_2) = lmd(ATS_2) = \infty$.

Note 8. When $n = 2$, DTS_2 (or $DATS_2$) is the graph $K_4 - e$. Since DTS_2 contains a subgraph C_3, DTS_2 is not a bipartite graph (from [8], a graph DTS_2 is not bipartite if it contains a cycle of odd length), and in consequence, DTS_2 is not 2-chromatic. From Note 4, $|\mathscr{B}| \ge 2$. But since taking any two vertices in $\mathscr{B}$, does not resolve the vertices v_1 and v_2 or are adjacent to each other and does not resolve itself, $|\mathscr{B}| \ge 3$. Now, suppose $\mathscr{B} = \{v_1, u_1, w_1\}$, then the resolving multiset generated for each vertex is given by $r_m(v_1|\mathscr{B}) = \{0, 1, 1\}, r_m(v_2|\mathscr{B}) = \{1, 1, 1\}, r_m(u_1|\mathscr{B}) = \{1, 0, 2\}$ and $r_m(w_1|\mathscr{B}) = \{1, 2, 0\}$. Thus, $lmd(DTS_2) = lmd(DATS_2) = 3$.

Theorem 9. *Let G be a graph such that $G \in \{TS_n, DTS_n, ATS_n, DATS_n\}, n \ge 2$. Then,* $lmd(G) = \begin{cases} \infty, & \textit{if } n = 2 \\ 2, & \textit{if } n \equiv 1 \pmod 2 \\ 3, & \textit{if } n \equiv 0 \pmod 2 \end{cases}$.

Proof. Proof follows from Lemmas 4, 5, 6 and 7. □

5. LMD for general snake graph

All general snake graphs defined here are based on the path P_n with the vertex set $V_1 = \{v_i \mid 1 \leq i \leq n\}$ and the edge set $E_1 = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. The general snake graph $S_n(n \geq 2)$ is a graph obtained by replacing each edge of a path $P_n(n \in \mathbb{Z}^+, n \geq 2)$, by the cycle $C_m, (m \in \mathbb{Z}^+, m \geq 3)$. Let $V(S_n) = V_1 \cup V_2$ where $V_2 = \{u_{i,j} \mid 1 \leq j \leq m-2, 1 \leq i \leq n-1\}$ such that edge set is $E(S_n) = E_1 \cup E_2$ where $E_2 = \{u_{i,1}v_i, u_{i,m-2}v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_{i,j}u_{i,j+1} \mid 1 \leq i \leq n-1, 1 \leq j \leq m-3\}$ are the remaining edges, with $|V| = (n-1)(m-2)+n$ and $|E| = 3(n-1)$.
In this section, LMD for general snake graphs is discussed for any $n \in \mathbb{Z}^+, m \geq 4$.

Theorem 10. *Let ES_n be the snake graph constructed by replacing each edge of a path P_n with an even cycle $C_m, m \geq 4$ and $m \equiv 0 \pmod 2$. Then, $lmd(ES_n) = 1$.*

Proof. From Note 4, since ES_n is a bipartite graph, $lmd(ES_n) = 1$. □

Theorem 11. *Let OS_n be the snake graph constructed by replacing each edge of a path P_n with an odd cycle $C_m, m \geq 5$, and $m \equiv 1 \pmod 2$. Then, $lmd(OS_n) = 2$.*

Proof. Let OS_n be an odd cycle snake graph. Since OS_n contains at least one odd cycle, it is not a bipartite graph. Thus, from Note 4, $lmd(OS_n) \geq 2$.
Now, let $V(OS_n) = V_1 \cup V_2$ where

$$V_1 = \{v_i | 1 \leq i \leq n\} \cup \{u_{i,j} \mid 1 \leq i \leq n-1, 1 \leq j \leq m-2\}$$

be the vertex set and

$$\begin{aligned} E(OS_n) = &\{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_{i,1}v_i, u_{i,m-2}v_{i+1} | 1 \leq i \leq n-1\} \\ &\cup \{u_{i,j}u_{i,j+1} | 1 \leq j \leq m-3\} \end{aligned}$$

be the edge set of OS_n. To prove the upper bound, consider $\mathscr{B}$, depending on the value of n.

