Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-21
https://doi.org/10.22049/cco.2026.31439.2851

CCO
Commun. Comb. Optim.

Research Article

Weak signed total double Roman k-domination number of graphs

Jafar Amjadi[1,*], Zahra Nayebi[1,‡], Lutz Volkmann[2,†]

[1]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran
*j-amjadi@azaruniv.ac.ir
‡z.nayebi78@gmail.com

[2]Institute for Geometry and Practical Mathematics, RWTH Aachen University, 52056 Aachen,Germany
†volkm@math2.rwth-aachen.de

Received: 20 March 2026; Accepted: 5 September 2026
Published Online: 9 September 2026

Abstract: Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. A weak signed total double Roman k-dominating function (WSTDRkDF) on a graph G is defined as a function $f\colon V(G) \to \{-1, 1, 2, 3\}$ satisfying the condition that $\sum_{x\in N(v)} f(x) \geq k$ for each vertex $v \in V(G)$, where $N(v)$ is the neighborhood of v. The weight of a WSTDRkDF f is $\omega(f) = \sum_{v\in V(G)} f(v)$. The weak signed total double Roman k-domination number $\gamma^k_{wstdR}(G)$ of G is the minimum weight among all WSTDRkDF on G. In this paper, we initiate the study of the weak signed total double Roman k-domination number of graphs, and we present different sharp bounds on $\gamma^k_{wstdR}(G)$. In particular, we establish lower bounds on weak signed total double Roman k-domination number of trees for $k \in \{1, 2, 3\}$. In addition, we determine the weak signed total double Roman k-domination number of some classes of graphs.

Keywords: weak signed total double Roman k-domination number, signed total double Roman k-domination number

AMS Subject Classification: 05C69

1. Terminology and introduction

For notation and graph theory terminology, we in general follow the books [7, 8]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The

---

* *Corresponding Author*
† *Corresponding Author*

integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) \colon uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. If $X \subseteq V(G)$, then $G[X]$ is the subgraph induced by X. For a set $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) = N(X) = \bigcup_{v \in X} N(v)$, and its *closed neighborhood* is the set $N_G[X] = N[X] = N(X) \cup X$. For disjoint subsets X and Y of vertices of a graph G, we denote by (X, Y) the set of edges between X and Y. The *complement* of a graph G is denoted by $\overline{G}$. Let K_n be the complete graph of order n and $K_{p,q}$ the complete bipartite graph with one partite set of size p and the other one of size q.
In this paper we continue the study of signed Roman dominating functions in graphs and digraphs, introduced and investigated by Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1]. Since then several related variations and generalizations have been studied (see for example [2, 3] and the survey articles [4–6]). For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f \colon V(G) \to \mathbb{R}$, we define

$$f(S) = \sum_{x \in S} f(x).$$

Let $k \geq 1$ be an integer. The *weak signed total double Roman k-dominating function*, abbreviated WSTDRkDF, on a graph G is defined as a function $f \colon V(G) \to \{-1, 1, 2, 3\}$ such that $f(N(v)) = \sum_{x \in N(v)} f(x) \geq k$ for each vertex $v \in V(G)$. The weight of a WSTDRkDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The *weak signed total double Roman k-domination number* $\gamma^k_{wstdR}(G)$ of G is the minimum weight of a WSTDRkDF on G. A $\gamma^k_{wstdR}(G)$-function of G is a WSTDRkDF of weight $\gamma^k_{wstdR}(G)$.
The *signed total double Roman k-dominating function*, abbreviated STDRkDF, on a graph G is defined in [10] as a WSTDRkDF f such that every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 3$ or adjacent to two vertices x and y with $f(x) = f(y) = 2$, and every vertex u with $f(u) = 1$ is adjacent to vertex v with $f(v) \geq 2$. The weight of an STDRkDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The *signed total double Roman k-domination number* $\gamma^k_{stdR}(G)$ of G is the minimum weight of an STDRkDF on G. A $\gamma^k_{stdR}(G)$-function of G is an STDRkDF of weight $\gamma^k_{stdR}(G)$.
For a WSTDRkDF or STDRkDF f on G, let

$$V_i = V_i(f) = \{v \in V(G) \colon f(v) = i\} \quad \text{for} \quad i \in \{-1, 1, 2, 3\}.$$

A WSTDRkGF or an STDRkDF $f \colon V(G) \to \{-1, 1, 2, 3\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2, V_3) of $V(G)$. In the special case $k = 1$ we also write $\gamma^1_{stdR}(G) = \gamma_{stdR}(G)$ and $\gamma^1_{wstdR}(G) = \gamma_{wstdR}(G)$. The parameter $\gamma_{stdR}(G)$ was introduced and investigated in [9, 11]. The weak signed total double Roman k-domination number or signed total double Roman k-domination exist when $\delta(G) \geq$

$\frac{k}{3}$. Thus we assume throughout this paper that $\delta \geq \frac{k}{3}$. The definitions lead to $\gamma^k_{wstdR}(G) \leq \gamma^k_{stdR}(G)$.
In this paper we initiate the study of the weak signed total double Roman k-domination number of graphs. We present different bounds on this parameter. In particular, we will see that many lower bounds on $\gamma^k_{stdR}(G)$ are also valid for $\gamma^k_{wstdR}(G)$. In addition, we show that the difference $\gamma_{stdR}(G) - \gamma_{wstdR}(G)$ can be arbitrarily large, and we determine the weak signed total double Roman k-domination number of some classes of graphs.
We make use of the of the following known results.

Example 1. [12] Let $n, k \geq 1$ be integers such that $k \leq 3n - 3$ and $n \geq 5$. Then $\gamma^k_{stdR}(K_n) = k + 3$ when $k \geq 2n - 1$ and $\gamma^k_{stdR}(K_n) = k + 2$ when $k \leq 2n - 2$.

Proposition 1. [11] *If $n \geq 3$, then $\gamma_{stdR}(C_n) = n$ when $n \neq 5$ and $\gamma_{stdR}(C_5) = 6$.*

Proposition 2. [12] *If $n \geq 3$ and $2 \leq k \leq 6$ are integers, then it holds:*

(a) *If $n = 3$, then $\gamma^2_{stdR}(C_3) = 4$. If $n \geq 4$, then $\gamma^2_{stdR}(C_n) = n$ when $n \equiv 0 \pmod 4$. If $n \geq 5$, then $\gamma^2_{stdR}(C_n) = n + 2$ when $n \equiv 1, 3 \pmod 4$. If $n \equiv 2 \pmod 4$, then $\gamma^2_{stdR}(C_6) = 8$ and $\gamma^2_{stdR}(C_n) = n + 4$ when $n \geq 10$.*

(b) *$\gamma^3_{stdR}(C_n) = \lceil \frac{3n}{2} \rceil + 1$ when $n \equiv 2 \pmod 4$ and $\gamma^3_{stdR}(C_n) = \lceil \frac{3n}{2} \rceil$ otherwise.*

(c) *$\gamma^4_{stdR}(C_n) = 2n$.*

(d) *$\gamma^5_{stdR}(C_n) = \lceil \frac{5n}{2} \rceil + 1$ when $n \equiv 2 \pmod 4$ and $\gamma^5_{stdR}(C_n) = \lceil \frac{5n}{2} \rceil$ otherwise.*

