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Research Article

Some remarks on the Roman domination number of the complement of the zero-divisor graph of a commutative ring

S. Visweswaran

Department of Mathematics, Saurashtra University, Rajkot, 360005, India
s_visweswaran2006@yahoo.co.in

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Abstract: Let R be commutative ring with identity that admits at least one nonzero zero-divisor. We denote the set of all zero divisors of R, the zero-divisor graph of R, and the complement of the zero-divisor graph of R by $Z(R), \Gamma(R)$, and $(\Gamma(R))^c$, respectively. Recall that $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)\backslash\{0\}$ and distinct vertices x and y are adjacent if and only if $xy = 0$. For a graph G, we denote its domination number by $\gamma(G)$ and its Roman domination number by $\gamma_{\mathcal{R}}(G)$. This paper aims to compute $\gamma_{\mathcal{R}}((\Gamma(R))^c)$ with the assumption that $\gamma((\Gamma(R))^c) < \infty$. If $Z(R)$ is an ideal of R, then we provide some necessary (respectively, sufficient) conditions such that $\gamma((\Gamma(R))^c) < \infty$ and we compute $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. If $Z(R)$ is not an ideal of R, then it is known that $\gamma((\Gamma(R))^c) = 2$, and we have verified that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{2, 3, 4\}$ and we characterize R such that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 2, 3$ or 4.

Keywords: reduced ring, complement of the zero-divisor graph, domination number, Roman domination number.

AMS Subject Classification: 13A70, 05C25

1. Introduction

The rings considered in this paper are commutative with identity that admit at least one nonzero zero-divisor. Throughout this paper, unless otherwise specified, we use R to denote a ring. The graphs considered in this paper are simple and undirected and we do not put any restriction on the size of the vertex set of G. We use $G = (V, E)$ to denote a graph. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Motivated by the research work of Beck [8], Anderson and Nasser [1], and Anderson and Livingston [3], several researchers have associated graphs with commutative rings and investigated the interplay between graph theory and commutative ring theory. For a detailed and well-written book which studied several graphs constructed over commutative rings, we refer to [2].

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Let $G = (V, E)$ be a graph. The definitions of a dominating set of G, a γ-set of G, and the domination number of G are well-known, see [6, Chapter 10, Section 2]. The domination number of graphs associated with commutative rings has been studied by several authors, for example, see [4, 11, 15–17, 21].

The concept of a Roman domination number of a graph $G = (V, E)$ was introduced and studied by Cockayne et al., see [9]. Recall that a *Roman dominating function* (RDF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ [9, p.12]. Let $i \in \{0, 1, 2\}$. Let us denote $\{v \in V \mid f(v) = i\}$ by V_i. Then, $V = \bigcup_{i=0}^{2} V_i$ with $V_i \cap V_j = \emptyset$ for all distinct $i, j \in \{0, 1, 2\}$. We denote f by $f = (V_0, V_1, V_2)$.

As $Z(R)\backslash\{0\}$ can be infinite, we allow graphs to be infinite in this paper. For an infinite graph $G = (V, E)$, proceeding as in the proof of [9, Proposition 1], one can show that G admits at least one Roman dominating function $f = (V_0, V_1, V_2)$ such that both V_1 and V_2 are finite if and only if $\gamma(G) < \infty$.

Let $f = (V_0, V_1, V_2)$ be a Roman dominating function on G such that both V_1 and V_2 are finite. Then, the *weight* of f is $f(V) = |V_1| + 2|V_2|$. Let us denote $\{f = (V_0, V_1, V_2) \mid f \text{ is an RDF on } G \text{ such that both } V_1 \text{ and } V_2 \text{ are finite}\}$ by $\mathcal{F}(G)$. The *Roman domination number* of G, denoted by $\gamma_{\mathcal{R}}(G)$, is the minimum of $\{f(V) \mid f \in \mathcal{F}(G)\}$. Since for any $f \in \mathcal{F}(G)$, $f(V) \in \mathbb{N}$, there exists at least one $f \in \mathcal{F}(G)$ such that $f(V) = \gamma_{\mathcal{R}}(G)$. We say that a Roman dominating function f on G is a *$\gamma_{\mathcal{R}}$-function* on G if $f(V) = \gamma_{\mathcal{R}}(G)$ [9, p.13]. In the case $\gamma(G) < \infty$, proceeding as in the proof of [9, Proposition 1], it can be shown that $\gamma(G) \leq \gamma_{\mathcal{R}}(G) \leq 2\gamma(G)$. If G has at least one edge, then proceeding as in the proof of [9, Proposition 2], one can verify that $\gamma_{\mathcal{R}}(G) \geq \gamma(G) + 1$.

Let T be a ring (T can be an integral domain). For basic definitions and notations from commutative ring theory, the reader can refer to any of the standard textbooks, for example, [5, 10, 12]. However, we mention the notations which are often used in this paper. We denote the set of all prime ideals, the set of all maximal ideals, and the set of all minimal prime ideals of T by $\mathrm{Spec}(T)$, $\mathrm{Max}(T)$, and $\mathrm{Min}(T)$, respectively. We denote the nilradical of T by $\mathrm{Nil}(T)$. We say that T is *reduced* if $\mathrm{Nil}(T) = (0)$. We denote the Krull dimension of T by $\dim T$. For any $n \in \mathbb{N}\backslash\{1\}$, we denote the ring of integers modulo n by $\mathbb{Z}_n$. For a subset A of T, we denote $A\backslash\{0\}$ by A^*. We denote the polynomial ring in one variable X over T by $T[X]$ and the polynomial ring in n ($n \geq 2$) variables $X_1, X_2, \ldots, X_n$ over T by $T[X_1, X_2, \ldots, X_n]$. We denote the cardinality of a set A by $|A|$. We say that T is *quasi-local* if $|\mathrm{Max}(T)| = 1$. A Noetherian quasi-local ring is called a *local ring*. If A is a proper subset of a set B, then we denote it by $A \subset B$.

Let T be a ring. We denote the Jacobson radical of T by $\mathrm{J}(T)$. Let T be such that $|\mathrm{Max}(T)| \geq 2$. The comaximal ideal graph of T is denoted by $\mathscr{C}(T)$, see [22]. Khojasteh and Heydari [13] computed the Roman domination number of $\mathscr{C}(T)$ with the assumption that $\gamma(\mathscr{C}(T)) < \infty$. For a ring R with $|Z(R)^*| \geq 2$ and $\gamma(\Gamma(R)) < \infty$, $\gamma_{\mathcal{R}}(\Gamma(R))$ is determined in [19]. Motivated by the research work [13, 19], this paper aims to determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$ with the assumption that $\gamma((\Gamma(R))^c) < \infty$.

This paper consists of five sections. Section 1 is on introduction and Section 5 is on conclusion. In Section 2, we mention some preliminary results that are known in the literature. If $Z(R)$ is an ideal of R with $|Z(R)^*| \geq 2$, then we do not know any necessary and sufficient condition such that $\gamma((\Gamma(R))^c) < \infty$. However, in Section 3, in some special cases, we are able to provide a necessary and sufficient condition such that $\gamma((\Gamma(R))^c) < \infty$ and we determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. If $Z(R)$ is not an ideal of R, then $\gamma((\Gamma(R))^c) = 2$ by [21, Lemma 2.3]. In Section 4, we observe that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{2, 3, 4\}$ and we are able to characterize R such that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 2$. If R is reduced, then necessary and sufficient conditions are determined such that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$. If R is not reduced, then with an additional condition, we are able to characterize R such that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ and we illustrate the results obtained by us with suitable examples.

2. Some preliminary results

In this section, we mention some preliminary results that are needed for proving the main results of this paper.

Lemma 1. *Let $G = (V, E)$ be a graph such that $|V| \geq 2$. If G has no isolated vertex, then $\gamma(G^c) \geq 2$.*

Proof. Let $v \in V$. By the assumptions on G, there exists $u \in V \backslash \{v\}$ such that v and u are adjacent in G. Therefore, $\{v\}$ cannot be a dominating set of G^c, so $\gamma(G^c) \geq 2$. □

Corollary 1. *If $|Z(R)^*| \geq 2$, then $\gamma((\Gamma(R))^c) \geq 2$.*

Proof. Since $V(\Gamma(R)) = Z(R)^*$ and $\Gamma(R)$ is connected by [3, Theorem 2.3], it follows that $\Gamma(R)$ has no isolated vertex. Hence, $\gamma((\Gamma(R))^c) \geq 2$ by Lemma 1. □

Let $G = (V, E)$ be a graph. For any $u, v \in V$, we denote the distance between u and v in G by $d_G(u, v)$ or $d(u, v)$. If G is connected, then we denote the diameter of G by $\mathrm{diam}(G)$.

Lemma 2. *If a graph $G = (V, E)$ is such that G^c is connected and $\mathrm{diam}(G^c) \geq 3$, then $\gamma(G) = 2$ and $3 \leq \gamma_{\mathcal{R}}(G) \leq 4$.*

Proof. As G^c is connected and $\mathrm{diam}(G^c) \geq 3$ by assumption, we can find distinct u and v from V such that $d_{G^c}(u, v) \geq 3$. Then, $\{u, v\}$ is a dominating set of G by [21, Lemma 2.11], so $\gamma(G) \leq 2$. As $|V| \geq 3$ and G^c is connected, G^c has no isolated vertex. From $(G^c)^c = G$, we obtain from Lemma 1 that $\gamma(G) \geq 2$, so $\gamma(G) = 2$. Since u and v are adjacent in G, we obtain that G has at least one edge. Hence, $3 = \gamma(G) + 1 \leq \gamma_{\mathcal{R}}(G) \leq 2\gamma(G) = 4$ by [9, Propositions 1 and 2]. □

We know from [3, Theorem 2.3] that $\Gamma(R)$ is connected and $\mathrm{diam}(\Gamma(R)) \leq 3$. There are rings R such that $\mathrm{diam}(\Gamma(R)) = 3$, see [14, Section 5].

Remark 1. If $\mathrm{diam}(\Gamma(R)) = 3$, then by applying Lemma 2 with $G = (\Gamma(R))^c$, we obtain that $\gamma((\Gamma(R))^c) = 2$ and $3 \leq \gamma_{\mathcal{R}}((\Gamma(R))^c) \leq 4$.

Lemma 3. *Let $G = (V, E)$ be a graph such that $|V| \geq 3$. If $\gamma(G) = 2$, then $3 \leq \gamma_{\mathcal{R}}(G) \leq 4$.*

Proof. For a proof of this lemma, see the proof of [18, Lemma 2.5]. □

Lemma 4. *Let $G = (V, E)$ be a graph such that $|V| \geq 3$. If $\gamma(G) = 2$, then $\gamma_{\mathcal{R}}(G) = 3$ if and only if there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ on G such that either $V_2 = V_0 = \emptyset$, and $|V_1| = 3$ or $|V_1| = |V_2| = 1$.*

Proof. For a proof of this lemma, one can refer to the proof of [18, Lemma 2.6]. □

Let $G = (V, E)$ be a graph. For any $v \in V$, we denote the degree of v in G by $d_G(v)$ or $d(v)$, see [6, Definition 1.4.1].