Case 1: $n \equiv 0 \pmod 2$,
In this case, let $\mathscr{B} = \{v_1, u_{(n-1),(m-2)}\}$. Then, $r_m(v_i|\mathscr{B}) = \{i-1, n-i+1\}$ and
$$r_m(u_{i,j}|\mathscr{B}) = \begin{cases} \{i-1+j, n-i+j+1\}, & \text{if } 1 \leq j \leq \left\lfloor \frac{m-2}{2} \right\rfloor \\ \{i-1+j, n-i+j-1\}, & \text{if } j = \left\lfloor \frac{m-2}{2} \right\rfloor \\ \{i-1-j+m, n-i-j+m-2\}, & \text{if } \left\lfloor \frac{m-2}{2} \right\rfloor < j \leq m-2 \end{cases}.$$

Case 2: $n \equiv 1 \pmod 2$,
In this case, let $\mathscr{B} = \{v_1, v_n\}$. Then, $r_m(v_i|\mathscr{B}) = \{i-1, n-i\}$ and

$$r_m\left(u_{i,j}|\mathscr{B}\right)=\begin{cases}\{i-1+j,n-i+j\}, & \text{if } 1\leq j\leq\left\lfloor\frac{m-2}{2}\right\rfloor\\ \{i-1+j,n-i+j\}, & \text{if } j=\left\lceil\frac{m-2}{2}\right\rceil\\ \{i-1-j+m,n-i-j+m-1\}, & \text{if } \left\lceil\frac{m-2}{2}\right\rceil<j\leq m-2\end{cases}.$$

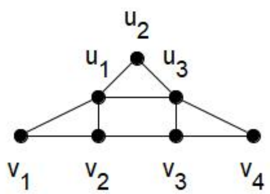
From the above, it can be seen that $r_m\left(v_i|\mathscr{B}\right)\neq r_m\left(v_{i+1}|\mathscr{B}\right), r_m\left(v_i|\mathscr{B}\right)\neq r_m\left(u_{i,1}|\mathscr{B}\right), r_m\left(v_i|\mathscr{B}\right)\neq r_m\left(u_{i,m-2}|\mathscr{B}\right)$ and $r_m\left(u_{i,j}|\mathscr{B}\right)\neq r_m\left(u_{i,j+1}|\mathscr{B}\right), 1\leq i\leq n-1, 1\leq j\leq m-2$. Thus, adjacent vertices in OS_n carry different multisets and $\mathscr{B}$ is a minimum local multiset basis of OS_n. Thus, $lmd(OS_n)\leq 2\Rightarrow lmd(OS_n)=2$. □

6. LMD of some truss structures

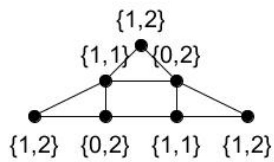
In this section, the *lmsb* of graphs of some common truss structures such as Queenpost truss graph, Kingpost truss graph, Howe truss graph, Pratt truss graph, Double fink truss graph, Diagonal pratt truss graph and Fink Diagonal Pratt truss graph are studied and their LMD is determined.

Theorem 12. *Let QPT be the Queenpost truss graph and KPT be the Kingost truss graph. Then, $lmd(QPT)=lmd(KPT)=2$.*

Proof. The graph, *lmsb* and the LMD of the Queenpost truss graph and the Kingpost truss graph are given in Figure 2 and Figure 3. □



(a) Queenpost truss graph modelled

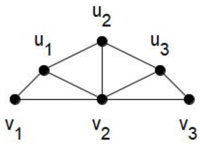


(b) Queenpost truss graph with multiset representation of each vertex

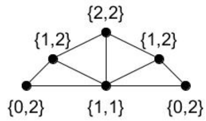
Figure 2: Queenpost truss graph

The Howe truss graph HT_n (Figure 4) is constructed on a path $P_n, n\geq 5, n\equiv 1 \pmod 2$ with vertex set $V(HT_n)=\{v_i|1\leq i\leq n\}\cup\{u_j|1\leq j\leq n-2\}$ and edge set $E(HT_n)=\{v_iv_{i+1}|1\leq i\leq n-1\}\cup\{u_iu_{i+1}|1\leq i\leq n-3\}\cup\left\{u_iv_i,u_iv_{i+1}|1\leq i\leq\left\lfloor\frac{j}{2}\right\rfloor\right\}\cup\left\{u_{\left\lceil\frac{n-2}{2}\right\rceil}v_j|\left\lfloor\frac{n-2}{2}\right\rfloor\leq j\leq\left\lceil\frac{n}{2}\right\rceil\right\}\cup\left\{u_iv_{i+1},u_iv_{i+2}|\left\lceil\frac{j}{2}\right\rceil\leq i\leq n-2\right\}$.