(e) *$\gamma^6_{stdR}(C_n) = 3n$.*

Proposition 3. [11] *Let $1 \leq p \leq q$ be integers. Then $\gamma_{stdR}(K_{p,q}) = 4$ when $q = p = 2$, $p = 2$ and $q = 4$ or $p = 1$ and $q \geq 2$, $\gamma_{stdR}(K_{p,q}) = 3$ when $p = 3$ and $q \neq 4$ or $p \geq 5$ and $\gamma_{stdR}(K_{p,q}) = 2$ otherwise.*

Proposition 4. [12] *If k, p and q are integers such that $1 \leq p \leq q$ and $2 \leq k \leq 3p$, then it holds:*

(a) *If $4 \leq p$, then $\gamma^k_{stdR}(K_{p,q}) = 2k$.*

(b) *Let $p = 3$. If $3 \leq k \leq 9$, then $\gamma^k_{stdR}(K_{3,q}) = 2k$. In addition, $\gamma^2_{tsdR}(K_{3,q}) = 5$ when $q \geq 4$ and $\gamma^2_{stdR}(K_{3,3}) = 6$.*

(c) *Let $p = 2$. If $2 \leq k \leq 6$ with $k \neq 3$ and $q \neq 3$, then $\gamma^k_{tsdR}(K_{2,q}) = 2k$. In addition, $\gamma^2_{stdR}(K_{2,3}) = 5$, $\gamma^3_{stdR}(K_{2,2}) = 6$, $\gamma^3_{stdR}(K_{2,q}) = 7$ for $q \geq 3$.*

(d) *Let $p = 1$ and $2 \leq k \leq 3$. Then $\gamma^3_{stdR}(K_{1,q}) = 2k = 6$, $\gamma^2_{stdR}(K_{1,q}) = 2k + 1$ when $q \geq 3$, $\gamma^2_{stdR}(K_{1,1}) = 2k = 4$ and $\gamma^2_{stdR}(K_{1,2}) = 2k = 4$.*

2. Preliminary results

In this section we present basic properties of the weak signed total double Roman k-dominating functions and the weak signed total double Roman k-domination numbers. The definitions immediately lead to our first proposition.

Proposition 5. *If $f = (V_{-1}, V_1, V_2, V_3)$ is a WSTDRkDF on a graph G of order n, then the following properties hold.*

(a) $|V_{-1}| + |V_1| + |V_2| + |V_3| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.

The proof of the next proposition is identically with the proof of Proposition 2.2 in [10] (see also [12]) and is therefore omitted.

Proposition 6. *Let $k \geq 1$ be an integer. If $f = (V_{-1}, V_1, V_2, V_3)$ is a WSTDRkDF on a graph G of order n where $\Delta = \Delta(G)$ and $\delta = \delta(G)$, then the following properties hold.*

(a) $(3\Delta - k)|V_3| + (2\Delta - k)|V_2| + (\Delta - k)|V_1| \geq (\delta + k)|V_{-1}|$.

(b) $(3\Delta + \delta)|V_3| + (2\Delta + \delta)|V_2| + (\Delta + \delta)|V_1| \geq (\delta + k)n$.

(c) $(\Delta + \delta)\omega(f) \geq (\delta + 2k - \Delta)n + (\delta - \Delta)|V_2| + 2(\delta - \Delta)|V_3|$.

(d) $\omega(f) \geq (\delta + 2k - 3\Delta)n/(3\Delta + \delta) + |V_2| + 2|V_3|$.

As an application of Proposition 6 (c), we obtain a lower bound on the weak signed total double Roman k-domination number for r-regular graphs.

Corollary 1. *Let $k \geq 1$ be an integer. If G is an r-regular graph of order n with $r \geq \frac{k}{3}$, then $\gamma^k_{wstdR}(G) \geq \lceil \frac{kn}{r} \rceil$.*

Multiplying both sides of the inequality in Proposition 6 (d) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 6 (c), we obtain the next lower bound.

Corollary 2. *Let $k \geq 1$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq \frac{k}{3}$ and maximum degree Δ. If $\delta < \Delta$, then*

$$\gamma^k_{wstdR}(G) \geq \left(\frac{-3\Delta + 3\delta + 4k}{3\Delta + \delta} \right) n.$$

Since $\gamma^k_{stdR}(G) \geq \gamma^k_{wstdR}(G)$, the next upper bound is immediate by Corollary 2.

Corollary 3. [10, 12] *Let $k \geq 1$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq \frac{k}{3}$ and maximum degree Δ. If $\delta < \Delta$, then*

$$\gamma^k_{stdR}(G) \geq \left(\frac{-3\Delta + 3\delta + 4k}{3\Delta + \delta}\right) n.$$

In [12], the authors present examples which demonstrate that the bound in Corollary 3 is sharp for all integers $k \geq 1$. These examples also show that Corollary 2 is sharp for all integers $k \geq 1$.

Theorem 1. *Let G be a graph and $k \geq 2$ be an integer.*

(a) *If $k \geq \Delta(G) + 2$, then $\gamma^k_{wstdR}(G) = \gamma^k_{stdR}(G)$.*

(b) *If $k = \Delta(G) + 1$ and there exists a $\gamma^k_{wstdR}(G)$-function g with $g(x) \neq -1$ for all $x \in V(G)$, then $\gamma^k_{wstdR}(G) = \gamma^k_{stdR}(G)$.*

Proof. (a) If f is a $\gamma^k_{wstdR}(G)$-function, then we show that f is an STDRkDF of G. The definition implies $f(N(v)) \geq k$ for each $v \in V(G)$. Now let w be a vertex such that $f(w) = 1$. If we assume that $f(x) \leq 1$ for all $x \in N(w)$, then the hypothesis $k \geq \Delta(G) + 2$ leads to the contradiction $f(N(w)) \leq \deg(w) \leq \Delta(G) \leq k - 2$. Thus there exists a neighbor z of w with $f(z) \geq 2$. Next let w be a vertex such that $f(w) = -1$. If we assume that $f(x) \leq 2$ for all $x \in N(w)$ and there exists at most one neighbor u with $f(u) = 2$, then we obtain the contradiction $f(N(w)) \leq \deg(w) + 1 \leq \Delta(G) + 1 \leq k - 1$. Thus there exists at least one neighbor z of w with $f(z) = 3$ or two neighbors u and v of w such that $f(u), f(v) \geq 2$. Consequetnly, f is an STDRkDF of G, and we deduce that

$$\gamma^k_{stdR}(G) \leq \omega(f) = \gamma^k_{wstsR}(G) \leq \gamma^k_{stdR}(G)$$

and therefore $\gamma^k_{wstdR}(G) = \gamma^k_{stdR}(G)$.
(b) Since $g(x) \neq -1$ for all $x \in V(G)$, we obtain the contradiction $f(N(w)) \leq \deg(w) \leq \Delta(G) = k - 1$, if we assume that $g(w) = 1$ and $g(y) = 1$ for all $y \in N(w)$. Thus there exists a neighbor z of w with $g(z) \geq 2$ for each vertex w with $f(w) = 1$. Hence g is an STDRkDF of G, and we deduce $\gamma^k_{wstdR}(G) = \gamma^k_{stdR}(G)$ as above. $\square$

3. Weak signed total double Roman k-domination number of K_n

Example 2. Let $n, k \geq 1$ be integers such that $k \leq 3n - 3$ and $n \geq 2$.