Corollary 2. *Let $G = (V, E)$ be a graph such that $|V| \geq 4$. If $\gamma(G) = 2$, then the following statements are equivalent:*
(1) $\gamma_{\mathcal{R}}(G) = 3$.
(2) *There exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ on G such that $|V_1| = |V_2| = 1$.*
(3) *There exists $x \in V$ such that $d(x) = 1$ in G^c.*

Proof. For a proof, refer to the proof of [18, Corollary 2.7]. □

3. Some results on $\gamma_{\mathcal{R}}((\Gamma(R))^c)$, where $Z(R)$ is an ideal of R

In this section, we assume that $Z(R)$ is an ideal of R, $\gamma((\Gamma(R))^c) < \infty$, and discuss some results on $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. If $|Z(R)^*| = 1$, then $\gamma((\Gamma(R))^c) = 1 = \gamma_{\mathcal{R}}((\Gamma(R))^c)$ and for a characterization of rings R such that $|Z(R)^*| = 1$, see [3, Example 2.1(a)]. Hence, we assume that $|Z(R)^*| \geq 2$. If $Z(R)$ is an ideal of R, then $Z(R) \in \mathrm{Spec}(R)$ and throughout this section, we denote $Z(R)$ by $\mathfrak{p}$. For examples of rings R with $Z(R)$ is an ideal of R for which $(\Gamma(R))^c$ does not admit any finite dominating set, the reader is referred to [21, Examples 2.4, 2.9, 2.14]. We do not know any necessary and sufficient condition such that $\gamma((\Gamma(R))^c) < \infty$. However, there are rings R such that $\gamma((\Gamma(R))^c) < \infty$. For such rings R, we want to determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. Hence, the case $Z(R)$ is an ideal of R needs to be studied. For any ring T with $|Z(T)^*| \geq 2$, necessary and sufficient conditions are known that $\mathrm{diam}(\Gamma(T)) = 1, 2$ or 3, see [14, Section 2]. In view of Remark 1, we are interested in studying whether such characterizations help us to characterize R such that $\gamma((\Gamma(R))^c) < \infty$.

First, we assume that $\mathrm{diam}(\Gamma(R)) = 1$. Then, we obtain from [3, Theorem 2.8] that $\mathfrak{p}^2 = (0)$. We begin with the following proposition.

Proposition 1. *[21, Remark 2.7(i)] Let $|Z(R)^*| \geq 2$. If $\mathfrak{p}^2 = (0)$, then the following statements are equivalent:*
(1) $\gamma((\Gamma(R))^c) < \infty$.
(2) $|\mathfrak{p}| < \infty$.
(3) R is finite.
If (1) holds, then $\gamma((\Gamma(R))^c) = |\mathfrak{p}^*| = \gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Proof. For a proof of this lemma, see [21, Remark 2.7(i)].
Assume that (1) holds. Then, by (1) $\Rightarrow$ (2) of this proposition, $\mathfrak{p}^*$ is finite. Since $\mathfrak{p}^2 = (0)$, $(\Gamma(R))^c$ has no edges. Thus, $(\Gamma(R))^c$ is a finite graph with no edges. So, we obtain from [9, Proposition 2] that $\gamma((\Gamma(R))^c) = |\mathfrak{p}^*| = \gamma_{\mathcal{R}}((\Gamma(R))^c)$. □

For any ring T and an ideal I of T, the *annihilator of I in T*, denoted by $\mathrm{Ann}_T(I)$ or $\mathrm{Ann}(I)$, is defined as $\mathrm{Ann}(I) = \{t \in T \mid tI = (0)\}$. It is not hard to verify that $\mathrm{Ann}(I)$ is an ideal of T. If $I = Tx$ for some $x \in T$, then $t \in T$ satisfies $tI = (0)$ if and only if $tx = 0$. We use to denote $\mathrm{Ann}(Tx)$ by $\mathrm{Ann}(x)$.

Remark 2. Let $r \in R^*$. We verify in this remark that $r \in (\mathrm{Ann}(\mathfrak{p}))^*$ if and only if $\mathfrak{p} = \mathrm{Ann}(r)$. If $r \in (\mathrm{Ann}(\mathfrak{p}))^*$, then $\mathfrak{p}r = (0)$, so $\mathfrak{p} \subseteq \mathrm{Ann}(r)$. As $r \neq 0$, we get that $\mathrm{Ann}(r) \subseteq Z(R) = \mathfrak{p}$. Hence, $\mathfrak{p} = \mathrm{Ann}(r)$. Conversely, if $\mathfrak{p} = \mathrm{Ann}(r)$, then $\mathfrak{p}r = (0)$, so $r \in (\mathrm{Ann}(\mathfrak{p}))^*$.

To proceed further, we first provide some necessary conditions such that $\gamma((\Gamma(R))^c) < \infty$.

Lemma 5. *If $\gamma((\Gamma(R))^c) < \infty$, then $(\mathrm{Ann}(\mathfrak{p}))^*$ is finite.*

Proof. If $\gamma((\Gamma(R))^c) < \infty$, then it follows from the proof of [21, Proposition 2.8(i)] that $(\mathrm{Ann}(\mathfrak{p}))^*$ is finite. □

Lemma 6. *If $(\mathrm{Ann}(\mathfrak{p}))^*$ is non-empty and $\gamma((\Gamma(R))^c) < \infty$, then $R/\mathfrak{p}$ is finite, so $\mathfrak{p} \in \mathrm{Max}(R)$.*

Proof. For a proof of this lemma, see the proof of [21, Lemma 2.15(i)]. □

We use f.g. for finitely generated. If $\mathfrak{p}^2 \neq (0)$ and $\mathfrak{p}$ is a f.g. ideal of R, then in the following proposition, we provide a necessary and sufficient condition such that $\gamma((\Gamma(R))^c) < \infty$ and in the case $\gamma((\Gamma(R))^c)$ is finite, we provide an upper bound for $\gamma((\Gamma(R))^c)$ (respectively, $\gamma_{\mathcal{R}}((\Gamma(R))^c)$).

Proposition 2. *[21, Proposition 2.8(i)] If $\mathfrak{p}^2 \neq (0)$ and $\mathfrak{p}$ is a f.g. ideal of R, then the following statements are equivalent:*
(1) $\gamma((\Gamma(R))^c) < \infty$.
(2) $(\mathrm{Ann}(\mathfrak{p}))^*$ is a finite subset of $Z(R)^*$.
If (2) holds, then $\gamma((\Gamma(R))^c) \leq |(\mathrm{Ann}(\mathfrak{p}))^*| + n$, where n is the least positive integer such that $\mathfrak{p}$ is generated by n elements, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) \leq |(\mathrm{Ann}(\mathfrak{p}))^*| + 2n$.

Proof. For the proof of (1) $\Leftrightarrow$ (2) of this proposition, refer to the proof of [21, Proposition 2.8(i)].
Assume that (2) holds. Let $n \in \mathbb{N}$ be least such that $\mathfrak{p}$ is generated by n elements. Let $p_1, \ldots, p_n \in \mathfrak{p}$ be such that $\mathfrak{p} = \sum_{i=1}^{n} Rp_i$. We know from the proof of [21, Proposition 2.8(i)] that $(\mathrm{Ann}(\mathfrak{p}))^* \cup \{p_1, \ldots, p_n\}$ is a dominating set of $(\Gamma(R))^c$. So, $\gamma((\Gamma(R))^c) \leq |(\mathrm{Ann}(\mathfrak{p}))^*| + n$. Let $|(\mathrm{Ann}(\mathfrak{p}))^* \cap \{p_1, \ldots, p_n\}| = m$. Since $\mathfrak{p}^2 \neq (0)$ by hypothesis, $\mathfrak{p}p_i \neq (0)$ for at least one $i \in \{1 \ldots, n\}$. Hence, $m < n$. We can assume without loss of generality that $\{p_1, \ldots, p_n\} \backslash (\mathrm{Ann}(\mathfrak{p}))^* = \{p_{i_1}, \ldots, p_{i_{n-m}}\}$. Let $V_2 = \{p_{i_1}, \ldots, p_{i_{n-m}}\}$, $V_1 = (\mathrm{Ann}(\mathfrak{p}))^*$, and $V_0 = Z(R)^* \backslash (V_1 \cup V_2)$. Then, $f = (V_0, V_1, V_2)$ is an RDF on $(\Gamma(R))^c$ and the weight of f equals $|V_1| + 2|V_2| = |(\mathrm{Ann}(\mathfrak{p}))^*| + 2(n-m)$. Therefore, $\gamma_{\mathcal{R}}((\Gamma(R))^c) \leq |(\mathrm{Ann}(\mathfrak{p}))^*| + 2(n-m) \leq |(\mathrm{Ann}(\mathfrak{p}))^*| + 2n$. □

Remark 3. We next assume that $\mathrm{diam}(\Gamma(R)) = 2$ and try to characterize R such that $\gamma((\Gamma(R))^c) < \infty$ and determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. Since $Z(R)$ is an ideal of R, it follows from [14, Theorem 2.6 (3)(ii)] that $\mathrm{diam}(\Gamma(R)) = 2$ if and only if $\mathfrak{p}^2 \neq (0)$ and for any distinct $x, y \in Z(R)^*$, $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) \neq (0)$.

If $k \geq 3$ is least such that $\mathfrak{p}^k = (0)$, then $\mathfrak{p}x = (0)$ for any nonzero $x \in \mathfrak{p}^{k-1}$. As $\mathfrak{p}^2 \neq (0)$ and for any $a, b \in Z(R)^*$, $ax = bx = 0$, it follows that $\mathrm{diam}(\Gamma(R)) = 2$. If $\mathfrak{p}$ is f.g., then in the following lemma, we characterize R such that $\gamma((\Gamma(R))^c) < \infty$.

Lemma 7. *If $\mathfrak{p}$ is a f.g. ideal of R and $k \geq 3$ is least such that $\mathfrak{p}^k = (0)$, then $\gamma((\Gamma(R))^c) < \infty$ if and only if R is finite.*

Proof. Assume that $\gamma((\Gamma(R))^c) < \infty$. We obtain from Lemma 6 that $R/\mathfrak{p}$ is finite, so $\mathfrak{p} \in \mathrm{Max}(R)$. Since $\mathfrak{p}^k = (0)$, it follows that $\mathrm{Spec}(R) = \mathrm{Max}(R) = \{\mathfrak{p}\}$. Since $\mathfrak{p}$ is a f.g. ideal of R, for each $i \in \{1, 2, \ldots, k-1\}$, $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ is a finite-dimensional vector space over the finite field $R/\mathfrak{p}$. So, $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ is finite for each $i \in \{1, 2, \ldots, k-1\}$. As $\mathfrak{p}^k = (0)$, $\mathfrak{p}^{k-1}, \mathfrak{p}^{k-2}/\mathfrak{p}^{k-1}, \ldots, \mathfrak{p}/\mathfrak{p}^2$ are finite. If a submodule N of a module M is such that both N and M/N are finite, then M is finite. From $\mathfrak{p}^{k-1}$ and $\mathfrak{p}^{k-2}/\mathfrak{p}^{k-1}$ are finite, we get that $\mathfrak{p}^{k-2}$ is finite. By applying this argument $k-2$ times, we arrive at $\mathfrak{p}$ is finite. Since $R/\mathfrak{p}$ is finite, it follows that R is finite.
Conversely, if R is finite, then $Z(R)^*$ is finite, so $\gamma((\Gamma(R))^c) < \infty$. □

We next mention an example of a finite local ring R due to Anderson and Naseer, see [1, p.501] and determine $\gamma((\Gamma(R))^c)$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Example 1. Let $T = \mathbb{Z}_4[X, Y, Z]$ and let I be the ideal of T generated by $\{X^2 - 2, Y^2 - 2, Z^2, 2X, 2Y, 2Z, XY, YZ - 2, XZ\}$. If $R = T/I$, then $\gamma((\Gamma(R))^c) = 4$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 6$.