Theorem 13. *Let HT_n be the Howe truss graph on a path $P_n, n\geq 5$ and $n\equiv 1 \pmod 2$. Then, $lmd(HT_n)=2$.*

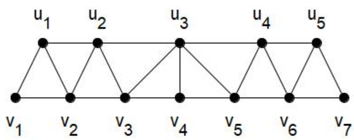


(a) Kingpost truss graph modelled

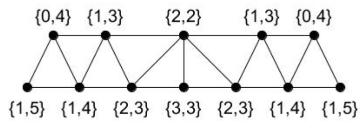


(b) Kingpost truss graph with multiset representation of each vertex

Figure 3: Kingpost truss graph



(a) Howe truss graph modelled



(b) Howe truss graph with multiset representation of each vertex

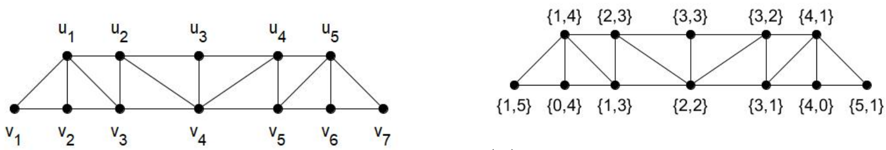
Figure 4: An example of Howe truss graph

Proof. From the definition of HT_n graph, C_3 is a subgraph of HT_n. Thus, HT_n is not a bipartite graph and from Note 4, $lmd(HT_n) \geq 2$.
Now, suppose $\mathscr{B} = \{u_1, u_{n-2}\}$, then, the multiset generated for each vertex is given by $r_m(v_1|\mathscr{B}) = \{1, n-2\}$; when $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, r_m(v_i|\mathscr{B}) = \{i-1, n-i-1\}; r_m\left(v_{\left\lceil \frac{n}{2} \right\rceil}|\mathscr{B}\right) = \{i-1, n-i\}$; when $\left\lceil \frac{n}{2} \right\rceil < i \leq n-1, r_m(v_i|\mathscr{B}) = \{i-2, n-i\}$ and $r_m(v_n|\mathscr{B}) = \{1, n-2\}$. From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B}), r_m(u_i|\mathscr{B}) \neq r_m(v_i|\mathscr{B}), r_m(u_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$ when $1 \leq i \leq \left\lfloor \frac{j}{2} \right\rfloor; r_m(u_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B}), r_m(u_i|\mathscr{B}) \neq r_m(v_{i+2}|\mathscr{B})$ when $\left\lceil \frac{j}{2} \right\rceil \leq i \leq n-2; r_m\left(u_{\left\lceil \frac{n-2}{2} \right\rceil}|\mathscr{B}\right) \neq r_m(v_j|\mathscr{B})$ when $\left\lfloor \frac{n-2}{2} \right\rfloor \leq j \leq \left\lceil \frac{n}{2} \right\rceil$. Thus, $\mathscr{B}$ is a local multiset basis of HT_n and $lmd(HT_n) \leq 2 \Rightarrow lmd(HT_n) = 2$. □

The Pratt truss graph PT_n (Figure 5) is constructed on a path $P_n, n \geq 5, ; n \equiv 1 \pmod 2$ with vertex set $V(PT_n) = \{v_i| \leq i \leq n\} \cup \{u_j|0 \leq j \leq n-1\}$ and edge set $E(PT_n) = \{v_i v_{i+1}|1 \leq i \leq n-1\} \cup \{u_i u_{i+1}|0 \leq i \leq n-2\} \cup \{u_0 v_1, u_1 v_1\} \cup \left\{u_i v_{i+1}, u_i v_{i+2}|1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor\right\} \cup \left\{u_{\left\lceil \frac{n-2}{2} \right\rceil} v_{\left\lceil \frac{n-2}{2} \right\rceil}\right\} \cup \left\{u_i v_i, u_i v_{i+1}| \left\lceil \frac{n-2}{2} \right\rceil \leq i \leq n-2\right\} \cup \{u_{n-2} v_n, u_{n-1} v_n\}$.

Theorem 14. *Let PT_n be the Pratt truss graph on a path $P_n, n \geq 5$ and $n \equiv 1 \pmod 2$. Then, $lmd(PT_n) = 2$.*

Proof. From the definition of PT_n graph, C_3 is a subgraph of PT_n. Thus, PT_n is



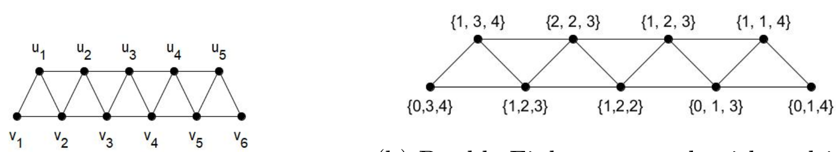
(a) Pratt truss graph modelled

(b) Pratt truss graph with multiset representation of each vertex

Figure 5: An example of Pratt truss graph

not a bipartite graph and from Note 4, $lmd(PT_n) \geq 2$.
From the definition of PT_n and HT_n, HT_n is an induced subgraph of PT_n. Thus, from Theorem 13 $\mathscr{B} = \{u_1, u_{n-2}\}$, and the vertices u_0 and u_{n-1} receive the multiset $r_m(u_0|\mathscr{B}) = r_m(u_{n-1}|\mathscr{B}) = \{1, n-2\}$. From this, the end vertices of each edge carry a distinct multiset. Thus, $\mathscr{B}$ is a local multiset basis of PT_n and $lmd(PT_n) \leq 2 \Rightarrow lmd(PT_n) = 2$. □