(a) If $2n - 1 \leq k \leq 3n - 3$, then $\gamma^k_{wstdR}(K_n) = k + 3$.

(b) If $n \leq k \leq 2n - 2$, then $\gamma^k_{wstdR}(K_n) = k + 2$.

(c) If $1 \leq k \leq n - 1$ and $n - k$ is odd, then $\gamma^k_{wstdR}(K_n) = k + 1$.

(d) If $1 \leq k \leq n - 1$ and $n - k$ is even, then $\gamma^k_{wstdR}(K_n) = k + 2$.

Proof. Let f be a $\gamma^k_{wstdR}(K_n)$-function. Since there exists at least on vertex w with $f(w) \geq 1$, we deduce that $\gamma^k_{wstdR}(K_n) = \omega(f) = f(w) + f(N(w)) \geq k + 1$.
(a) If $2n - 1 \leq k \leq 3n - 3$, then Theorem 1 (a) implies the desired result.
(b) Let $n \leq k \leq 2n - 2$. Then there exists at least one vertex w with $f(w) \geq 2$ and therefore $\omega(f) = f(w) + f(N(w)) \geq k + 2$. Hence Example 1 implies $k + 2 = \gamma^k_{stdR}(K_n) \geq \gamma^k_{wstdR}(K_n) \geq k + 2$ and so $\gamma^k_{wstdR}(K_n) = k + 2$.
(c) Let $1 \leq k \leq n - 1$ and $n - k$ odd. Let the function $g\colon V(K_n) \to \{-1, 1, 2, 3\}$ assign to $(n + k + 1)/2$ vertices the value 1 and to $(n - k - 1)/2$ vertices the value -1. Then g is a WSTDRkDF on K_n of weight $k + 1$ and hence $\gamma^k_{wstdR}(K_n) = k + 1$ in this case.
(d) Let $1 \leq k \leq n - 1$ and $n - k$ even. If there exists a vertex w with $f(w) \geq 2$, then $\gamma^k_{wstdR}(K_n) = \omega(f) = f(w) + f(N(w)) \geq k + 2$. If $f(x) \in \{-1, 1\}$ for all $x \in V(K_n)$, then $f(N(x))$ is even when n is odd and $f(N(x))$ is odd when n is even. Since $f(N(x)) \geq k$, we observe that $f(N(x)) \geq k + 1$ when $n - k$ is even. If w is a vertex with $f(w) = 1$, we deduce in this case that $\gamma^k_{wstdR}(K_n) = \omega(f) = f(w) + f(N(w)) \geq 1 + k + 1 = k + 2$. Now let the function $g\colon V(K_n) \to \{-1, 1, 2, 3\}$ assign to $(n+k+2)/2$ vertices the value 1 and to $(n-k-2)/2$ vertices the value -1. Then g is a WSTDRkDF on K_n of weight $k + 2$ and hence $\gamma^k_{wstdR}(K_n) = k + 2$ in the last case. □

4. Bounds

Observation 2. Let $k \geq 1$ be an integer. If G is a graph of order n with $\delta(G) \geq k$, then $\gamma^k_{wstdR}(G) \leq n$.

Proof. Define the function $f\colon V(G) \to \{-1, 1, 2, 3\}$ by $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G) \geq k$, the function f is a WSTDRkDF on G of weight n and thus $\gamma^k_{wstdR}(G) \leq n$. □

Example 3. Let $k \geq 1$ be an integer. If H is k-regular graph of order n, then it follows from Corollary 1 that $\gamma^k_{wstdR}(H) \geq n$ and so $\gamma^k_{wstdR}(H) = n$, according to Observation 2.

Example 3 shows that Corollary 1 and Observation 2 are both sharp.

A *packing* (also called a 2-*packing* in the literature) of a graph G is a set of vertices that are mutually distance at least 3 apart. Thus, a set S of vertices in a graph G is a packing of G if $N_G[u] \cap N_G[v] = \emptyset$ for every two distinct vertices $u, v \in S$. The *packing number* of G, denoted by $\rho(G)$, is the maximum size of a packing in G.
Analogously to Proposition 2.6 in [10] and Theorem 12 in [12], one can prove the following bounds.

Proposition 7. *If $k \geq 1$ is an integer and G a graph of order n with $\delta(G) \geq \frac{k}{3}$, then*

$$\gamma^k_{wstdR}(G) \geq k + \Delta(G) - n.$$

Proposition 8. *If $k \geq 1$ is an integer and G a graph of order n with minimum degree $\delta(G) \geq \frac{k}{3}$, then*

$$\gamma^k_{wstdR}(G) \geq \rho(G)(\delta(G) + k) - n.$$

In [12], the authors present infinite families of graphs H with $\gamma^k_{stdR}(H) = \rho(H)(\delta(H) + k) - n(H)$. Since $\gamma^k_{stdR}(H) \geq \gamma^k_{wstdR}(H)$, we also have equality in Proposition 8 for these families.

Theorem 3. *If $k \geq 1$ is an integer and G a graph of order n with minimum degree $\delta(G) \geq \frac{k}{3}$, then*

$$\gamma^k_{wstdR}(G) \geq k + 2 + \delta(G) - n.$$

If in addition $\delta(G) - k$ is odd, then $\gamma^k_{wstdR}(G) \geq k + 3 + \delta(G) - n$.

Proof. Let f be a $\gamma^k_{wstdR}(G)$-function. Then there exists a vertex w with $f(w) \geq 1$. It follows from the definitions that

$$\begin{aligned}
\gamma^k_{wstdR}(G) &= \sum_{x \in V(G)} f(x) = f(w) + \sum_{x \in N(w)} f(x) + \sum_{x \in V(G) \setminus N[w]} f(x) \\
&\geq 1 + k - (n - d(w) - 1) = k + 2 + d(w) - n \geq k + 2 + \delta(G) - n.
\end{aligned}$$

Now assume that $\delta(G) - k$ is odd. If there exists a vertex w with $f(w) \geq 2$ or a vertex v with $f(v) = 1$ and $d(v) \geq \delta(G) + 1$, then we obtain $\gamma^k_{wstdR}(G) \geq k + 3 + \delta(G) - n$ analogously to the inequality chain above. So assume next that $f(x) \in \{-1, 1\}$ for all $x \in V(G)$, and each vertex y with $f(y) = 1$ has degree $d(y) = \delta(G)$. Let now u be a vertex of minimum degree with $d(u) = 1$. Since $\delta(G) - k$ is odd, we observe that $\sum_{x \in N(u)} f(x) \geq k + 1$ and thus

$$\begin{aligned}
\gamma^k_{wstdR}(G) &= \sum_{x \in V(G)} f(x) = f(u) + \sum_{x \in N(u)} f(x) + \sum_{x \in V(G) \setminus N[u]} f(x) \\
&\geq 1 + k + 1 - (n - d(u) - 1) \\
&= k + 3 + d(u) - n \\
&= k + 3 + \delta(G) - n.
\end{aligned}$$

□

Example 1 (c) and (d) show that Theorem 3 is sharp for $k \leq n - 1$.
Using Corollary 1, one can prove analogously to Theorem 2.4 in [10] the following Nordhaus-Gaddum type result.