Proof. Anderson and Naseer provided the ring R as a counterexample to Beck's conjecture, see [8, Conjecture 1, p.208]. Let us denote $X + I$ by x, $Y + I$ by y, and $Z + I$ by z. As R contains a copy of $\mathbb{Z}_4$, we identify $n + I$ with n for any $n \in \mathbb{Z}_4$. We know that R is a finite local ring with $\mathfrak{m} = Rx + Ry + Rz$ as its unique maximal, $\mathfrak{m}^3 = (0)$ but $\mathfrak{m}^2 \neq (0)$, $|R| = 32$, $Z(R) = \mathfrak{m} = \{0, 2, x, y, z, x+2, y+2, z+2, x+y, y+z, z+x, x+y+2, y+z+2, z+x+2, x+y+z, x+y+z+2\}$, and $\mathfrak{m}^2 = \{0, 2\}$, see [1, p.501]. Since $2x = 2y = 2z = 0 + I$, it follows that $\mathfrak{m} \subseteq \text{Ann}(2)$. So, $\mathfrak{m} = \text{Ann}(2)$. From $\mathfrak{m}^2 \neq (0+I)$ and $2m = 0+I$ for any $m \in Z(R)^*$, it follows that $\text{diam}(\Gamma(R)) = 2$. We next proceed to determine $\gamma((\Gamma(R))^c)$. The arguments given below to determine the exact value of $\gamma((\Gamma(R))^c)$ depends on $\text{Ann}(m)$ for each $m \in Z(R)^*$, and for finding $\text{Ann}(m)$, one can refer to the multiplication table for $\mathfrak{m}$, see [1, p.503]. Let $m \in Z(R)^* \backslash \{2\}$. As 2 annihilates $\mathfrak{m}$, it follows that $\text{Ann}(m) = \text{Ann}(m+2)$. As $\text{Ann}(m)$ contains only 8 elements for any $m \in Z(R)^* \backslash \{2\}$, see [1, p.503], whereas $|\mathfrak{m}| = 16$, we get that $(\text{Ann}(\mathfrak{m}))^* = \{2\}$. From $x(x+2) = x(x+y) = x(x+z) = x(x+y+z) = x^2 = 2$, $z(y+z) = yz = 2$. $(z+2)x = (z+2)z = 0$, it follows that any $m \in Z(R)^* \backslash \{2, x, z, z+2\}$ is adjacent to either x or z in $(\Gamma(R))^c$. So, $\{2, x, z, z+2\}$ is a dominating set of $(\Gamma(R))^c$. Hence, $\gamma((\Gamma(R))^c) \leq 4$. By Corollary 1, $\gamma((\Gamma(R))^c) \geq 2$. For any distinct $m_1, m_2 \in Z(R)^* \backslash \{2\}$, we can find $m_3 \in Z(R)^* \backslash \{2, m_1, m_2\}$ such that $m_1 m_3 = 0 = m_2 m_3$, see [1, p.503]. Since 2 must be in any dominating set of $(\Gamma(R))^c$, it follows that $(\Gamma(R))^c$ cannot admit any dominating set D with $|D| = 3$. Therefore, $\gamma((\Gamma(R))^c) \geq 4$. Hence, $\gamma((\Gamma(R))^c) = 4$.

We next compute $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. From $yz = 2 \neq 0$, it follows that $(\Gamma(R))^c$ has at least one edge. Hence, $5 = \gamma((\Gamma(R))^c) + 1 \leq \gamma_{\mathcal{R}}((\Gamma(R))^c)$ by [9, Proposition 2]. We claim that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \neq 5$. If $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 5$, then there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ on $(\Gamma(R))^c$ such that either $|V_1| = 3$ and $|V_2| = 1$ or $|V_1| = 1$ and $|V_2| = 2$. If $|V_1| = 3$ and $|V_2| = 1$, then $|V_0| = 11$. Let $V_2 = \{m\}$. Since there is no edge of $(\Gamma(R))^c$ whose one end vertex is in V_1 and the other in V_2 by [9, Proposition 3(b)] and each element of V_0 is adjacent to m in $(\Gamma(R))^c$, it follows that $d(m) = 11$ in $(\Gamma(R))^c$. We arrive at a contradiction, since one can verify that $d(w) < 11$ in $(\Gamma(R))^c$ for each $w \in Z(R)^* \backslash \{2\}$, see [1, p.503]. If $|V_1| = 1$ and $|V_2| = 2$, then as $V_1 \cup V_2$ is a dominating set of $(\Gamma(R))^c$, we get that $\gamma((\Gamma(R))^c) \leq 3$, a contradiction. Therefore, $\gamma_{\mathcal{R}}((\Gamma(R))^c) \neq 5$, so $\gamma_{\mathcal{R}}((\Gamma(R))^c) \geq 6$. We have observed in the previous paragraph that any $m \in Z(R)^* \backslash \{2, x, z, z+2\}$ is adjacent to either x or z in $(\Gamma(R))^c$. Hence, $f' = (V_0' = Z(R)^* \backslash \{2, x, z, z+2\}, V_1' = \{2, z+2\}, V_2' = \{x, z\})$ is an RDF on $(\Gamma(R))^c$, and the weight of f' equals $|V_1'| + 2|V_2'| = 6$. Hence, $\gamma_{\mathcal{R}}((\Gamma(R))^c) \leq 6$, so $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 6$, and f' is a $\gamma_{\mathcal{R}}$-function on $(\Gamma(R))^c$. □

Remark 4. Recall that R is said to be a *chained ring* if the set of ideals of R is linearly ordered by inclusion, see [10, p.184]. If R is a chained ring, then for any $a, b \in Z(R)$, either $\text{Ann}(a) \subseteq \text{Ann}(b)$ or $\text{Ann}(b) \subseteq \text{Ann}(a)$. We can assume without loss of generality that $\text{Ann}(a) \subseteq \text{Ann}(b)$. There exists $x \in R^*$ such that $ax = 0$, so $(a+b)x = 0$. Thus,

$a+b \in Z(R)$. For any $a \in Z(R)$ and for any $c \in R$, $ca \in Z(R)$. Hence $Z(R)$ is an ideal of R, so $Z(R) \in \mathrm{Spec}(R)$. Let us denote $Z(R)$ by $\mathfrak{p}$. If $\gamma((\Gamma(R))^c) < \infty$, then $(\mathrm{Ann}(\mathfrak{p})^*) \neq \emptyset$ by [21, Lemma 2.13(i)]. For an example of a chained ring R such that $(\mathrm{Ann}(\mathfrak{p}))^* = \emptyset$, see [21, Example 2.14].

If $\mathfrak{p}$ is f.g. and if $\mathfrak{p}^2 \neq (0)$, then in the following proposition, we provide a necessary and sufficient condition such that $\gamma((\Gamma(R))^c) < \infty$ and determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Proposition 3. *Let R be a chained ring. Let $Z(R) = \mathfrak{p}$ be such that $\mathfrak{p}^2 \neq (0)$. If $\mathfrak{p}$ is f.g., then the following statements are equivalent:*
(1) $\gamma((\Gamma(R))^c) < \infty$.
(2) $|R/\mathfrak{p}| < \infty$.
If (2) *holds, then* $\gamma((\Gamma(R))^c) = |R/\mathfrak{p}|$, *and* $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |R/\mathfrak{p}| + 1$

Proof. Since $\mathfrak{p}$ is a f.g. ideal of a chained ring R by hypothesis, it follows that $\mathfrak{p} = Rp$ for some $p \in \mathfrak{p}$. As $\mathfrak{p}^2 \neq 0$, we get that $p^2 \neq 0$. If $r \in R^*$ is such that $pr = 0$, then $\mathfrak{p}r = (0)$, so $(\mathrm{Ann}(\mathfrak{p}))^* \neq \emptyset$. For a proof of (1) $\Leftrightarrow$ (2) of this proposition and (2) implies $\gamma((\Gamma(R))^c) = |R/\mathfrak{p}|$, see the proof of [21, Proposition 2.17(i)] and the proof of [21, Proposition 2.17(i)] shows that $(\mathrm{Ann}(\mathfrak{p}))^* \cup \{p\}$ is a dominating set of $(\Gamma(R))^c$ with minimum cardinality. Hence, $|(\mathrm{Ann}(\mathfrak{p}))^*| = |R/\mathfrak{p}| - 1$.
Assume that (2) holds. Then, $\gamma((\Gamma(R))^c) = |R/\mathfrak{p}|$. From $\mathfrak{p}^2 \neq (0)$, it follows that $(\Gamma(R))^c$ has at least one edge. So, $|R/\mathfrak{p}| + 1 \leq \gamma_{\mathcal{R}}((\Gamma(R))^c)$ by [9, Proposition 2]. As $r \in (\mathrm{Ann}(\mathfrak{p}))^*$ and $p \in Z(R)^* \backslash (\mathrm{Ann}(\mathfrak{p}))^*$, $Z(R)^* = (\mathrm{Ann}(\mathfrak{p}))^* \cup (Z(R)^* \backslash (\mathrm{Ann}(\mathfrak{p}))^*)$ is the disjoint union of two non-empty sets. Also, $pz \neq 0$ for each $z \in Z(R)^* \backslash (\mathrm{Ann}(\mathfrak{p}))^*$. The function $f = (V_0 = Z(R)^* \backslash ((\mathrm{Ann}(\mathfrak{p}))^* \cup \{p\}), V_1 = (\mathrm{Ann}(\mathfrak{p}))^*, V_2 = \{p\})$ is an RDF on $(\Gamma(R))^c$ and the weight of f equals $|V_1| + 2|V_2| = |R/\mathfrak{p}| - 1 + 2 = |R/\mathfrak{p}| + 1$. Therefore, $\gamma_{\mathcal{R}}((\Gamma(R))^c) \leq |R/\mathfrak{p}| + 1$. Hence, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |R/\mathfrak{p}| + 1$, and f is a $\gamma_{\mathcal{R}}$-function on $(\Gamma(R))^c$. $\square$

Remark 5. A principal ideal ring R called a *special principal ideal ring* (SPIR) if R has only one prime ideal. If $\mathrm{Spec}(R) = \{\mathfrak{p}\}$, then $\mathfrak{p}$ is principal and we obtain from [5, Proposition 1.8] that $\mathfrak{p} = \mathrm{Nil}(R)$. Let $p \in \mathfrak{p}$ be such that $\mathfrak{p} = Rp$. If $k \geq 2$ is least such that $p^k = 0$, then $\mathfrak{p}^k = (0)$, but $\mathfrak{p}^{k-1} \neq (0)$. We obtain from the proof of [5, Proposition 8.8 ((iii) $\Rightarrow$ (i))] that $\{\mathfrak{p} = Rp, \ldots, \mathfrak{p}^{k-1} = Rp^{k-1}\}$ is the set of all nonzero proper ideals of R. If R is an SPIR with $\mathfrak{p}$ as its only prime ideal, then we denote it by saying that $(R, \mathfrak{p})$ is an SPIR. As the set of ideals of R is linearly ordered by inclusion, R is a chained ring with $Z(R) = \mathfrak{p}$.

If $(R, \mathfrak{p})$ is an SPIR with $|Z(R)^*| \geq 2$, then in the following corollary, we characterize R such that $\gamma((\Gamma(R))^c) < \infty$ and in the case $\gamma((\Gamma(R))^c) < \infty$, we determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Corollary 3. *Let $(R, \mathfrak{p})$ be an SPIR with $|Z(R)^*| \geq 2$ and let $k \geq 2$ be least such that $\mathfrak{p}^k = (0)$. Then, the following statements hold.*

(i) If $k = 2$, then $\gamma((\Gamma(R))^c) < \infty$ if and only if R is finite and in the case R is finite, $\gamma((\Gamma(R))^c) = |\mathfrak{p}^*| = \gamma_{\mathcal{R}}((\Gamma(R))^c)$.
(ii) If $k \geq 3$, then $\gamma((\Gamma(R))^c) < \infty$ if and only if R is finite, and if $\gamma((\Gamma(R))^c) < \infty$, then $\gamma((\Gamma(R))^c) = |R/\mathfrak{p}|$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |R/\mathfrak{p}| + 1$.