The Double Fink truss graph DFT_n (Figure 6) is constructed on a path $P_n, n \geq 4$ with vertex set $V(DFT_n) = V \cup U$ where $V = \{v_i | 1 \leq i \leq n\}$ and $U = \{u_j | 1 \leq j \leq n-1\}$ and edge set $E(DFT_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-2\} \cup \{u_i v_i, u_i v_{i+1} | 1 \leq i \leq n-2\}$.



(a) Double Fink truss graph modelled

(b) Double Fink truss graph with multiset representation of each vertex

Figure 6: An example of Double Fink truss graph

Theorem 15. *Let DFT_n be the Double Fink truss graph on a path $P_n, n \geq 4$. Then, $lmd(DFT_n) = 3$.*

Proof. Let $\mathscr{B} \subset V(DFT_n)$ be a local resolving set of DFT_n.
When $n \geq 4$, since DFT_n contains at least one C_3, by Note 4, $lmd(DFT_n) \geq 2$. Now, the proof is discussed by contradiction. Assume that $lmd(DFT_n) < 3 \Rightarrow |\mathscr{B}| = 2$. Let $\mathscr{B} = \{b_1, b_2\}$. Depending on the possible vertex pairs in $\mathscr{B}$, there exists at least one pair of adjacent vertices that receive the same multiset as following.

Case 1: $b_1 \in V, b_2 \in U$,
Let $b_1 = v_i$ and $b_2 = u_j, (1 \leq i \leq j \leq n-1)$. For $i < k \leq j$, the vertices $r_m(v_k|\mathscr{B}) = r_m(u_{k-1}|\mathscr{B}) = \{k-i, j-k+1\}$ receive the same multiset.

Case 2: $b_1, b_2 \in V$,
Let $b_1 = v_i$ and $b_2 = v_j, (1 \leq i \leq n)$. For $k < i$, the vertices $r_m(v_k|\mathscr{B}) = r_m(u_k|\mathscr{B}) = \{i-k, j-k\}$ receive the same multiset.

Case 3: $b_1, b_2 \in U$,
Let $b_1 = u_i$ and $b_2 = u_j, (1 \leq i < j \leq n-1)$. For $k < i$, the vertices $r_m(u_k|\mathscr{B}) = r_m(v_{k+1}|\mathscr{B}) = \{i-k, j-k\}$ receive the same multiset.
From the above cases, it can be seen that, $\mathscr{B}$ is not an *lmsb* of DFT_n and two vertices are not sufficient to form an *lmsb* of DFT_n. Thus, $|\mathscr{B}| > 2 \Rightarrow lmd(DFT_n) \geq 3$.
Further, to obtain the upper bound when $n \geq 4$, suppose $\mathscr{B} = \{v_1, v_{n-1}, v_n\}$, then the resolving multiset generated for each vertex is given by

$$r_m(v_i|\mathscr{B}) = \begin{cases} \{i-1, n-i, n-i-1\}, & \text{if } 1 \leq i \leq n-1 \\ \{i-1, n-i, n-i+1\}, & \text{if } i = n \end{cases}.$$

Also,

$$r_m(u_i|\mathscr{B}) = \begin{cases} \{i, n-i, n-i-1\}, & \text{if } 1 \leq i \leq n-2 \\ \{i, n-i, n-i\}, & \text{if } i = n-1 \end{cases}.$$

From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$, $r_m(u_i|\mathscr{B}) \neq r_m(v_i|\mathscr{B})$, $r_m(u_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$ and $r_m(u_i|\mathscr{B}) \neq r_m(u_{i+1}|\mathscr{B})$. Thus, in this case, adjacent vertices in DFT_n carry different multisets. Hence, $\mathscr{B}$ is a minimum *lmsb* of DFT_n. Thus, $lmd(DFT_n) \leq 3 \Rightarrow lmd(DFT_n) = 3$, for $n \geq 4$. □

The Warren with Verticals truss graph WVT_n (Figure 7) is constructed on a path $P_n, n \geq 7, n \equiv 1 \pmod 2$ with the vertex set $V(WVT_n) = \{v_i | 1 \leq i \leq n\} \cup \{u_j | 1 \leq j \leq n-2\}$ and edge set $E(WVT_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-3\} \cup \{u_j v_j, u_j v_{j+1}, u_j v_{j+2} | j \equiv 1 \pmod 2, 1 \leq j \leq n-2\} \cup \{u_j v_{j+1} | j \equiv 0 \pmod 2, 1 \leq j \leq n-2\}$.