Proposition 9. *If $k \geq 1$ is an integer and G an r-regular graph of order n such that* $\min\{r, n-r-1\} \geq k/3$, *then*

$$\gamma^k_{wstdR}(G) + \gamma^k_{wstdR}(\overline{G}) \geq \frac{4kn}{n-1}.$$

If n is even, then $\gamma^k_{wstdR}(G) + \gamma^k_{wstdR}(\overline{G}) \geq 4k(n-1)/(n-2)$.

Let $k \geq 2$ be an even integer, and let H and $\overline{H}$ be k-regular of order $n = 2k+1$. Then Example 3 implies $\gamma^k_{wstdR}(H) = \gamma^k_{wstdR}(\overline{H}) = n$. Consequently,

$$\gamma^k_{wstdR}(H) + \gamma^k_{wstdR}(\overline{H}) = 2n = \frac{4kn}{n-1}.$$

Therefore the Nordhaus-Gaddum bound of Proposition 9 is sharp for even $k \geq 2$.

5. Weak signed total double Roman k-domination number of cycles

Proposition 10. *Let C_n be a cycle of length $n \geq 3$. If $1 \leq k \leq 6$ is an integer, then it holds:*

(a) $\gamma_{wstdR}(C_n) = n/2$ *when* $n \equiv 0 \pmod 4$, $\gamma_{wstdR}(C_n) = (n+3)/2$ *when* $n \equiv 1, 3 \pmod 4$ *and* $\gamma_{wstdR}(C_n) = (n+6)/2$ *when* $n \equiv 2 \pmod 4$.

(b) $\gamma^2_{wstdR}(C_n) = n$.

(c) $\gamma^3_{stdR}(C_n) = \lceil \frac{3n}{2} \rceil + 1$ *when* $n \equiv 2 \pmod 4$ *and* $\gamma^3_{stdR}(C_n) = \lceil \frac{3n}{2} \rceil$ *otherwise.*

(d) $\gamma^4_{stdR}(C_n) = 2n$.

(e) $\gamma^5_{stdR}(C_n) = \lceil \frac{5n}{2} \rceil + 1$ *when* $n \equiv 2 \pmod 4$ *and* $\gamma^5_{stdR}(C_n) = \lceil \frac{5n}{2} \rceil$ *otherwise.*

(f) $\gamma^6_{stdR}(C_n) = 3n$.

Proof. Let $C_n = v_1 v_2 \ldots v_n v_1$. It follows from Corollary 1 that

$$\gamma^k_{wstdR}(C_n) \geq \left\lceil \frac{kn}{2} \right\rceil. \tag{5.1}$$

If f is a WSTDRkDF on C_n, then we note that $f(v_i) + f(v_{i+2}) \geq k$ for all $1 \leq i \leq n$ modulo n.

(a) Let $k = 1$. Assume first that $n = 4p$ with an integer $p \geq 1$. Then (1) implies $\gamma_{wstdR}(C_n) \geq n/2$. Now define the function g by $g(v_{4i-1}) = g(v_{4i}) = 2$ for $1 \leq i \leq p$ and $g(x) = -1$ otherwise. Then g is a WSTDR1DF on C_n of weight $n/2$ and thus $\gamma_{wstdR}(C_n) \leq n/2$. This leads to $\gamma_{wstdR}(C_n) = n/2$ when $n \equiv 0 \pmod 4$.

Assume second that $n = 4p+1$ with an integer $p \geq 1$, and let f be a WSTDR1DF on C_n. If $f(v_i) \geq 1$ for all $1 \leq i \leq n$, then $\omega(f) \geq n \geq (n+3)/2$. Hence assume now, without loss of generality, that $f(v_2) = -1$. It follows that $f(v_{4p+1}) \geq 2$ and so

$$\omega(f) = f(v_{4p+1}) + \sum_{i=0}^{p-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) \geq 2 + 2p = \frac{n+3}{2}.$$

Conversely, define the function g by $g(v_{4i-1}) = g(v_{4i}) = 2$ for $1 \leq i \leq p$, $g(v_{4p+1}) = 2$ and $g(x) = -1$ otherwise. Then g is a WSTDR1DF on C_n of weight $2p+2 = (n+3)/3$ and thus $\gamma_{wstdR}(C_n) \leq (n+3)/2$. This leads to $\gamma_{wstdR}(C_n) = (n+3)/2$ when $n \equiv 1 \pmod 4$.

Assume third that $n = 4p+2$ with an integer $p \geq 1$, and let f be a WSTDR1DF on C_n. If $f(v_i) \geq 1$ for all $1 \leq i \leq n$, then $\omega(f) \geq n \geq (n+6)/2$. Hence assume now, without loss of generality, that $f(v_2) = -1$. It follows that $f(v_{4p+2}) \geq 2$ and $f(v_4) \geq 2$. If $f(v_1) \geq 2$ or $f(v_{4p+1}) \geq 2$, say $f(v_{4p+1}) \geq 2$, then

$$\omega(f) = f(v_{4p+2})+f(v_{4p+1})+\sum_{i=0}^{p-1}(f(v_{4i+1})+f(v_{4i+2})+f(v_{4i+3})+f(v_{4i+4})) \geq 4+2p = \frac{n+6}{2}.$$

Hence assume now that $f(v_1), f(v_{4p+1}) \leq 1$. This implies $f(v_1) = f(v_{4p+1}) = 1$ and $f(v_3) \geq 1$. Therefore, we deduce that

$$\begin{aligned}\omega(f) &= f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_{4p+2}) + f(v_{4p+1}) \\ &+ \sum_{i=1}^{p-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) \geq 6 + 2(p-1) = \frac{n+6}{2}.\end{aligned}$$

Conversely, define the function g by $g(v_{4i-1}) = g(v_{4i}) = 2$ for $1 \leq i \leq p$, $g(v_{4p+1}) = g(v_{4p+2}) = 2$ and $g(x) = -1$ otherwise. Then g is a WSTDR1DF on C_n of weight $2p+4 = (n+6)/2$ and thus $\gamma_{wstdR}(C_n) \leq (n+6)/2$. This leads to $\gamma_{wstdR}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.