Proof. (i) Assume that $k = 2$. We obtain from Proposition 1 that $\gamma((\Gamma(R))^c) < \infty$ if and only if R is finite and if R is finite, then $\gamma((\Gamma(R))^c) = |\mathfrak{p}^*| = \gamma_{\mathcal{R}}((\Gamma(R))^c)$.
(ii) Since $(R, \mathfrak{p})$ is an SPIR, $\mathfrak{p}$ is principal. Assume that $k \geq 3$. Then, $\gamma((\Gamma(R))^c) < \infty$ if and only if R is finite by Lemma 7. Assume that $\gamma((\Gamma(R))^c) < \infty$. Since R is a chained ring with $Z(R) = \mathfrak{p}$ and $\mathfrak{p}^2 \neq (0)$, it follows from Proposition 3 that $\gamma((\Gamma(R))^c) = |R/\mathfrak{p}|$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |R/\mathfrak{p}| + 1$. □

The following example illustrates Corollary 3. We denote the finite field containing exactly p^n elements (p is a prime number and $n \geq 1$) by $\mathbb{F}_{p^n}$.

Example 2. (1) Let p be a prime number and let $R = \mathbb{Z}_{p^k}$ $(k \geq 3)$. Then, $(R, \mathfrak{p} = pR)$ is a finite SPIR with $k \geq 3$ is least such that $p^k = 0$, so $\gamma((\Gamma(R))^c) = p$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = p+1$.
(2) Let $T = \mathbb{F}_{p^n}[X]$ and let $R = T/TX^K$ $(k \geq 3)$. Then, $(R, \mathfrak{p} = TX/TX^k)$ is a finite SPIR, $k \geq 3$ is least such that $\mathfrak{p}^k = (0 + TX^k)$, so $\gamma((\Gamma(R))^c) = p^n$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = p^n + 1$.

Proof. (i) The ring $R = \mathbb{Z}_{p^k}$ $(k \geq 3)$ is a principal ideal ring with $|R| = p^k$, $pR \in \mathrm{Max}(R)$ and $(pR)^k = (0)$, but $(pR)^{k-1} \neq (0)$. Hence, $\mathrm{Spec}(R) = \{pR\}$. So, (R, pR) is a finite SPIR. From $|R/pR| = p$, we obtain from Corollary 3 (ii) that $\gamma((\Gamma(R))^c) = p$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = p + 1$.
(ii) Since $T = \mathbb{F}_{p^n}[X]$ is a principal ideal domain (P.I.D.), we obtain that $R = T/TX^k$ $(k \geq 3)$ is a principal ideal ring. As $TX \in \mathrm{Max}(T)$, it follows that $\mathfrak{p} = TX/TX^k \in \mathrm{Max}(R)$. If we denote $X + TX^k$ by x, then $\mathfrak{p} = Rx$ with $(Rx)^k = (0 + TX^k)$, but $(Rx)^{k-1} \neq (0 + TX^k)$. Since $R/\mathfrak{p}$ is ring-isomorphic T/TX and T/TX is ring-isomorphic to $\mathbb{F}_{p^n}$, it follows that $|R/\mathfrak{p}| = p^n$. Proceeding as in the proof of Lemma 7, one can show that $|R| = p^{kn}$. Thus, $(R, \mathfrak{p})$ is a finite SPIR and $k \geq 3$ is least such that $\mathfrak{p}^k = (0 + TX^k)$. From Corollary 3 (ii), we obtain that $\gamma((\Gamma(R))^c) = p^n$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = p^n + 1$. □

Let T be a ring. For elements $t_1, \ldots, t_n \in T$, we denote the ideal of T generated by $\{t_1, \ldots, t_n\}$ by $(t_1, \ldots, t_n)$. We denote the set of all proper ideals of T by $\mathbb{I}(T)$ and $\mathbb{I}(T)\backslash\{(0)\}$ by $\mathbb{I}(T)^*$. For any field F, the quotient field of $F[X]$ is called the *field of rational functions in one variable X over F* and is denoted by $F(X)$.
Recall that an integral domain V is called a *valuation ring* if the set of ideals of V is linearly ordered by inclusion [10, p.181]. In the following example, we provide an infinite chained ring R such that $\gamma((\Gamma(R))^c) < \infty$ to illustrate that Corollary 3 may fail to hold for a chained ring that is not an SPIR.

Example 3. Let $K = \mathbb{Z}_2(X)$ and let $V = K[Y]_{(Y)}$. Let $W = \mathbb{Z}_2[X]_{(X)} + VY$. If $R = W/WY$, then R is an infinite chained ring such that $Z(R) = WX/WY$ is principal, $R/Z(R)$ is ring-isomorphic to $\mathbb{Z}_2$, $\gamma((\Gamma(R))^c) = 2$, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$.

Proof. The ring V is a local principal ideal domain with VY as its unique maximal ideal. Hence, $\dim V = 1$. Therefore, $\mathbb{I}(V)^* = \{VY^n \mid n \in \mathbb{N}\}$ by [5, Theorem 9.2 $((iii) \Rightarrow (v))$]. Hence, V is a valuation ring and it is called a discrete valuation ring, see [5, Theorem 9.2]. The valuation ring V can be expressed in the form $V = K + VY$, where VY is the unique maximal ideal of V. The ring $D = \mathbb{Z}_2[X]_{(X)}$ is a local principal ideal domain with DX as its unique maximal ideal. Thus, $\dim D = 1$. So, D is a discrete valuation ring. As $W = D + VY$ and the quotient field of D equals K, we obtain from [7, Theorem 2.1(h)] that W is a valuation ring. Observe that $D = \mathbb{Z}_2 + DX$. Thus, $W = \mathbb{Z}_2 + DX + VY$. Since the ideal $DX + VY$ of W is such that $W/DX + VY$ is ring-isomorphic to the field $\mathbb{Z}_2$, $DX + VY$ is the unique maximal ideal of W. Let us denote the ideal $DX + VY$ of W by $\mathfrak{m}$. By hypothesis, $R = W/WY$. As the set of ideals of W is linearly ordered by inclusion, the set of ideals of R is linearly ordered by inclusion. So, R is a chained ring. Since $\mathfrak{m}$ is the unique maximal of W, it follows that $\mathfrak{m}/WY$ is the unique maximal ideal of R. Let us denote it by $\mathfrak{p}$. We claim that $\mathfrak{p} = R(X + WY)$. From $X + WY \in \mathfrak{p}$, we get that $R(X + WY) \subseteq \mathfrak{p}$. Let $r \in \mathfrak{p}$. Then, $r = m + WY$ for some $m \in \mathfrak{m} = DX + VY$. Hence, $m = dX + vY$ for some $d \in D$ and $v \in V$. As $V = K + VY$, we can express v in the form $v = \alpha + v_1Y$ for some $\alpha \in K$ and $v_1 \in V$. Thus, $m = dX + (\alpha + v_1Y)Y = dX + \alpha Y + v_1Y^2$. Therefore, $r = m + WY = (dX + \alpha Y + Y(v_1Y)) + WY$. Since $\alpha Y = X(\alpha/X)Y$ and $VY \subset W$, we obtain that $\alpha Y \in WX$ and $v_1Y^2 \in WY$. Hence, $r = m + WY = (dX + wX) + WY$ for some $w \in W$. From $D \subset W$, it follows that $r = (d + w)X + WY \in R(X + WY)$. This proves that $\mathfrak{p}$ is principal and $\mathfrak{p} = R(X + WY)$. We next verify that $\mathfrak{p} = Z(R)$. It is clear that $Z(R) \subseteq \mathfrak{p}$. We have $Y/X \in VY \subset W$ and $(X + WY)(Y/X + WY) = Y + WY = 0 + WY$. As $1/X \notin D$, $1/X$ cannot be in W, so $Y/X \notin WY$. Hence, $X + WY \in Z(R)$. From $\mathfrak{p} = R(X + WY)$, it follows that $\mathfrak{p} \subseteq Z(R)$. Therefore, $Z(R) = \mathfrak{p}$. We next verify that $X^i \notin WY$ for all $i \in \mathbb{N}$ and $X^i - X^j \notin WY$ for all distinct $i, j \in \mathbb{N}$. Let $i \in \mathbb{N}$. If $X^i \in WY$, then $X^i \in VY$, as $WY \subset VY$. Hence, $X^i \in K \cap VY = (0)$, a contradiction. Therefore, $X^i \notin WY$. Let $i, j \in \mathbb{N}$ be distinct. We can assume that $i < j$. If $X^i - X^j \in WY$, then $X^i - X^j \in K \cap VY = (0)$, so $X^i = X^j$, a contradiction. Therefore, $X^i - X^j \notin WY$. Thus, $X^i + WY \in Z(R)^*$ for all $i \in \mathbb{N}$, and $X^i + WY \neq X^j + WY$ for all distinct $i, j \in \mathbb{N}$. This shows that $Z(R)$ is not finite, so R is not finite. Since $R = W/WY$ and $Z(R) = \mathfrak{p} = \mathfrak{m}/WY$, we obtain that $R/\mathfrak{p}$ is ring-isomorphic to $W/\mathfrak{m}$. From $|W/\mathfrak{m}| = 2$, it follows that $|R/\mathfrak{p}| = 2$. Thus, (2) of Proposition 3 holds. So, we obtain from Proposition 3 that $\gamma((\Gamma(R))^c) = |R/\mathfrak{p}| = 2$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |R/\mathfrak{p}| + 1 = 3$. □

Remark 6. Assume that $\mathrm{diam}(\Gamma(R)) = 2$. If $\mathfrak{p}$ is not principal, but $\mathfrak{p} = Ra + Rb$ for some $a, b \in \mathfrak{p}$. Then, $\mathfrak{p}^2 \neq (0)$ and there exists $r \in R^*$ such that $ra = rb = 0$ by Remark 3. Thus, $(\mathrm{Ann}(\mathfrak{p}))^* \neq \emptyset$.

In the following example, we provide a ring R to illustrate Remark 6 and characterize R such that $\gamma((\Gamma(R))^c) < \infty$, and when $\gamma((\Gamma(R))^c) < \infty$, we determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Example 4. Let $T = K[X, Y]$, where K is a field, and let $I = (X^2, XY)$. If $R = T/I$, then $\gamma((\Gamma(R))^c) < \infty$ if and only if K is finite, and in the case K is finite, $\gamma((\Gamma(R))^c) = |K|$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |K| + 1$.

Proof. For a proof of $\gamma((\Gamma(R))^c) < \infty$ if and only if K is finite and in the case $|K| < \infty$, $\gamma((\Gamma(R))^c) = |K|$, see the proof of [21, Example 2.9]. Let us denote $X + I$ by x and $Y + I$ by y. Let us denote $Rx + Ry$ by $\mathfrak{p}$. It was shown in the proof of [21, Example 2.9] that $Z(R) = \mathfrak{p} \in \mathrm{Max}(R)$ and $(\mathrm{Ann}(\mathfrak{p}))^* = \{\alpha x \mid \alpha \in K^*\}$.
Assume that K is finite. Then, $|(\mathrm{Ann}(\mathfrak{p}))^*| = |K^*| = |K| - 1$. We verify that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |K| + 1$. Since $y \neq y^2$ and $y^3 \neq 0 + I$, it follows that $(\Gamma(R))^c$ has at least one edge. Hence by [9, Proposition 2], $\gamma_{\mathcal{R}}((\Gamma(R))^c) \geq \gamma((\Gamma(R))^c) + 1 = |K| + 1$. Let $V_2 = \{y\}, V_1 = (\mathrm{Ann}(\mathfrak{p}))^*$, and $V_0 = Z(R)^* \backslash ((\mathrm{Ann}(\mathfrak{p}))^* \cup \{y\})$. If $f = (V_0, V_1, V_2)$, then f is an RDF on $(\Gamma(R))^c$ and the weight of f equals $|V_1| + 2|V_2| = |K| - 1 + 2 = |K| + 1$. Therefore, $\gamma_{\mathcal{R}}((\Gamma(R))^c) \leq |K| + 1$, so $\gamma_{\mathcal{R}}((\Gamma(R))^c) = |K| + 1$ and f is a $\gamma_{\mathcal{R}}$-function on $(\Gamma(R))^c$. $\square$

We next assume that $\mathrm{diam}(\Gamma(R)) = 3$ and try to determine $\gamma((\Gamma(R))^c)$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. For a characterization of a ring T such that $\mathrm{diam}(\Gamma(T)) = 3$, see [14, Theorem 2.6(4)].