(a) Warren with verticals truss graph modelled

(b) Warren with verticals truss graph with multiset representation of each vertex

Figure 7: An example of Warren with verticals truss graph

Theorem 16. *Let WVT_n be the Warren with Verticals truss graph on a path $P_n, n \geq 7$ and $n \equiv 1 \pmod 2$. Then, $lmd(WVT_n) = \frac{n-3}{2}$.*

Proof. From the definition of WVT_n graph, C_3 is a subgraph of WVT_n. Thus, WVT_n is not a bipartite graph and from Note 4, $lmd(WVT_n) \geq 2$. Now, by the definition of WVT_n, at least one such K_3 with another vertex adjacent to one vertex of K_3 is a subgraph of WVT_n. From Lemma 2, at least two vertices must be there in $\mathscr{B}$, to resolve the vertices of each such subgraph. Also, the minimum number of such subgraphs adjacent to one another is $n-3$. From Lemma 2, at least every alternate vertex of the $n-3$ such subgraphs must be in $\mathscr{B}$. Thus, $lmd(WVT_n) \geq \frac{n-3}{2}$.
To prove the upper bound, let $\mathscr{B}$ be a minimum *lmsb* of WVT_n. Depending on the vertices in $\mathscr{B}$, there exist the following two cases.

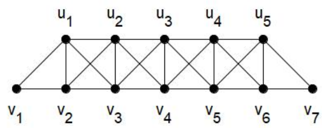
Case 1: $\{v_k, v_{k+2}\} \in \mathscr{B}$, for some $2 \leq k \leq n-3$,
Then, the vertices of the induced subgraph on the vertex set $\{u_j, v_i | k-2 \leq i \leq k+4, k-2 \leq j \leq k+2, 4 \leq k \leq n-4\}$ are resolved. Extending this to the graph WVT_n, every such induced subgraph requires two vertices in $\mathscr{B}$. If $v_k \in \mathscr{B}$, for all $2 \leq k \leq n-2, n \equiv 0 (mod\, 2)$, then, $V(WVT_n)$ is resolved and $|\mathscr{B}| \leq \frac{n-1}{2}$.

Case 2: $\{u_k, u_{k+2}\} \in \mathscr{B}$, for some $2 \leq k \leq n-5$,
Then, the vertices $\{u_j, v_i | k-1 \leq i \leq k+5, k-1 \leq j \leq k+3, 2 \leq k \leq n-5\}$ are resolved. Thus, if $u_k \in \mathscr{B}$ for all $2 \leq k \leq n-2, n \equiv 0 (mod\, 2)$, then, $V(WVT_n)$ is resolved and $|\mathscr{B}| \leq \frac{n-3}{2}$.
From the above two cases, $|\mathscr{B}| \leq \frac{n-3}{2}$. Consider $\mathscr{B} = \{u_i | 1 \leq i \leq n-2, i \equiv 0 \pmod 2\}$. Then, the multiset generated for each vertex is given by $r_m(v_1|\mathscr{B}) = r_m(v_n|\mathscr{B}) = \{|1-i+1|\,|2 \leq i \leq n-1, i \equiv 0 \pmod 2\}, r_m(v_k|\mathscr{B}) = \{1, |k-i-1|\,|1 \leq i \leq k-3, i \equiv 0 \pmod 2\} \cup \{|k-i+1|\,|k+1 \leq i \leq n-1, i \equiv 0 \pmod 2\}$ when $3 \leq k \leq n-1, k \equiv 1 \pmod 2$; $r_m(v_2|\mathscr{B}) = r_m(v_{n-1}|\mathscr{B}) = \{2, |1-i+1|\,|k+2 \leq i \leq n-1, i \equiv 0 \pmod 2\}, r_m(v_k|\mathscr{B}) = \{2, 2, |k-i-1|\,|1 \leq i \leq k-4, i \equiv 0 \pmod 2\} \cup \{|k-i+1|\,|k+2 \leq i \leq n-1, i \equiv 0 \pmod 2\}$ when $4 \leq k \leq n-1, k \equiv 0 \pmod 2$ and $r_m(u_k|\mathscr{B}) = \{|k-i||1 \leq i \leq n-2, i \equiv 0 \pmod 2\}, 1 \leq k \leq n-2$.
From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$ when $1 \leq i \leq n-1$, $r_m(u_i|\mathscr{B}) \neq r_m(u_{i+1}|\mathscr{B})$, when $1 \leq i \leq n-3$ and $r_m(u_i|\mathscr{B}) \neq r_m(v_i|\mathscr{B}), r_m(u_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B}), r_m(u_i|\mathscr{B}) \neq r_m(v_{i+2}|\mathscr{B})$, when $1 \leq i \leq n-2$. Thus, $\mathscr{B}$ is a minimum *lmsb* of WVT_n and $lmd(WVT_n) \leq \frac{n-3}{2} \Rightarrow lmd(WVT_n) = \frac{n-3}{2}$. □