Finally, assume that $n = 4p+3$ with an integer $p \geq 0$, and let f be a WSTDR1DF on C_n. If $f(v_i) \geq 1$ for all $1 \leq i \leq n$, then $\omega(f) \geq n \geq (n+3)/2$. Hence assume now, without loss of generality, that $f(v_2) = -1$. It follows that $f(v_{4p+3}) \geq 2$. If $f(v_1) \geq 2$ or $f(v_{4p+2}) \geq 2$, say $f(v_{4p+2}) \geq 2$, then

$$\omega(f) = f(v_{4p+1})+f(v_{4p+2})+f(v_{4p+3})+\sum_{i=0}^{p-1}(f(v_{4i+1})+f(v_{4i+2})+f(v_{4i+3})+f(v_{4i+4})) \geq 3+2p = \frac{n+3}{2}.$$

Hence assume next that $f(v_1), f(v_{4p+2}) \leq 1$. This implies $f(v_1) = f(v_{4p+2}) = 1$ and so

$$\omega(f) = f(v_1)+f(v_{4p+2})+f(v_{4p+3})+\sum_{i=1}^{p}(f(v_{4i-2})+f(v_{4i-1})+f(v_{4i})+f(v_{4i+1})) \geq 4+2p = \frac{n+5}{2}.$$

Conversely, define the function g by $g(v_{4i-1}) = g(v_{4i}) = 2$ for $1 \le i \le p$, $g(v_{4p+2}) - g(v_{4p+3}) = 2$ and $g(x) = -1$ otherwise. Then g is a WSTDR1DF on C_n of weight $2p+3 = (n+3)/3$ and thus $\gamma_{wstdR}(C_n) \le (n+3)/2$. This yields to $\gamma_{wstdR}(C_n) = (n+3)/2$ when $n \equiv 3 \pmod 4$, and the proof of (a) is complete.
(b) Let $k = 2$. Inequality (1) implies $\gamma^2_{wstdR}(C_n) \ge n$. Conversely, the function g with $g(v_i) = 1$ for $1 \le i \le n$, is a WSTDR2DF on C_n of weight n and thus $\gamma^2_{wstdR}(C_n) = n$.
(c) Let $k = 3$. Because of $f(v_i) + f(v_{i+2}) \ge k = 3$ for all $1 \le i \le n$ modulo n, we note that $f(v_i) \ne -1$ for all $1 \le i \le n$. Therefore Theorem 1 (b) and Proposition 2 (b) lead to the desired result.
Finally, (d), (e) and (f) follow from Theorem 1 and Proposition 2 (c), (d) and (e) immediately.

□

If C_n is a cycle of length $n \ge 6$, then Propsotion 1 and Proposition 10 (a) lead to $\gamma_{stdR}(C_n) - \gamma_{wstdR}(C_n) \ge n - (n+6)/2 = (n-6)/2$. Thus the difference $\gamma_{stdR}(G) - \gamma_{wstdR}(G)$ can be arbitrarily large.

6. Weak signed total double Roman k-domination number of $K_{p,q}$

Proposition 11. *If k, p and q are integers such that $1 \le p \le q$ and $1 \le k \le 3p$, then $\gamma^k_{wstdR}(K_{p,q}) = 2k$.*

Proof. Let X and Y be the partite sets of $K_{p,q}$, and let g be a $\gamma^k_{wtsdR}(K_{p,q})$-function. If $u \in X$ and $v \in Y$, then it follows that

$$\gamma^k_{wstdR}(K_{p,q}) = g(X) + g(Y) = g(N(v)) + g(N(u)) \ge 2k.$$

For the converse we start with $k = 1$. Let $X = \{v_1, v_2, \ldots, v_d\}$ or $Y = \{v_1, v_2, \ldots, v_d\}$ with $d = p$ or $d = q$. We distinguish two cases. If $d = 2t$ with an integer $t \ge 1$, then define $f(v_1) = 2$, $f(v_2) = f(v_3) = \ldots = f(v_t) = 1$ and $f(v_{t+1}) = f(v_{t+2}) = \ldots = f(v_{2t}) = -1$. If $d = 2t+1$ with an integer $t \ge 0$, then define $f(v_1) = f(v_2) = \ldots = f(v_{t+1}) = 1$ and $f(v_{t+2}) = f(v_{t+3}) = \ldots = f(v_{2t+1}) = -1$. These two cases show that there exists a function $f\colon V(K_{p,q}) \to \{-1, 1, 2, 3\}$ satisfying $f(X) = f(Y) = 1$. Therefore f is a WSTDR1DF on $K_{p,q}$ of weight 2 and so $\gamma_{wtsdR}(K_{p,q}) \le 2$. Therfore $\gamma_{wtsdR}(K_{p,q}) = 2 = 2k$ in this case.
For $2 \le k \le 3p$, we mainly use Proposition 4.
If $p \ge 4$, then Proposition 4 (a) implies $\gamma^k_{wstdR}(K_{p,q}) \le \gamma^k_{stdR}(K_{p,q}) = 2k$. Therefore $\gamma^k_{wstdR}((K_{p,q}) = 2k$ in this case.
If $p = 3$, then Proposition 4 (b) leads to $\gamma^k_{wstdR}(K_{3,q}) = 2k$ for $3 \le k \le 9$ as before. In the case $k = 2$ let $X = \{x_1, x_2, x_3\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$. Define f by $f(x_1) = 2$, $f(x_2) = 1$ and $f(x_3) = -1$. If $q = 2t$ with an integer $t \ge 2$, then define $f(v_1) = f(v_2) = \ldots = f(v_{t+1}) = 1$ and $f(v_{t+2}) = f(v_{t+3}) = \ldots = f(v_{2t}) = -1$. If $q = 2t+1$ with an integer $t \ge 1$, then define $f(v_1) = 2$, $f(v_2) = f(v_3) = \ldots = f(v_{t+1}) = 1$ and

$f(v_{t+2}) = f(v_{t+3}) = \ldots = f(v_{2t+1}) = -1$. In both cases $f\colon V(K_{3,q}) \to \{-1,1,2,3\}$ is a function with $f(X) = f(Y) = 2$. Therefore f is a WSTDR2DF on $K_{3,q}$ of weight 4 and so $\gamma^2_{wtsdR}(K_{3,q}) \leq 4$. Therfore $\gamma^2_{wtsdR}(K_{3,q}) = 4 = 2k$ in this case.
Let now $p = 2$ and $2 \leq k \leq 6$. If $k \neq 3$, then it follows easily from Proposition 4 (c) that $\gamma^k_{wstdR}(K_{2,q}) = 2k$. In the case $k = 3$ let $X = \{x_1, x_2\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$. Define f by $f(x_1) = 2$ and $f(x_2) = 1$. If $q = 2t$ with an integer $t \geq 1$, then define $f(v_1) = 2$, $f(v_2) = f(v_3) = \ldots = f(v_{t+1}) = 1$ and $f(v_{t+2}) = f(v_{t+3}) = \ldots = f(v_{2t}) = -1$. If $q = 2t + 1$ with an integer $t \geq 1$, then define $f(v_1) = 3$, $f(v_2) = f(v_3) = \ldots = f(v_{t+1}) = 1$ and $f(v_{t+2}) = f(v_{t+3}) = \ldots = f(v_{2t+1}) = -1$. In both cases $f\colon V(K_{2,q}) \to \{-1,1,2,3\}$ is a function with $f(X) = f(Y) = 3$. Therefore f is a WSTDR3DF on $K_{2,q}$ of weight 6 and so $\gamma^3_{wtsdR}(K_{2,q}) \leq 6$. Thus $\gamma^3_{wtsdR}(K_{2,q}) = 6 = 2k$.
Finally, $p = 1$ and $2 \leq k \leq 3$. If $k = 3$, then Proposition 4 (d) yields the desired result. In the case $k = 2$ let $X = \{x\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$. Define f by $f(x) = 2$. If $q = 2t$ with an integer $t \geq 1$, then define $f(v_1) = f(v_2) = \ldots = f(v_{t+1}) = 1$ and $f(v_{t+2}) = f(v_{t+3}) = \ldots = f(v_{2t}) = -1$. If $q = 2t + 1$ with an integer $t \geq 0$, then define $f(v_1) = 2$, $f(v_2) = f(v_3) = \ldots = f(v_{t+1}) = 1$ and $f(v_{t+2}) = f(v_{t+3}) = \ldots = f(v_{2t+1}) = -1$. Clearly, $f(X) = f(Y) = 2$, and so f is a WSTDR2DF on $K_{1,q}$ of weight 4 and thus $\gamma^2_{wtsdR}(K_{1,q}) \leq 4$. Hence $\gamma^2_{wtsdR}(K_{1,q}) = 4 = 2k$ also in the last case. □