Lemma 8. *If* $\mathrm{diam}(\Gamma(R)) = 3$, *then* $\gamma((\Gamma(R))^c) = 2$ *and* $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.

Proof. By hypothesis, $\mathrm{diam}(\Gamma(R)) = 3$. Hence, we obtain by Remark 1 that $\gamma((\Gamma(R))^c) = 2$ and $3 \leq \gamma_{\mathcal{R}}((\Gamma(R))^c) \leq 4$. Let us denote $Z(R)$ by $\mathfrak{p}$. By [14, Theorem 2.6(4)], we can find $a, b \in \mathfrak{p}^*$ such that $\mathrm{Ann}(Ra + Rb) = (0)$. So, there is no $r \in R^*$ such that $\mathfrak{p}r = (0)$.
Let $r \in \mathfrak{p}^*$. We claim that $d(r)$ in $\Gamma(R)$ is not finite. Since $\Gamma(R)$ is connected and $|Z(R)^*| \geq 4$, we can find $s \in Z(R)^* \backslash \{r\}$ such that $rs = 0$. Since $(\mathrm{Ann}(\mathfrak{p}))^* = \emptyset$, we obtain that $\mathfrak{p} \not\subseteq \mathrm{Ann}(s)$. Therefore, there exists $s_1 \in \mathfrak{p}$ such that $ss_1 \neq 0$. Let $n \geq 1$. Assume that there are elements $s_1, \ldots, s_n \in \mathfrak{p}$ such that $s(\prod_{i=1}^n s_i) \neq 0$. Since $\mathfrak{p} \not\subseteq \mathrm{Ann}(s\prod_{i=1}^n s_i)$, there exists $s_{n+1} \in \mathfrak{p}$ such that $s(\prod_{i=1}^{n+1} s_i) \neq 0$. Thus, there exists a sequence $(s_n)_{n \in \mathbb{N}}$ from $\mathfrak{p}$ such that $s(\prod_{i=1}^n s_i) \neq 0$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Let us denote $s(\prod_{i=1}^n s_i)$ by t_n. From $rs = 0$, it follows that $rt_n = 0$. Let $n, m \in \mathbb{N}$ be distinct. We assert that $t_n \neq t_m$. We can assume without loss of generality that $n < m$. If $t_n = t_m$, then $s(\prod_{i=1}^n s_i) = s(\prod_{i=1}^n s_i)(\prod_{j=n+1}^m s_j)$. Therefore, $t_n(1 - \prod_{j=n+1}^m s_j) = 0$. So, $1 - \prod_{j=n+1}^m s_j \in Z(R) = \mathfrak{p}$. As $\prod_{j=n+1}^m s_j \in \mathfrak{p}$, it follows that $1 = 1 - \prod_{j=n+1}^m s_j + \prod_{j=n+1}^m s_j \in \mathfrak{p}$, a contradiction. Thus, $d(r)$ is not finite in $\Gamma(R)$ for each $r \in \mathfrak{p}^*$. By applying (1) $\Rightarrow$ (3) of Corollary 2 with $G = (\Gamma(R))^c$, we obtain that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \neq 3$, so $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$. $\square$

The following example illustrates Lemma 8.

Example 5. Let $T = K[X, Y]_{(X,Y)}$, where K is a field. Let $\mathcal{P}$ denote the set of all non-associate prime elements of T and let M be the T-module, $M = \bigoplus_{p \in \mathcal{P}} \frac{T}{Tp}$. If $R = T(+)M$

is the ring obtained by using Nagata's principle of idealization, then $Z(R)$ is an ideal of R, $\text{diam}(\Gamma(R)) = 3$, $\gamma((\Gamma(R))^c) = 2$, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.

Proof. Let us denote (X, Y), the ideal of $K[X, Y]$ generated by $\{X, Y\}$ by $\mathfrak{m}$. We know from the proof of [20, Example 2.8] that $Z(R) = \mathfrak{m}T(+)M$ is an ideal of R. Let us denote the zero element of M by 0_M. We know from the proof of [21, Example 2.12] that for any distinct $p, q \in \mathcal{P}$, $d_{\Gamma(R)}(a, b) = 3$, where $a = (p, 0_M)$ and $b = (q, 0_M)$. So, $\text{diam}(\Gamma(R)) = 3$. Hence, $\gamma((\Gamma(R))^c) = 2$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$ by Lemma 8. □

4. Some results on $\gamma_{\mathcal{R}}((\Gamma(R))^c)$, where $Z(R)$ is not an ideal of R

In this section, we assume that $Z(R)$ is not an ideal of R and discuss some results on $\gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Proposition 4. *If $Z(R)$ is not an ideal of R, then $\gamma((\Gamma(R))^c) = 2$. If $|Z(R)^*| = 2$, then R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 2$. If $|Z(R)^*| \geq 3$, then $2 < \gamma_{\mathcal{R}}((\Gamma(R))^c) \leq 4$*

Proof. If $Z(R)$ is not an ideal of R, then $\gamma((\Gamma(R))^c) = 2$ by [21, Lemma 2.3].
If $|Z(R)^*| = 2$, then $|Z(R)| = 3$. Since $Z(R)$ is not an ideal of R, we obtain from [3, Example 2.1(a)] that R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The function $f = (V_0 = \emptyset, V_1 = Z(R)^*, V_2 = \emptyset)$ is a $\gamma_{\mathcal{R}}$-function on $(\Gamma(R))^c$ such that the weight of f equals $|V_1| = 2$, so $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 2$. If $|Z(R)^*| \geq 3$, then $3 \leq \gamma_{\mathcal{R}}((\Gamma(R))^c) \leq 4$ by applying Lemma 3 with $G = (\Gamma(R))^c$. □

Remark 7. If a ring T is such that $|Z(T)^*| = 3$, then T is ring-isomorphic to one of the rings from the collection $\mathcal{C} = \{\mathbb{Z}_6, \mathbb{Z}_8, \frac{\mathbb{Z}_2[X]}{(X^3)}, \frac{\mathbb{Z}_2[X,Y]}{(X^2,XY,Y^2)}, \frac{\mathbb{F}_4[X]}{(X^2)}\}$, see [3, Example 2.1($a$)].

If $|Z(R)^*| \geq 3$, then the following proposition provides necessary and sufficient conditions such that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$.

Proposition 5. *If $|Z(R)^*| \geq 3$, then the following statements are equivalent:*
(1) $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$.
(2) $|(\text{Ann}(x))^*| = 1$ *for some* $x \in Z(R)^*$.
(3) *There exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ on $(\Gamma(R))^c$ such that $|V_1| = |V_2| = 1$.*

Proof. By Proposition 4, $\gamma((\Gamma(R))^c) = 2$ and $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{3, 4\}$.
(1) $\Rightarrow$ (2) Assume that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$. By Lemma 4, there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ on $(\Gamma(R))^c$ such that either $V_2 = V_0 = \emptyset$, and $|V_1| = 3$ or $|V_2| = 1 = |V_1|$. If $V_2 = V_0 = \emptyset$, and $|V_1| = 3$, then $|Z(R)^*| = 3$. Since $Z(T)$ is an ideal of T for any $T \in \mathcal{C}\backslash\{\mathbb{Z}_6\}$, where $\mathcal{C}$ is as mentioned in Remark 7, we obtain that R is ring-isomorphic to $\mathbb{Z}_6$. For the ring $\mathbb{Z}_6$, $(Z(\mathbb{Z}_6))^* = \{2, 4, 3\}$, and $(\text{Ann}_{\mathbb{Z}_6}(2))^* = \{3\}$.

Hence, there exists $x \in Z(R)^*$ such that $|(\mathrm{Ann}(x))^*| = 1$. Let $f = (V_0, V_1, V_2)$ be such that $|V_1| = |V_2| = 1$. Let $V_i = \{a_i\}$ for each $i \in \{1, 2\}$. Then, $V_0 \neq \emptyset$ and $ca_2 \neq 0$ for each $c \in V_0$. Since $\{a_2\}$ cannot be a dominating set of $(\Gamma(R))^c$, $a_1a_2 = 0$. Hence, $a_1 \in (\mathrm{Ann}(a_2))^*$. We claim that $a_2^2 \neq 0$. If $a_1 + a_2 \notin Z(R)$, then $a_2^2 = (a_1 + a_2)a_2 \neq 0$. If $a_1 + a_2 \in Z(R)$, then $a_1 + a_2 \notin \{a_1, a_2\}$. Let $c \in V_0$. Since $\Gamma(R)$ is connected by [3, Theorem 2.3], there exists a path between a_2 and c in $\Gamma(R)$. Let P be a path of shortest length between a_2 and c in $\Gamma(R)$. As $\mathrm{diam}(\Gamma(R)) \leq 3$, see [3, Theorem 2.3] and $a_2c \neq 0$, P is of length 2 or 3. Either P is of the form $a_2 - a_1 - c$ or of the form $a_2 - a_1 - c' - c$ for some $c' \in V_0$, as any path between a_2 and c in $\Gamma(R)$ must pass through a_1. Therefore, $a_1 + a_2 \neq 0$. So, $a_1 + a_2 \in Z(R)^* \backslash (V_1 \cup V_2) = V_0$. Therefore, $a_2^2 = (a_1 + a_2)a_2 \neq 0$. From $Z(R)^* = \bigcup_{i=0}^{2} V_i$, it follows that $(\mathrm{Ann}(a_2))^* = \{a_1\}$, so $|(\mathrm{Ann}(a_2))^*| = 1$.

(2) $\Rightarrow$ (3) Assume that there exists $x \in Z(R)^*$ such that $|(\mathrm{Ann}(x))^*| = 1$. Let $(\mathrm{Ann}(x))^* = \{y\}$. If $y = x$, then $xz \neq 0$ for any $z \in Z(R)^* \backslash \{x\}$. Hence, $\{x\}$ is a dominating set of $(\Gamma(R))^c$, a contradiction. So, $y \neq x$. Let $V_1 = \{y\}$, $V_2 = \{x\}$, and let $V_0 = Z(R)^* \backslash (V_1 \cup V_2)$. Then, $xy = 0$ and $zx \neq 0$ for any $z \in V_0$. Hence, $f = (V_0, V_1, V_2)$ is a $\gamma_{\mathcal{R}}$-function on $(\Gamma(R))^c$ such that $|V_1| = 1 = |V_2|$.

(3) $\Rightarrow$ (1) Assume that there exists a $\gamma_{\mathcal{R}}$-function $f = (V_0, V_1, V_2)$ such that $|V_1| = 1 = |V_2|$. Since the weight of f equals 3, we obtain that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$. □

If $|Z(R)^*| \geq 3$, then we try to characterize R such that there exists $x \in Z(R)^*$ with $|(\mathrm{Ann}(x))^*| = 1$.