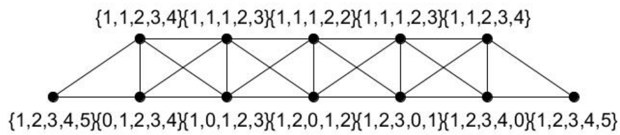
The Diagonal Pratt truss graph DPT_n (Figure 8) is constructed on a path $P_n, n \geq 4$ with vertex set $V(DPT_n) = \{v_i | 1 \leq i \leq n\} \cup \{u_j | 1 \leq j \leq n-2\}$ and edge set $E(DPT_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-3\} \cup \{u_i v_i, u_i v_{i+1}, u_i v_{i+2} | 1 \leq i \leq n-2\}$.

Theorem 17. *Let DPT_n be the Diagonal Pratt truss graph on a path $P_n, n \geq 4$. Then,*
$$lmd(DPT_n) = \begin{cases} n-1, & if \ n \equiv 0 \pmod 2 \\ n-2, & if \ n \equiv 1 \pmod 2 \end{cases}.$$

Proof. From the definition of DPT_n graph, C_3 is a subgraph of DPT_n. Thus, DPT_n is not a bipartite graph and from Note 4, $lmd(DPT_n) \geq 2$. Also, $DPT_n, (n \geq 4)$



(a) Diagonal Pratt truss graph with $n = 7$



(b) Diagonal Pratt truss graph with $n = 7$ with multiset representation of each vertex

Figure 8: An example of a Diagonal Pratt truss graph

contains at least one subgraph K_4 with another vertex adjacent to two vertices of K_4. Thus, from Lemma 3, $lmd(DPT_n) \geq 3$.
The lower bound is proved by contradiction. Let $lmd(DPT_n) < n-2 \Rightarrow$ there exists a $\mathscr{B}$ such that $|\mathscr{B}| = n-3$, be a minimum *lmsb* of DPT_n. Let the vertices $v_k, u_{k-1} \notin \mathscr{B}, 2 \leq k \leq n-1$, then $r_m(u_{k-1}|\mathscr{B}) = r_m(v_k|\mathscr{B})$, a contradiction to $\mathscr{B}$ being a minimum *lmsb* of DPT_n. Thus, to resolve the vertices $v_i, 2 \leq i \leq n-1$ and $u_j, 1 \leq i \leq n-2$, at least one of either $v_k \in \mathscr{B}$ or $u_{k-1} \in \mathscr{B}$. Thus, to resolve the graph DPT_n, at least $n-2$ vertices are necessary and $|\mathscr{B}| \geq n-3 \Rightarrow lmd(DPT_n) \geq \frac{n-3}{2}$.
When $n \equiv 0 \pmod 2$, the following points arise, depending on the vertices in $\mathscr{B}$.

(i) $v_{\frac{n}{2}}, v_{\frac{n}{2}+1} \notin \mathscr{B}$ or $u_{\frac{n-2}{2}}, u_{\frac{n-2}{2}+1} \notin \mathscr{B}$,
Then, $r_m\left(v_{\frac{n}{2}}|\mathscr{B}\right) = r_m\left(v_{\frac{n}{2}+1}|\mathscr{B}\right)$ and $r_m\left(u_{\frac{n-2}{2}}|\mathscr{B}\right) = r_m\left(u_{\frac{n-2}{2}+1}|\mathscr{B}\right)$.

(ii) $v_{\frac{n}{2}}, u_{\frac{n}{2}} \notin \mathscr{B}$,
Then, $r_m\left(v_{\frac{n}{2}}|\mathscr{B}\right) = r_m\left(u_{\frac{n}{2}}|\mathscr{B}\right)$ and $r_m\left(v_{\frac{n+1}{2}}|\mathscr{B}\right) = r_m\left(u_{\frac{n-1}{2}}|\mathscr{B}\right)$.

Both the above points are contradictions to $\mathscr{B}$ being a minimum *lmsb* of DPT_n. To resolve this, another vertex is necessary to be in $\mathscr{B}$. Thus, when $n \equiv 0 \pmod 2), lmd(DPT_n) \geq n-2+1 = n-1$.
To prove the upper bound, consider the following cases depending on the value of n.