7. Weak signed total double Roman k-domination number of complete p-partite graphs

We start with a useful lemma.

Lemma 1. *If $k \geq 1$ is an integer and $G = K_{n_1,n_2,\ldots,n_p}$ a complete p-partite graph with $p \geq 2$, then*

$$\gamma^k_{wstdR}(G) \geq \left\lceil \frac{kp}{p-1} \right\rceil = k + \left\lceil \frac{k}{p-1} \right\rceil.$$

Proof. Let $X_1, X_2, \ldots, X_p$ be the partite sets of G, and let f be a $\gamma^k_{wstdR}(G)$-function. If $x_i \in X_i$ for $1 \leq i \leq p$, then the definition implies $f(V(G) \setminus X_i) = f(N(x_i)) \geq k$. Therefore

$$(p-1)\omega(f) = (p-1)f(V(G)) = \sum_{i=1}^{p} f(V(G) \setminus X_i) \geq pk$$

and thus

$$\gamma^k_{wstdR}(G) = \omega(f) \geq \left\lceil \frac{kp}{p-1} \right\rceil = k + \left\lceil \frac{k}{p-1} \right\rceil.$$

□

Proposition 12. *Let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $1 \le n_1 \le n_2 \le \dots \le n_p$, $n_p \ge 2$ and $p \ge 3$. If $1 \le k \le p-1$ is an intger, then $\gamma^k_{wstdR}(G) = k+1$.*

Proof. Lemma 1 leads to $\gamma^k_{wstdR}(G) \ge k+1$.
Conversely, let $X_1, X_2, \dots, X_p$ be the partite sets of G, and let $X_i = \{v_i^1, v_i^2, \dots, v_i^{n_i}\}$ for $1 \le i \le p$.
Case 1. Assume that $n_1 \ge 2$. We shall define a function $f\colon V(G) \to \{-1,1,2,3\}$ such that $f(X_1) = f(X_2) = \dots = f(X_{k+1}) = 1$ and $f(X_i) = 0$ for $k+2 \le i \le p$.
First let $1 \le i \le k+1$. If $X_i = \{v_i^1, v_i^2, \dots, v_i^{2t}\}$ for an integer $t \ge 1$, then define $f(v_i^1) = 2$, $f(v_i^2) = f(v_i^3) = \dots, f(v_i^t) = 1$ and $f(v_i^{t+1}) = f(v_i^{t+2}) = \dots, f(v_i^{2t}) = -1$. If $X_i = \{v_i^1, v_i^2, \dots, v_i^{2t+1}\}$ for an integer $t \ge 1$, then define $f(v_i^1) = f(v_i^2) = \dots, f(v_i^{t+1}) = 1$ and $f(v_i^{t+2}) = f(v_i^{t+3}) = \dots, f(v_i^{2t+1}) = -1$.
Second let $k+2 \le i \le p$. If $X_i = \{v_i^1, v_i^2, \dots, v_i^{2t}\}$ for an integer $t \ge 1$, then define $f(v_i^1) = f(v_i^2) = \dots, f(v_i^t) = 1$ and $f(v_i^{t+1}) = f(v_i^{t+2}) = \dots, f(v_i^{2t}) = -1$. If $X_i = \{v_i^1, v_i^2, \dots, v_i^{2t+1}\}$ for an integer $t \ge 1$, then define $f(v_1^1) = 2$, $f(v_i^2) = f(v_i^3) = \dots, f(v_i^t) = 1$ and $f(v_i^{t+1}) = f(v_i^{t+2}) = \dots, f(v_i^{2t+1}) = -1$.
We observe that f is a WSTDRkDF on G of weight $k+1$ and therefore $\gamma^k_{wstdR}(G) \le k+1$. This leads to $\gamma^k_{wstdR}(G) = k+1$ in the first case.
Case 2. Assume that $n_1 = n_2 = \dots = n_q = 1$ for an integer $1 \le q \le p-1$. If $k \ge q-1$, then it is no problem to define a function $f\colon V(G) \to \{-1,1,2,3\}$ such that $f(X_1) = f(X_2) = \dots = f(X_{k+1}) = 1$ and $f(X_i) = 0$ for $k+2 \le i \le p$. Let now $1 \le k \le q-2$. If $q-k-1 = 2j$ is even, then define $f\colon V(G) \to \{-1,1,2,3\}$ such that $f(X_1) = f(X_2) = \dots = f(X_{k+1}) = 1$, $f(X_{k+2}) = f(X_{k+3}) = \dots = f(X_{k+j+1}) = 1$, $f(X_{k+j+2}) = f(X_{k+j+3}) = \dots = f(X_q) = -1$ and $f(X_i) = 0$ for $q+1 \le i \le p$. If $q-k-1 = 2j+1$ is odd, then $f\colon V(G) \to \{-1,1,2,3\}$ such that $f(X_1) = f(X_2) = \dots = f(X_{k+1}) = 1$, $f(X_{k+2}) = f(X_{k+3}) = \dots = f(X_{k+j+1}) = 1$, $f(X_{k+j+2}) = f(X_{k+j+3}) = \dots = f(X_q) = -1$, $f(X_p) = 1$ and $f(X_i) = 0$ for $q+1 \le i \le p-1$.
It is straightforward to veriy that f is a WSTDRkDF on G of weight $k+1$ in each of these cases and so $\gamma^k_{wstdR}(G) \le k+1$. Hence $\gamma^k_{wstdR}(G) = k+1$ also in the second case. □

Example 2 (d) shows that Proposition 12 is not valid in general when $n_p = 1$.