Lemma 9. *If there exists $x \in Z(R)^*$ such that $|(\mathrm{Ann}(x))^*| = 1$, then $\mathrm{Ann}(\mathrm{Ann}(x)) \in \mathrm{Max}(R)$, $\mathrm{Ann}(\mathrm{Ann}(x)) \subset Z(R)$, and $|R/\mathrm{Ann}(\mathrm{Ann}(x))| = 2$.*

Proof. As $|(\mathrm{Ann}(x))^*| = 1$ for some $x \in Z(R)^*$ by assumption, there exists $y \in Z(R)^*$ such that $\mathrm{Ann}(x) = \{0, y\}$. Thus, $\mathrm{Ann}(x) = Ry$. The mapping $f : R \to Ry$ defined by $f(r) = ry$ is an onto R-module homomorphism with $\mathrm{Ker}(f) = \mathrm{Ann}(y) = \mathrm{Ann}(\mathrm{Ann}(x))$. We obtain from the fundamental theorem of homomorphism of modules that $R/\mathrm{Ann}(\mathrm{Ann}(x)) \cong Ry$ as R-modules. Hence, $|R/\mathrm{Ann}(\mathrm{Ann}(x))| = |Ry| = 2$. Therefore, $\mathrm{Ann}(\mathrm{Ann}(x)) \in \mathrm{Max}(R)$. Let us denote $\mathrm{Ann}(\mathrm{Ann}(x))$ by $\mathfrak{m}$. Then, $|R/\mathfrak{m}| = 2$. From $(\mathrm{Ann}(\mathrm{Ann}(x)))\mathrm{Ann}(x) = (0)$, it follows that for any $m \in \mathfrak{m}, my = 0$. So, any element of $\mathfrak{m}$ is a zero-divisor of R. Since $Z(R)$ is not an ideal of R, we get that $\mathfrak{m} \subset Z(R)$. □

Let e be any non-trivial idempotent element of a ring T. Then, it can be shown that $\mathrm{Ann}(e) = T(1 - e)$. As $1 - e$ is also a non-trivial idempotent element of T, it follows that $\mathrm{Ann}(e) \neq (0)$. If $e \in (\mathrm{Ann}(t))^*$ for some $t \in T$, then from $e = e^2$, it follows that $e \in \mathrm{Ann}(t)\mathrm{Ann}(t)$, so $(\mathrm{Ann}(t))^2 \neq (0)$.

Proposition 6. *If $|Z(R)^*| \geq 3$, then the following statements are equivalent:*
(1) *There exists $x \in Z(R)^*$ such that x satisfies the following conditions:* $\mathrm{Ann}(\mathrm{Ann}(x)) \in \mathrm{Max}(R)$, $\mathrm{Ann}(\mathrm{Ann}(x)) \subset Z(R)$, $(\mathrm{Ann}(x))^2 \neq (0)$, *and* $|R/\mathrm{Ann}(\mathrm{Ann}(x))| = 2$.

(2) *R is ring-isomorphic to $\mathbb{Z}_2 \times T$ for some ring T such that $|T| \geq 3$.*
(3) *There exists $x \in Z(R)^*$ such that $|(\mathrm{Ann}(x))^*| = 1$ and $(\mathrm{Ann}(x))^2 \neq (0)$.*
(4) *There exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathfrak{m} \subset Z(R)$, $\mathfrak{m}$ is principal, $(\mathrm{Ann}(\mathfrak{m}))^2 \neq (0)$, and $|R/\mathfrak{m}| = 2$.*
If (2) holds, then $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$.

Proof. (1) $\Rightarrow$ (2) Let us denote $\mathrm{Ann}(\mathrm{Ann}(x))$ by $\mathfrak{m}$. Since $(\mathrm{Ann}(x))^2 \neq (0)$ by assumption, $\mathrm{Ann}(x) \not\subseteq \mathfrak{m}$. Hence, $\mathfrak{m} + \mathrm{Ann}(x) = R$. From $\mathfrak{m}(\mathrm{Ann}(x)) = (0)$, we obtain that $\mathfrak{m} \cap \mathrm{Ann}(x) = (0)$ by [5, Proposition 1.10(i)]. Therefore, the mapping $f : R \to R/\mathfrak{m} \times R/\mathrm{Ann}(x)$ defined by $f(r) = (r + \mathfrak{m}, r + \mathrm{Ann}(x))$ is an isomorphism of rings by [5, Proposition 1.10(ii) and (iii)]. Let us denote $R/\mathrm{Ann}(x)$ by T. Since $|R/\mathfrak{m}| = 2$ by assumption, we get that R is ring-isomorphic to $\mathbb{Z}_2 \times T$. From $|Z(R)^*| \geq 3$, we get that $|T| \geq 3$.
(2) $\Rightarrow$ (3) Let us denote the ring $\mathbb{Z}_2 \times T$ by R_1. Let $x_1 = (0,1)$. Then, $x_1 \in Z(R_1)^*$, and $\mathrm{Ann}_{R_1}(x_1) = \{(0,0),(1,0)\}$. So, $|(\mathrm{Ann}_{R_1}(x_1))^*| = 1$. As $(1,0) = (1,0)^2$, and $(1,0) \in \mathrm{Ann}_{R_1}(x_1)$, it follows that $(1,0) \in (\mathrm{Ann}_{R_1}(x_1))^2$. Therefore, $(\mathrm{Ann}_{R_1}(x_1))^2 \neq (0) \times (0)$. Since R is ring-isomorphic to R_1, there exists $x \in Z(R)^*$ such that $|(\mathrm{Ann}(x))^*| = 1$, and $(\mathrm{Ann}(x))^2 \neq (0)$.
(3) $\Rightarrow$ (4) Assume that there exists $x \in Z(R)^*$ such that $|(\mathrm{Ann}(x))^*| = 1$ and $(\mathrm{Ann}(x))^2 \neq (0)$. Let $\mathrm{Ann}(x) = \{0, y\}$. From $(\mathrm{Ann}(x))^2 \neq (0)$, it follows that $y^2 \neq 0$. From $y^2 \in (\mathrm{Ann}(x))^*$, we get that $y = y^2$. Hence, y is a non-trivial idempotent element of R. The mapping $f : R \to Ry \times R(1-y)$ defined by $f(r) = (ry, r(1-y))$ is an isomorphism of rings. Since $|Ry| = 2$, it follows that R is ring-isomorphic to $\mathbb{Z}_2 \times T$ for some ring T with $|T| \geq 3$. Let us denote the ring $\mathbb{Z}_2 \times T$ by R_1. Let $\mathfrak{m}_1 = (0) \times T$. As $R_1/\mathfrak{m}_1$ is ring-isomorphic to $\mathbb{Z}_2$, it follows that $\mathfrak{m}_1 \in \mathrm{Max}(R_1)$, and $|R_1/\mathfrak{m}_1| = 2$. Since $\mathfrak{m}_1(1,0) = (0) \times (0)$ and $(1,0) \notin \mathfrak{m}_1$, it follows that $(1,0) \in Z(R_1)\backslash\mathfrak{m}_1$. Thus, $\mathfrak{m}_1 \subset Z(R_1)$. The ideal $\mathfrak{m}_1$ of R_1 is principal, since $\mathfrak{m}_1 = R_1(0,1)$. From $\mathrm{Ann}_{R_1}(\mathfrak{m}_1) = \mathbb{Z}_2 \times (0)$, we obtain that $(\mathrm{Ann}_{R_1}(\mathfrak{m}_1))^2 = \mathbb{Z}_2 \times (0) \neq (0) \times (0)$. Since R is ring-isomorphic to R_1, there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathfrak{m} \subset Z(R)$, $\mathfrak{m}$ is principal, $(\mathrm{Ann}(\mathfrak{m}))^2 \neq (0)$, and $|R/\mathfrak{m}| = 2$.
(4) $\Rightarrow$ (1) Assume that there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathfrak{m} \subset Z(R)$, $\mathfrak{m}$ is principal, $(\mathrm{Ann}(\mathfrak{m}))^2 \neq (0)$, and $|R/\mathfrak{m}| = 2$. Let $\mathfrak{m} = Rx$. From $x \neq 0$ and $\mathfrak{m} \subset Z(R)$, it follows that $x \in Z(R)^*$. Thus, $\mathrm{Ann}(x) \neq (0)$. Hence, $\mathrm{Ann}(\mathrm{Ann}(x)) \neq R$. Since $\mathfrak{m}\mathrm{Ann}(x) = (0)$, we obtain that $\mathfrak{m} \subseteq \mathrm{Ann}(\mathrm{Ann}(x))$, so $\mathfrak{m} = \mathrm{Ann}(\mathrm{Ann}(x))$. Therefore, for some $x \in Z(R)^*$, $\mathrm{Ann}(\mathrm{Ann}(x)) \in \mathrm{Max}(R)$, $\mathrm{Ann}(\mathrm{Ann}(x)) \subset Z(R)$, $(\mathrm{Ann}(x))^2 \neq (0)$, and $|R/\mathfrak{m}| = 2$.
Assume that (2) holds. Then, (3) holds, so $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ by (2) $\Rightarrow$ (1) of Proposition 5. $\square$

If R is reduced, then for any nonzero ideal I of R, $I^2 \neq (0)$. So, for any $x \in Z(R)^*$, $(\mathrm{Ann}(x))^2 \neq (0)$. Hence, we have the following theorem.

Theorem 1. *If R is reduced with $|Z(R)^*| \geq 3$, then the following statements are equivalent:*
(1) *$\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$.*

(2) *There exists* $x \in Z(R)^*$ *such that* $|(\mathrm{Ann}(x))^*| = 1$.
(3) *For some* $x \in Z(R)^*$, $\mathrm{Ann}(\mathrm{Ann}(x)) \in \mathrm{Max}(R)$, $\mathrm{Ann}(\mathrm{Ann}(x)) \subset Z(R)$, *and* $|R/\mathrm{Ann}(\mathrm{Ann}(x))| = 2$.
(4) R *is ring-isomorphic to* $\mathbb{Z}_2 \times T$ *for some reduced ring* T *such that* $|T| \geq 3$.
(5) *There exists* $\mathfrak{m} \in \mathrm{Max}(R)$ *such that* $\mathfrak{m} \subset Z(R)$, $\mathfrak{m}$ *is principal, and* $|R/\mathfrak{m}| = 2$.

Proof. (1) $\Leftrightarrow$ (2) This follows from (1) $\Leftrightarrow$ (2) of Proposition 5.
(2) $\Rightarrow$ (3) This follows from Lemma 9.
(3) $\Rightarrow$ (4) Assume that (3) holds. Since R is reduced, $(\mathrm{Ann}(x))^2 \neq (0)$. Hence, R is ring-isomorphic to $\mathbb{Z}_2 \times T$, where T is a ring such that $|T| \geq 3$ by (1) $\Rightarrow$ (2) of Proposition 6. Since R is reduced, we obtain that T is reduced.
(4) $\Rightarrow$ (5) This follows from (2) $\Rightarrow$ (4) of Proposition 6.
(5) $\Rightarrow$ (1) Assume that (5) holds. Since $\mathrm{Ann}(\mathfrak{m}) \neq (0)$ and R is reduced, we obtain that $(\mathrm{Ann}(\mathfrak{m}))^2 \neq (0)$. Hence, (4) of Proposition 6 holds. Therefore, R is ring-isomorphic to $\mathbb{Z}_2 \times T$ for some ring T with $|T| \geq 3$ by (4) $\Rightarrow$ (2) of Proposition 6. If the statement (2) of Proposition 6 holds, then it is verified in the proof of Proposition 6 that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$. □

Let T be a ring. We denote the power series ring in one variable X over T by $T[[X]]$ and the power series ring in n variables $X_1, X_2, \ldots, X_n$ ($n \geq 2$) over T by $T[[X_1, X_2, \ldots, X_n]]$. We use the following lemma in the proof of Example 6.