Case 1: $n \equiv 1 \pmod 2$,
Suppose $\mathscr{B} = \{v_i | 2 \leq i \leq n-1\}$. Then, the multiset generated for each vertex is given by $r_m(v_k|\mathscr{B}) = \{|k-i| \,|2 \leq i \leq n-1\}, 1 \leq k \leq n, r_m(u_k|\mathscr{B}) = \{|k-i+1|, 1|2 \leq i \leq n-1, i \neq k+1\}$ if $1 \leq k \leq n-2$.

Case 2: $n \equiv 0 \pmod 2$,
Suppose $\mathscr{B} = \{v_i | 2 \leq i \leq n\}$. Then, the multiset generated for each vertex is given by $r_m(v_k|\mathscr{B}) = \{|k-i|\,|2 \leq i \leq n\}, 1 \leq k \leq n, r_m(u_k|\mathscr{B}) = \{|k-i+1|, 1|2 \leq i \leq n, i \neq k+1\}$ if $1 \leq k \leq n-2$.
From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B})$ when $1 \leq i \leq n-1$, $r_m(u_i|\mathscr{B}) \neq r_m(u_i|\mathscr{B})$, when $1 \leq i \leq n-3$ and $r_m(u_i|\mathscr{B}) \neq r_m(v_i|\mathscr{B}), r_m(u_i|\mathscr{B}) \neq r_m(v_{i+1}|\mathscr{B}), r_m(u_i|\mathscr{B}) \neq r_m(v_{i+2}|\mathscr{B})$, when $1 \leq i \leq n-2$. Thus, $\mathscr{B}$ is a minimum *lmsb* of DPT_n and $lmd(DPT_n) \leq n-2 \Rightarrow lmd(DPT_n) = n-2, n \equiv 1 \pmod 2$ and $lmd(DPT_n) \leq n-1 \Rightarrow lmd(DPT_n) = n-1, n \equiv 0 \pmod 2$. □

Line graph of a graph G, denoted by $L(G)$, is constructed on the vertex set $V(L(G)) = E(G)$ and two vertices are adjacent in $(L(G))$ if they are incident in G. The following truss, forms a $L(TS_n), n \equiv 0 \pmod 2$.
The Fink Diagonal Pratt truss graph $FDPT_n$ (Figure 9, Figure 10) is constructed on a path $P_n, n \geq 4, n \equiv 0 (mod\ 4)$ with vertex set $V(FDPT_n) = \{v_i | 1 \leq i \leq n\} \cup \left\{u_j | 1 \leq j \leq \frac{n}{2}\right\}$ and edge set $E(FDPT_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \left\{u_j u_{j+1} | 1 \leq j \leq \frac{n}{2} - 1\right\} \cup \{u_j v_{2j-1}, u_j v_{2j}, u_j v_{2j+1} | 1 \leq i \leq n-2\} \cup \left\{u_j, v_{2j-2} | 2 \leq j \leq \frac{n}{2}\right\}$.

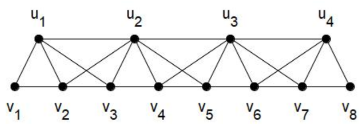


Figure 9: Fink Diagonal Pratt truss graph with $n = 8$

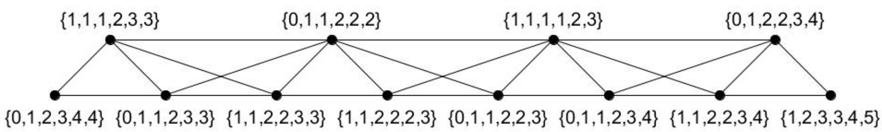


Figure 10: Fink Diagonal Pratt truss graph with $n = 8$ with multiset representation of each vertex

Theorem 18. *Let $FDPT_n$ be the Fink Diagonal Pratt truss graph on a path $P_n, n \geq 4$, and $n \equiv 0 (mod\ 4)$. Then, $lmd(FDPT_n) = \left\lceil \frac{3n}{4} \right\rceil$.*

Proof. From the definition of $FDPT_n$ graph, C_3 is a subgraph of $FDPT_n$. Thus, $FDPT_n$ is not a bipartite graph and from Note 4, $lmd(FDPT_n) \geq 2$. Also, $FDPT_n$ contains at least one subgraph K_4 with a fifth vertex adjacent to two vertices of K_4. Thus, from Lemma 3, $lmd(FDPT_n) \geq 3$. The set of vertices $\{v_{i+j} | i \equiv 1$