Proposition 13. *Let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $1 \le n_1 \le n_2 \le \dots \le n_p$ and $p \ge 3$. If $p \le k \le 2p-2$ is an intger, then $\gamma^k_{wstdR}(G) = k+2$.*

Proof. Lemma 1 implies $\gamma^k_{wstdR}(G) \ge k+2$.
Conversely, let $X_1, X_2, \dots, X_p$ be the partite sets of G, and let $X_i = \{v_i^1, v_i^2, \dots, v_i^{n_i}\}$ for $1 \le i \le p$. Since $p \le k \le 2p-2$, we observe that $k = p+s$ for an integer $0 \le s \le p-2$. Now we shall define a function $f\colon V(G) \to \{-1,1,2,3\}$ such that $f(X_1) = f(X_2) = \dots = f(X_{s+2}) = 2$ and $f(X_i) = 1$ for $s+3 \le i \le p$.
Case 1. Assume that $n_1 \ge 2$. First let $1 \le i \le s+2$. If $X_i = \{v_i^1, v_i^2, \dots, v_i^{2t}\}$ for an integer $t \ge 1$, then define $f(v_i^1) = f(v_i^2) = \dots, f(v_i^{t+1}) = 1$ and $f(v_i^{t+2}) = f(v_i^{t+3}) =$

$\ldots, f(v_i^{2t}) = -1$. If $X_i = \{v_i^1, v_i^2, \ldots, v_i^{2t+1}\}$ for an integer $t \geq 1$, then define $f(v_i^1) = 2$, $f(v_i^2) = f(v_i^3) = \ldots, f(v_i^{t+1}) = 1$ and $f(v_i^{t+2}) = f(v_i^{t+3}) = \ldots, f(v_i^{2t+1}) = -1$.
Second let $s+2 \leq i \leq p$. If $X_i = \{v_i^1, v_i^2, \ldots, v_i^{2t}\}$ for an integer $t \geq 1$, then define $f(v_i^1) = 2$, $f(v_i^2) = f(v_i^3) = \ldots, f(v_i^t) = 1$ and $f(v_i^{t+1}) = f(v_i^{t+2}) = \ldots, f(v_i^{2t}) = -1$. If $X_i = \{v_i^1, v_i^2, \ldots, v_i^{2t+1}\}$ for an integer $t \geq 1$, then define $f(v_i^1) = f(v_i^2) = \ldots, f(v_i^{t+1}) = 1$ and $f(v_i^{t+2}) = f(v_i^{t+3}) = \ldots, f(v_i^{2t+1}) = -1$.
We observe that f is a WSTDRkDF on G of weight $k+2$ and so $\gamma_{wstdR}^k(G) \leq k+2$. This leads to $\gamma_{wstdR}^k(G) = k+2$.
Case 2. Assume that $n_1 = n_2 = \ldots = n_q = 1$ for an integer $1 \leq q \leq p-1$. Since it is no problem to define $f(X_i) = 1$ or $f(X_i) = 2$ for $1 \leq i \leq q$, we obtain the desired result with the help of Case 1 immediately. □

Example 2 (b) is a special case of Proposition 13.

Proposition 14. *Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $1 \leq n_1 \leq n_2 \leq \ldots \leq n_p$ and $p \geq 3$. If $2p-1 \leq k \leq 3p-3$ is an intger, then $\gamma_{wstdR}^k(G) = k+3$.*

Proof. We deduce from Lemma 1 that $\gamma_{wstdR}^k(G) \geq k+3$.
Conversely, let $X_1, X_2, \ldots, X_p$ be the partite sets of G, and let $X_i = \{v_i^1, v_i^2, \ldots, v_i^{n_i}\}$ for $1 \leq i \leq p$. Since $2p-1 \leq k \leq 3p-3$, we observe that $k = 2p+s$ for an integer $-1 \leq s \leq p-3$. Now we shall define a function $f\colon V(G) \to \{-1,1,2,3\}$ such that $f(X_1) = f(X_2) = \ldots = f(X_{s+3}) = 3$ and $f(X_i) = 2$ for $s+4 \leq i \leq p$.
Case 1. Assume that $n_1 \geq 2$. First let $1 \leq i \leq s+3$. If $X_i = \{v_i^1, v_i^2, \ldots, v_i^{2t}\}$ for an integer $t \geq 1$, then define $f(v_i^1) = 2$, $f(v_i^2) = f(v_i^3) = \ldots, f(v_i^{t+1}) = 1$ and $f(v_i^{t+2}) = f(v_i^{t+3}) = \ldots, f(v_i^{2t}) = -1$. If $X_i = \{v_i^1, v_i^2, \ldots, v_i^{2t+1}\}$ for an integer $t \geq 1$, then define $f(v_i^1) = 3$, $f(v_i^2) = f(v_i^3) = \ldots, f(v_i^{t+1}) = 1$ and $f(v_i^{t+2}) = f(v_i^{t+3}) = \ldots, f(v_i^{2t+1}) = -1$.
Second let $s+4 \leq i \leq p$. Define f as in the proof of Proposition 13 for $1 \leq i \leq s+2$. We see that f is a WSTDRkDF on G of weight $k+3$, and we obtain $\gamma_{wstdR}^k(G) = k+3$.
Case 2. Assume that $n_1 = n_2 = \ldots = n_q = 1$ for an integer $1 \leq q \leq p-1$. Since it is no problem to define $f(X_i) = 2$ or $f(X_i) = 3$ for $1 \leq i \leq q$, Case 1 leads to the desired result. □

Propositions 12, 13 and 14, Example 2 and Proposition 11 yield the weak signed total double Roman k-domination number for all complete split graphs.

Corollary 4. *Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete split graph with $n_1 = n_2 = \ldots = n_{p-1} = 1$, $n_p \geq 2$ and $p \geq 3$. Then it holds:*

(a) *If $1 \leq k \leq p-1$, then $\gamma_{wstdR}^k(G) = k+1$.*

(b) *If $p \leq k \leq 2p-2$, then $\gamma_{wstdR}^k(G) = k+2$.*

(c) *If $2p-1 \leq k \leq 3p-3$, then $\gamma_{wstdR}^k(G) = k+3$.*

Since $\delta(G) = p - 1$ in Corollay 4, and $\delta(G) \geq \frac{k}{3}$, we note that only $k \leq 3p - 3$ is admissible in Corollary 4. Thus Corollary 4 presents the weak signed total double Roman k-domination number for complete split graphs for all possible integers $k \geq 1$ when $n_p \geq 2$ and $p \geq 3$. In the cases $n_p = 1$ or $p = 2$, Example 2 or Proposition 11 lead to the desired result.

8. Trees

In this section, we establish lower bounds on the weak signed total double Roman k-domination number for trees, considering separately the cases $k = 1, 2, 3$. The discussion will be divided into three subsections, each devoted to a specific case.

8.1. $k = 1$

In this subsection, we prove that every nontrivial tree T satisfies the lower bound $\gamma_{wstdR}(T) \geq \frac{-n(T)+8}{2}$. To characterize the trees that attain equality, we introduce the following family. For any positive integer $r \geq 2$, let T_r be the tree obtained from r disjoint copies of P_3 centered at $x_1, x_2, \ldots, x_r$ by adding a star $K_{1,r-1}$ centered at x and the edges $xx_1, xx_2, \ldots, xx_r$ (see Figure 1). Let $\mathcal{T}_1 = \{T_r \mid r \geq 2\} \cup \{K_{1,3}, P_4, H\}$, where H is the tree obtained from P_5 by attaching a leaf to its central vertex.