Lemma 10. *Let* $n \in \mathbb{N}\backslash\{1\}$. *Let* $T = K[[X_1, X_2, \ldots, X_n]]$, *where* K *is a field. If* $R = T/T(\prod_{i=1}^{n} X_i)$, *then* R *is a reduced ring,* $|\mathrm{Min}(R)| = n$, $Z(R)^*$ *is not finite, and* R *cannot be expressed as the direct product of two non-trivial rings.*

Proof. Since $X_1, X_2, \ldots, X_n$ are pairwise non-associate prime elements of T, it follows that $TX_i \in \mathrm{Spec}(T)$ for each $i \in \{1, 2, \ldots, n\}$, $TX_i \neq TX_j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$, and $T(\prod_{i=1}^{n} X_i) = \bigcap_{i=1}^{n} TX_i$. Let us denote $T(\prod_{i=1}^{n} X_i)$ by I and $X_i + I$ by x_i for each $i \in \{1, 2, \ldots, n\}$. Hence, $R = T/I$ is reduced. One can verify that $\mathrm{Min}(R) = \{\mathfrak{p}_i = Rx_i \mid i \in \{1, 2, \ldots, n\}\}$. Thus, $|\mathrm{Min}(R)| = n$. Let $i \in \{1, 2, \ldots, n\}$. Then, $x_i^k \in Z(R)^*$ for all $k \in \mathbb{N}$, and for all distinct $k, k' \in \mathbb{N}$, $x_i^k \neq x_i^{k'}$. So, $Z(R)^*$ is not finite.
Since K is a field, $\mathrm{Max}(K) = \{(0)\}$. Hence, by applying [5, Exercise 5(iv), p.11] n times, we obtain that $\mathrm{Max}(T) = \{\sum_{i=1}^{n} TX_i\}$. As $R = T/I$, $\mathrm{Max}(R) = \{\sum_{i=1}^{n} TX_i/I\}$. Thus, $|\mathrm{Max}(R)| = 1$, so R has no non-trivial idempotent element by [5, Exercise 12, p.11]. Hence, R cannot be expressed as the direct product of two non-trivial rings. □

For any ring T and any $\mathfrak{p} \in \mathrm{Min}(T)$, then $\mathfrak{p} \subseteq Z(T)$ by [12, Theorem 84]. If T is reduced, then it follows from [5, Proposition 1.8] and [12, Theorem 10] that $\bigcap_{\mathfrak{p} \in \mathrm{Min}(T)} \mathfrak{p} = (0)$, and it is not hard to show that $Z(T) = \bigcup_{\mathfrak{p} \in \mathrm{Min}(T)} \mathfrak{p}$. If a ring T is the direct product of two non-trivial rings, then $Z(T)$ cannot be an ideal of T. The following example illustrates Theorem 1.

Example 6. (1) Let $n \in \mathbb{N}\backslash\{1\}$. If $R = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$, and in the case $n = 2$, let $|F_j| \geq 3$ for some $j \in \{1, 2\}$, then $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ if and only if $|F_i| = 2$ for some $i \in \{1, 2, \ldots, n\}$.
(2) Let $n \in \mathbb{N}\backslash\{1\}$. Let T, R be as in the statement of Lemma 10. Then, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.
(3) Let $n \in \mathbb{N}$ be such that $n \geq 3$. Let $T = K[[X_1, X_2, \ldots, X_{n-1}]]$, where K is a field. If $R = \mathbb{Z}_2 \times T/T(\prod_{i=1}^{n-1} X_i)$, then $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$.

Proof. (1). Since R is the direct product of n fields, R is a reduced ring and each ideal of R is principal. As $|F_j| \geq 3$ for some $j \in \{1, 2\}$ in the case $n = 2$, it follows that $|Z(R)^*| \geq 3$. Let $i \in \{1, 2, \ldots, n\}$ and let $\mathfrak{p}_i = I_{1i} \times \cdots \times I_{ii} \times \cdots \times I_{ni}$ with $I_{ii} = (0)$ and $I_{ji} = F_j$ for each $j \in \{1, 2, \ldots, n\}\backslash\{i\}$. Then, $\mathrm{Spec}(R) = \mathrm{Max}(R) = \mathrm{Min}(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Thus, $|\mathrm{Min}(R)| = n$. By Proposition 4, $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{3, 4\}$. For each $i \in \{1, 2, 3, \ldots, n\}$, $\mathfrak{p}_i \subset Z(R)$, $\mathfrak{p}_i$ is principal, and $R/\mathfrak{p}_i$ is ring-isomorphic to F_i. Hence, it follows from (1) $\Leftrightarrow$ (5) of Theorem 1 that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ if and only if $|R/\mathfrak{p}_i| = 2$ for some $i \in \{1, 2, \ldots, n\}$, if and only if $|F_i| = 2$ for some $i \in \{1, 2, \ldots, n\}$.
(2) If T, R are as in the statement of Lemma 10, then we know from the proof of Lemma 10 that R is a reduced ring with $|\mathrm{Min}(R)| = n$, $Z(R)^*$ is infinite, and R cannot be expressed as the direct product of two non-trivial rings. The reduced ring R is such that $Z(R)$ is not an ideal of R. As $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{3, 4\}$ by Proposition 4, we obtain from (1) $\Rightarrow$ (4) of Theorem 1 that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.
(3) Let $n \in \mathbb{N}$ be such that $n \geq 3$. Let $T = K[[X_1, X_2, \ldots, X_{n-1}]]$, where K is a field and let $R = \mathbb{Z}_2 \times T/T(\prod_{i=1}^{n-1} X_i)$. We know from the proof of Lemma 10 that $T/T(\prod_{i=1}^{n-1} X_i)$ is an infinite reduced ring with $|\mathrm{Min}(T)| = n - 1$. From (4) $\Rightarrow$ (1) of Theorem 1, we get that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$. □

The following example illustrates that (3) $\Rightarrow$ (2) of Theorem 1 can fail to hold for a non-reduced ring.

Example 7. Let $T = \mathbb{Z}_2[X_1, X_2, X_3, \ldots]$ be the polynomial ring in an infinite number of variables $X_1, X_2, X_3, \ldots$ over $\mathbb{Z}_2$. Let $\mathfrak{m} = \sum_{n \in \mathbb{N}} TX_n$. If $R = T/\mathfrak{m}^2 \times T/\mathfrak{m}^2$, then R is not reduced, $Z(R)$ is not an ideal of R, $Z(R)^*$ is not finite, the statement (3) of Theorem 1 holds, but the statement (2) of Theorem 1 does not hold, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.

Proof. From $T/\mathfrak{m}$ is ring-isomorphic to $\mathbb{Z}_2$, we obtain that $\mathfrak{m} \in \mathrm{Max}(T)$. Let us denote the ring $T/\mathfrak{m}^2$ by T_1. For each $i \in \mathbb{N}$, let us denote $X_i + \mathfrak{m}^2$ by x_i. As $x_1 \neq 0 + \mathfrak{m}^2$ but $x_1^2 = 0 + \mathfrak{m}^2$, it follows that T_1 is not reduced. Hence, $R = T_1 \times T_1$ is not reduced. As $\mathfrak{m} \in \mathrm{Max}(T)$,it follows that $\mathfrak{m}/\mathfrak{m}^2 \in \mathrm{Max}(T_1)$. Since $(\mathfrak{m}/\mathfrak{m}^2)^2 = (0 + \mathfrak{m}^2)$, we obtain that T_1 is quasi-local with $\mathfrak{m}/\mathfrak{m}^2$ as its unique maximal ideal. Let us denote $\mathfrak{m}/\mathfrak{m}^2$ by $\mathfrak{p}$. Then, $\mathfrak{p}^2 = (0 + \mathfrak{m}^2)$. Thus, $\mathrm{Spec}(T_1) = \{\mathfrak{p}\}$, and $Z(T_1) = \mathfrak{p}$. Therefore, $\mathrm{Spec}(R) = \{\mathfrak{P}_1 = \mathfrak{p} \times T_1, \mathfrak{P}_2 = T_1 \times \mathfrak{p}\} = \mathrm{Max}(R)$, and $Z(R) = \bigcup_{j=1}^{2} \mathfrak{P}_j$ is not an ideal of R. For all distinct $i, j \in \mathbb{N}$, x_i and x_j both belong to $Z(T_1)$ and $x_i \neq x_j$. So, $Z(T_1)^*$ is not finite. Hence, $Z(R)^*$ is not finite. If $r = (x_1, 1 + \mathfrak{m}^2)$, then $r \in Z(R)^*$. As $Z(T_1) = \mathfrak{p}$, $\mathfrak{p}^2 = (0 + \mathfrak{m}^2)$, it follows that for any $z \in \mathfrak{p}^*$, $\mathrm{Ann}_{T_1}(z) = \mathfrak{p}$ by Remark 2. Hence, $\mathrm{Ann}_R(r) = \mathfrak{p} \times (0 + \mathfrak{m}^2)$, so $\mathrm{Ann}_R(\mathrm{Ann}_R(r)) = \mathrm{Ann}_R(\mathfrak{p} \times (0 + \mathfrak{m}^2)) = \mathfrak{p} \times T_1$.

Thus, $\mathrm{Ann}_R(\mathrm{Ann}_R(r)) = \mathfrak{P}_1 \in \mathrm{Max}(R)$. Observe that $\mathfrak{P}_1 \subset Z(R)$, and $R/\mathfrak{P}_1$ is ring-isomorphic to $T_1/\mathfrak{p}$. From $T_1/\mathfrak{p}$ is ring-isomorphic to $T/\mathfrak{m}$, and $|T/\mathfrak{m}| = 2$, we obtain that $|R/\mathrm{Ann}(\mathrm{Ann}(r))| = 2$. Thus, the element $r \in Z(R)^*$ satisfies the statement (3) of Theorem 1.
We claim that the statement (2) of Theorem 1 does not hold for R. Let $s \in Z(R)^*$. Then, $s = (t_1, t_2)$ for some $t_1, t_2 \in T_1$ with at least one of t_1 and t_2 different from $0 + \mathfrak{m}^2$. We verify that $(\mathrm{Ann}_R(s))^*$ is not finite. If $t_1 = 0 + \mathfrak{m}^2$, then $\mathrm{Ann}_R(s) = \mathrm{Ann}_{T_1}(t_1) \times \mathrm{Ann}_{T_1}(t_2) = T_1 \times \mathrm{Ann}_{T_1}(t_2)$ is not finite, since T_1 is not finite. Similarly, if $t_2 = 0+\mathfrak{m}^2$, then $\mathrm{Ann}_R(s) = \mathrm{Ann}_{T_1}(t_1) \times T_1$ is not finite. Suppose that $t_i \neq 0+\mathfrak{m}^2$ for each $i \in \{1, 2\}$. Then, $t_i \in Z(T_1)^*$ for some $i \in \{1, 2\}$. If $t_1 \in Z(T_1)^*$, then $\mathrm{Ann}_{T_1}(t_1) = \mathfrak{p}$, so $\mathrm{Ann}_R(s) = \mathfrak{p} \times \mathrm{Ann}_{T_1}(t_2)$ is not finite, since $\mathfrak{p}$ is not finite. If $t_2 \in Z(T_1)^*$, then $\mathrm{Ann}_R(s) = \mathrm{Ann}_{T_1}(t_1) \times \mathfrak{p}$ is not finite. Thus, for any $s \in Z(R)^*$, $\mathrm{Ann}_R(s)$ is not finite, so the statement (2) of Theorem 1 does not hold.
We know from Proposition 4 that $3 \le \gamma_{\mathcal{R}}((\Gamma(R))^c) \le 4$. For any $s \in Z(R)^*$, $d(s)$ is not finite in $\Gamma(R)$. So, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$ by (1) $\Rightarrow$ (3) of Corollary 2. □

In the following example, we mention some non-reduced rings R such that $Z(R)$ is not an ideal of R and compute $\gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Example 8. (1) If R is a non-reduced ring of the form $R = \mathbb{Z}/2^k\mathbb{Z} \times T$, where $k \ge 2$, and T is a ring which can be reduced, then $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$.
(2) Let K be a field, and let $T = K[[X_1, X_2, \ldots, X_n]]$, where $n \ge 2$. Let $I = T(\prod_{i=1}^n X_i^2)$. If $R = T/I$, then R is not reduced, $|\mathrm{Min}(R)| = n$, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.
(3) Let $p_1, p_2, \ldots, p_n$ $(n \ge 2)$ be distinct prime numbers with $p_1 < p_2 < \cdots < p_n$. Let $k_1, k_2, \ldots, k_n$ be positive integers such that $k_i \ge 2$ for some $i \in \{1, 2, \ldots, n\}$. If $m = \prod_{j=1}^n p_j^{k_j}$ and $R = \mathbb{Z}_m$, then R is not reduced, $|\mathrm{Min}(R)| = n$, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$ if and only if p_1 is odd.