$\pmod 2), 0 \le j \le 3\} \cup \{u_{\frac{i+1}{2}}, u_{\frac{i+1}{2}+1}\}$ forms an induced graph K_4 with a fifth vertex adjacent to two vertices of K_4. From Lemma 3, each such set of four vertices require at least three vertices in $\mathscr{B}$ to resolve them. Thus, $lmd(FDPT_n) \ge \lceil \frac{3n}{4} \rceil$.
Now, suppose $\mathscr{B} = \{v_i | i \equiv 1, 2 (mod\ 4)\} \cup \{u_j | j \equiv 0 \pmod 2\}$, then, the multiset generated for each vertex is unique as follows. When $i \equiv 1 \pmod 2$ and $i \le \frac{n}{2}, d(u_{\frac{n}{2}}, v_i) \ne d(u_{\frac{n}{2}}, v_{i+1}); i \equiv 1 \pmod 2$ and $i > \frac{n}{2}, d(v_1, v_i) \ne d(v_1, v_{i+1}); i \equiv 0 \pmod 2, 0 \in r_m(v_i|\mathscr{B}), 0 \notin r_m(v_{i+1}|\mathscr{B}); j \equiv 0 \pmod 2, 0 \in r_m(u_j|\mathscr{B}), 0 \notin r_m(u_{j-1}|\mathscr{B}), r_m(u_{j+1}|\mathscr{B}); 0 \in r_m(u_j|\mathscr{B}), 0 \notin r_m(v_{2j-1}), r_m(v_{2j}|\mathscr{B})$ or vice versa; $j \le \frac{n}{4}, d(u_{\frac{n}{2}}, u_j) \ne d(u_{\frac{n}{2}}, v_{2j+1}); j > \frac{n}{4}, d(v_1, u_j) \ne d(v_1, v_{2j-2})$.
From this, the end vertices of each edge carry a distinct multiset, $r_m(v_i|\mathscr{B}) \ne r_m(v_{i+1}|\mathscr{B}), r_m(u_j|\mathscr{B}) \ne r_m(u_{j+1}|\mathscr{B}), r_m(u_j|\mathscr{B}) \ne r_m(v_{2j-1}|\mathscr{B}), r_m(u_j|\mathscr{B}) \ne r_m(v_{2j}|\mathscr{B}), r_m(u_j|\mathscr{B}) \ne r_m(v_{2j+1}|\mathscr{B})$ when $1 \le i \le n-1, 1 \le j \le \frac{n}{2} - 1$ and $r_m(u_j|\mathscr{B}) \ne r_m(v_{2j-2}|\mathscr{B})$ when $2 \le j \le \frac{n}{2}$. Thus, $\mathscr{B}$ is a minimum *lmsb* of $FDPT_n$ and $lmd(FDPT_n) \le |\mathscr{B}| = \lceil \frac{n}{2} \rceil + \lceil \frac{n}{4} \rceil = \lceil \frac{3n}{4} \rceil \Rightarrow lmd(FDPT_n) = \lceil \frac{3n}{4} \rceil$. □

7. Conclusion

In conclusion, our study on the LMD of truss structures, following our discussion of LMD in triangular snake graphs, has illuminated promising avenues for structural monitoring and assessment. Using the *lmsb* of the Triangular Snake graphs, LMD of the various truss structures can be determined and $lmd(TS_n)$ is lesser than or equal to the LMD of Truss structures, in general, where the exact values depend on their individual structure. In comparison, the Double Fink truss and the Diagonal Pratt truss can generally take higher loads due to better load distribution, resistance to bending. The Diagonal Pratt truss is better for longer spans without intermediate support, offering greater geometric strength, while the Double Fink truss can be used for moderate to longer spans with high stiffness and stability, and the Fink Diagonal Pratt truss is better suited for shorter spans and has moderate load capacity. In the meanwhile, Fink Diagonal Pratt Truss is simpler and potentially less expensive to construct, has an easier design and analysis due to straightforward geometry and requires less height clearance than the Diagonal Pratt Truss and the Double Fink truss. Also, our analysis has shown us that $lmd(TS_n) \le lmd(DFT_n) \le lmd(DPT_n) \le lmd(FDPT_n)$ making it more advantageous in terms of the number of reference points needed to locate each joint in the truss, so as to monitor the overall health of the truss.

By applying the principles of graph theory to triangular snake graphs and truss systems, we have gained valuable insights into their local redundancy and robustness, facilitating enhanced monitoring strategies. Our research has highlighted the potential of LMD analysis in optimizing sensor placement, and enabling real-time monitoring of truss structures. Furthermore, the synergistic relationship between LMD analysis and the triangular snake graph structure has opened up new possibilities for advancing the field of structural health monitoring. This concept, based on triangular snake graph, can be extended to a variety of combination trusses, which will further help structural engineers to analyse the safety and stability of new complex trusses or help them plan and design bridges with longer span, with higher load carrying capacity, or design bridges with other constraints. With further research and development, this interdisciplinary approach holds great promise for improving the safety, reliability, and performance of truss systems across a wide range of engineering applications.

Conflict of Interest: The authors declare no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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