Proof. (1) Since $\mathbb{Z}/2^k\mathbb{Z}$ $(k \ge 2)$ is not reduced, it follows that $R = \mathbb{Z}/2^k\mathbb{Z} \times T$ is not reduced. If $r = (2 + 2^k\mathbb{Z}, 1)$, then $r \in Z(R)^*$, and $\mathrm{Ann}(r) = \{(0 + 2^k\mathbb{Z}, 0), (2^{k-1} + 2^k\mathbb{Z}, 0)\}$. Thus, $|(\mathrm{Ann}(r))^*| = 1$. Hence, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ by (2) $\Rightarrow$ (1) of Proposition 5.
(2) As $X_1, X_2, \ldots, X_n$ are prime elements of T and are pairwise non-associate, $TX_i \in \mathrm{Spec}(T)$ for each $i \in \{1, 2, \ldots, n\}$, and $TX_i \neq TX_j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. For each $i \in \{1, 2, \ldots, n\}$, TX_i^2 is TX_i-primary and it can be easily verified that $I = T(\prod_{i=1}^n X_i^2) = \bigcap_{i=1}^n TX_i^2$. Let us denote $X_i + I$ by x_i for each $i \in \{1, 2, \ldots, n\}$. Since $\prod_{i=1}^n x_i \neq 0 + I$, but $(\prod_{i=1}^n x_i)^2 = 0 + I$, we obtain that $R = T/I$ is not reduced. Since $\mathrm{Max}(T) = \{\sum_{i=1}^n TX_i\}$, it follows that $\mathrm{Max}(R) = \{\sum_{i=1}^n TX_i/I\}$. Let $i \in \{1, 2, \ldots, n\}$. Let us denote Rx_i by $\mathfrak{p}_i$ and Rx_i^2 by $\mathfrak{q}_i$. Then, $\mathfrak{p}_i \in \mathrm{Spec}(R)$, $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal of R, and $(0 + I) = \bigcap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition of the zero ideal in R. Hence, $Z(R) = \bigcup_{i=1}^n \mathfrak{p}_i$ by [5, Proposition 4.7]. If we denote x_1 by a and $\prod_{j=2}^n x_j$ by b, then $a, b \in Z(R)^*$ but $a + b \notin Z(R)$. So, $Z(R)$ is not an ideal of R. As $\mathrm{Nil}(R) = \bigcap_{i=1}^n \mathfrak{p}_i$, $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are not comparable under inclusion for all distinct $i, j \in \{1, 2, \ldots, n\}$, it follows that $\mathrm{Min}(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Thus, $|\mathrm{Min}(R)| = n$. As x_1, x_2, x_1x_2 are pairwise distinct elements of $Z(R)^*$, we

get that $|Z(R)^*| \geq 3$. By Proposition 4, it follows that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{3,4\}$. We claim that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \neq 3$. If $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$, then there exists $r \in Z(R)^*$ such that $|(\mathrm{Ann}(r))^*| = 1$ by (1) $\Rightarrow$ (2) of Proposition 5, and from Lemma 9, we get that $\mathrm{Ann}(\mathrm{Ann}(r)) \in \mathrm{Max}(R)$, $\mathrm{Ann}(\mathrm{Ann}(r)) \subset Z(R)$, and $|R/\mathrm{Ann}(\mathrm{Ann}(r))| = 2$. From $\mathrm{Max}(R) = \{\sum_{i=1}^{n} TX_i/I\}$, we obtain that $\mathrm{Ann}(\mathrm{Ann}(r)) = \sum_{i=1}^{n} TX_i/I = \sum_{i=1}^{n} Rx_i$. Let us denote $\sum_{i=1}^{n} Rx_i$ by $\mathfrak{m}$. From $Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$, and $\mathfrak{m} \subset Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$, it follows that $\mathfrak{m} \subseteq \mathfrak{p}_i$ for some $i \in \{1,2,\ldots,n\}$ by [5, Proposition 1.11(i)]. Hence, $\mathfrak{m} = \mathfrak{p}_i \in \mathrm{Max}(R)$ for some $i \in \{1,2,\ldots,n\}$, a contradiction, since $\mathfrak{p}_k = Rx_k \subset \mathfrak{m}$ for each $k \in \{1,2,\ldots,n\}$. Therefore, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.

(2) By hypothesis, $m = \prod_{j=1}^{n} p_j^{k_j}$, where $n \geq 2$, $p_1, p_2, \ldots, p_n$ are prime numbers such that $p_1 < p_2 < \cdots < p_n$, and $k_i \geq 2$ for some $i \in \{1,2,\ldots,n\}$. Let $j \in \{1,2,\ldots,n\}$. If we denote the ring $\mathbb{Z}/p_j^{k_j}\mathbb{Z}$ by R_j, then $R = \mathbb{Z}_m$ is ring-isomorphic to $R_1 \times R_2 \times \cdots \times R_n$. Since $k_i \geq 2$, R_i is not reduced. Hence, R is not reduced. It is clear that $Z(R)$ is not an ideal of R and $|Z(R)^*| \geq 3$. We know from Proposition 4 that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{3,4\}$. The ring R is such that $\mathrm{Max}(R) = \{\mathfrak{m}_j = p_jR \mid j \in \{1,2,\ldots,n\}\}$, $Z(R) = \bigcup_{j=1}^{n} \mathfrak{m}_j$, and $R/\mathfrak{m}_j$ is ring-isomorphic to $\mathbb{Z}/p_j\mathbb{Z}$ for each $j \in \{1,2,\ldots,n\}$.

Assume that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$. If p_1 is not odd, then $p_1 = 2$. Either $k_1 = 1$ or $k_1 \geq 2$. If $k_1 = 1$, then R is ring-isomorphic to $\mathbb{Z}_2 \times T$, where T is a non-reduced ring. Hence, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ by Proposition 6, a contradiction. If $k_1 \geq 2$, then R is ring-isomorphic to $\mathbb{Z}/2^{k_1}\mathbb{Z} \times T_1$ for some ring T_1. Hence, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ by (1), a contradiction. Therefore, p_1 is odd.

Conversely, assume that p_1 is odd. As $p_1 < p_2 < \cdots < p_n$, it follows that p_j is odd for each $j \in \{1,2,\ldots,n\}$. Hence, R contains no $\mathfrak{m} \in \mathrm{Max}(R)$ such that $|R/\mathfrak{m}| = 2$. If $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$, then we obtain from (1) $\Rightarrow$ (2) of Proposition 5 that there exists $x \in Z(R)^*$ such that $|(\mathrm{Ann}(x))^*| = 1$. Hence, there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $|R/\mathfrak{m}| = 2$ by Lemma 9, a contradiction. Therefore, $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$. $\square$

Recall that a ring R is *von Neumann regular* if for each element a of R, there is an element b in R such that $a = a^2b$, see [10, Exercise 16, p.111]. We know that R is von Neumann regular if and only if $\dim R = 0$ and R is reduced, see [10, Exercise 16 ((a) $\Leftrightarrow$ (d)), p.111].

In the following example, we provide a von Neumann regular ring R such that $\mathrm{Min}(R)$ is not finite, $Z(R)$ is not an ideal of R, and compute $\gamma_{\mathcal{R}}((\Gamma(R))^c)$.

Example 9. Let Λ be an infinite set. Let $R = \prod_{\lambda \in \Lambda} F_\lambda$, where F_λ is a field for each $\lambda \in \Lambda$. Then, R is von Neumann regular, $\mathrm{Max}(R)$ is infinite, and $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ if and only if $|F_\lambda| = 2$ for some $\lambda \in \Lambda$.

Proof. For any field F and for any $\alpha \in F$, $\alpha = \alpha^2\beta$ for some $\beta \in F$ with $\beta = 1$ if $\alpha = 0$, and $\beta = \alpha^{-1}$ if $\alpha \neq 0$. Hence, it follows that for each $r \in R$, there exists $s \in R$ such that $r = r^2s$. Therefore, R is von Neumann regular. So, $\mathrm{Spec}(R) = \mathrm{Max}(R) = \mathrm{Min}(R)$. Let $\lambda \in \Lambda$. Let $f_\lambda \in R$ be defined as follows. The λth coordinate of f_λ equals 0, whereas the λ'th coordinate of f_λ equals 1 for all $\lambda' \in \Lambda \backslash \{\lambda\}$. Let $e_\lambda \in R$ be defined as follows. The λth coordinate of e_λ equals 1, whereas the λ'th coordinate of e_λ equals 0 for all $\lambda' \in \Lambda\backslash\{\lambda\}$. Then, $f_\lambda e_\lambda$ is the zero element of R. Therefore, $Rf_\lambda \subseteq Z(R)$. As R/Rf_λ is ring-isomorphic to F_λ, we get that $Rf_\lambda \in \mathrm{Max}(R)$. If λ_1 and λ_2 are distinct elements of Λ, then $Rf_{\lambda_1} \neq Rf_{\lambda_2}$. Since Λ is infinite, it follows that $\mathrm{Max}(R)$ is infinite. Let $r \in Z(R)^*$. Then, the λth coordinate of r equals 0 for some $\lambda \in \Lambda$. So, $r \in Rf_\lambda$. Therefore, $Z(R) = \bigcup_{\lambda\in\Lambda} Rf_\lambda$, so $Z(R)^*$ is infinite. By Proposition 4, $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{3, 4\}$. It can be easily verified that the set of all principal maximal ideals of R equals $\{Rf_\lambda \mid \lambda \in \Lambda\}$. As for each $\lambda \in \Lambda$, R/Rf_λ is ring-isomorphic to F_λ, we obtain that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$ if and only if $|F_\lambda| = 2$ for some $\lambda \in \Lambda$ by (1) $\Leftrightarrow$ (5) of Theorem 1. $\square$

Let $F_n = \mathbb{Z}_3$ for each $n \in \mathbb{N}$. If $R = \prod_{n\in\mathbb{N}} F_n$, then it follows from Proposition 4 and Example 9 that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 4$.

5. Conclusion

For a commutative ring R with identity that admits at least two nonzero zero-divisors with $\gamma((\Gamma(R))^c) < \infty$, we have tried to find $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. If $Z(R)$ is an ideal of R, then we have given some necessary (respectively, sufficient) conditions such that $\gamma((\Gamma(R))^c) < \infty$ and determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. If $\mathrm{diam}(\Gamma(R)) = 1$ (respectively, $\mathrm{diam}(\Gamma(R)) = 3$), then for the results obtained by us, see Proposition 1 (respectively, Lemma 8). If $\mathrm{diam}(\Gamma(R)) = 2$, then we are able to characterize R such that $\gamma((\Gamma(R))^c) < \infty$ in some special cases and determine $\gamma_{\mathcal{R}}((\Gamma(R))^c)$. If $Z(R)$ is not an ideal of R, then we have noted that $\gamma((\Gamma(R))^c) = 2$. If $|Z(R)^*| \geq 3$, then we have verified that $\gamma_{\mathcal{R}}((\Gamma(R))^c) \in \{3, 4\}$ and we are able to characterize reduced rings R such that $\gamma_{\mathcal{R}}((\Gamma(R))^c) = 3$, see (Theorem 1).

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