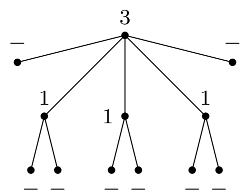

Figure 1. **The tree T_3 in family $\mathcal{T}_1$**

Theorem 4. *Let T be a tree of order $n \geq 4$. Then $\gamma_{wstdR}(T) \geq \frac{-n+8}{2}$, with equality if and only if $T \in \mathcal{T}_1$.*

Proof. The proof is carried out using induction on the order n. Note that the base step deals with trees with few vertices or small diameter. If $\mathrm{diam}(T) = 2$, then T is a star and we have $\gamma_{wstdR}(T) = 2 \geq \frac{-n+8}{2}$ with equality if and only if $T = K_{1,3} \in \mathcal{T}_1$. Similarly, if $\mathrm{diam}(T) = 3$, then T is a double star $S_{r,s}$ for some positive integers $r \geq s \geq 1$, and we have $\gamma_{wstdR}(T) = 2 \geq \frac{-n+8}{2}$, with equality if and only if $r = s = 1$, that is $T = P_4 \in \mathcal{T}_1$.

This proves the base case. Let the statement holds for any tree T' of order $4 \leq n' < n$. Let T be a tree of order n and let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{wstdR}(T)$-function. By the observations above, we can assume that $\mathrm{diam}(T) \geq 4$. Suppose that there exists a non-pendant edge uv, which satisfies that both endpoints u and v belong to V_{-1}.

Removing this edge splits T into two components, T_1 and T_2 of order at least two. Clearly, the restriction $f_i = f|_{T_i}$ is a WSTDRDF on T_i for each $i = 1, 2$. Since $f(u) = f(v) = -1$, it is easy to see that $|V(T_i)| = 2$ is not possible for $1 \le i \le 2$. Note that if $|V(T_i)| = 3$ for some $1 \le i \le 2$, then $\omega(f|_{T_i}) = 3 > \frac{-|V(T_i)|+8}{2}$. Using this fact or the induction hypothesis, we obtain

$$\gamma_{wstdR}(T) = \omega(f_1) + \omega(f_2) \ge \frac{-|V(T_1)| + 8}{2} + \frac{-|V(T_2)| + 8}{2} = \frac{-n + 16}{2} > \frac{-n + 8}{2}.$$

Consequently, we may assume that V_{-1} as an independent set. We distinguish the following claims.

Claim 1. If there exists a vertex v with $\deg_T(v) \ge 2$ such that $f(v) = -1$, then $\gamma_{wstdR}(T) > \frac{-n+8}{2}$.

Proof of Claim 1. By the definition of a WSTDRDF, v cannot be a support vertex. Let $T_1, T_2, \ldots, T_r$ be the components of $T - v$. Obviously, the function $f|_{T_i}$ is a WSTDRDF of T_i for each $i \in \{1, 2, \ldots, r\}$. Clearly $r \ge 2$. If $|V(T_i)| = 2$ for some $1 \le i \le r$, then $\omega(f|_{T_i}) = \frac{-|V(T_i)|+8}{2} = 3$. Using the induction hypothesis for T_j of order at least 3, we obtain $\gamma_{wstdR}(T) = \sum_{i=1}^{r} \omega(f_i) + f(v) \ge \sum_{i=1}^{r} \frac{-|V(T_i)|+8}{2} - 1 \ge \frac{\sum_{i=1}^{r} |V(T_i)|+8r-2}{2} = \frac{-(n-1)+8r-2}{2} > \frac{-n+8}{2}$. ♦

According to Claim 1, we can now suppose that each vertex belonging to V_{-1} is a leaf. It follows that every vertex of T with degree at least two is assigned either 1, 2 or 3 under f.

Claim 2. If there exists a support vertex v with two leaf neighbors u, w such that $f(u) = -1$ and $f(w) \ge 1$, then $\gamma_{wstdR}(T) > \frac{-n+8}{2}$.

Proof of Claim 2. First assume that $f(w) = 1$ and let $T' = T - \{u, w\}$. Clearly, the function $f|_{T'}$ is a WSTDRDF of T'. Since $\mathrm{diam}(T) \ge 4$, the order of T is at least four. Using the induction hypothesis on T', it follows that $\gamma_{wstdR}(T) = \omega(f|_{T'}) \ge \frac{-(n-2)+8}{2} > \frac{-n+8}{2}$. If $f(w) \ge 2$, then consider the tree $T' = T - \{u\}$. Now reassigning w the value $f(w) - 1$ provides a WSTDRDF of T' with weight $\omega(f)$. Using the induction hypothesis on T', it follows that $\gamma_{wstdR}(T) = \omega(f) \ge \gamma_{wstdR}(T') \ge \frac{-(n-1)+8}{2} > \frac{-n+8}{2}$. ♦

According to Claim 2, we can now suppose that each leaf of T belongs to V_{-1}. Let $v_1 v_2 \ldots v_d$ be a diametral path in T, and root the tree at v_d. Since $\mathrm{diam}(T) \ge 4$, we have $d \ge 5$. Let $v_1 = u_1, u_2, \ldots, u_s$ denote the leaves adjacent to v_2.

By definition we have $f(v_2) = f(N(v_1)) \ge 1$ and $f(v_3) + \sum_{i=1}^{s} f(u_i) = f(N(v_2)) \ge 1$. Since $f(u_i) = -1$ for each i, the last inequality leads to $f(v_3) \ge s + 1$ and $1 \le s \le 2$. If v_3 is not a support vertex, then consider the tree $T' = T - T_{v_2}$. If T' has order three, then T is path P_5 or is a tree obtained from P_5 by attaching a leaf to one of its support vertices. In either case, it is easy to see that $\gamma_{wstdR}(T) > \frac{-n+8}{2}$. Hence we can assume that the order of T' is at least 4. Clearly, the function $f' = f|_{T'}$ is a WSTDRDF. Using the induction hypothesis on T', we have

$$\gamma_{wstdR}(T) \ge \frac{-(n - 1 - s) + 8}{2} + (-s + 1) > \frac{-n + 8}{2}.$$

Now let v_3 be a support vertex and $z_1, z_2, \ldots, z_m$ be the leaf-neighbors of v_3. By our earlier assumption, $f(z_1) = f(z_2) = \cdots = f(z_m) = -1$. If $f(v_2) \geq 2$, then consider the tree $T' = T - z_1$. Now by reassigning v_2 the value $f(v_2) - 1$, we obtain a WSTDRDF of T' with weight $\omega(f)$. Since $|V(T')| \geq 4$, the induction hypothesis on T' leads to $\gamma_{wstdR}(T) \geq \frac{-(n-1)+8}{2} > \frac{-n+8}{2}$. Hence we can assume that $f(v_2) = 1$. If $s = 1$, consider the tree $T' = T - \{v_2, u_1, z_1\}$ of order $n - 3$. Since $\mathrm{diam}(T) \geq 4$, we have $|V(T')| \geq 3$. If $|V(T')| = 3$, then $T = H \in \mathcal{T}_1$, and it is easy to see that $\gamma_{wstdR}(T) = 1 = \frac{-n+8}{2}$, as desired. Now let $|V(T')| \geq 4$. Clearly, the function $f|_{T'}$ is a WSTDRDF with weight $\omega(f) + 1$. Using the induction hypothesis on T', we have $\gamma_{wstdR}(T) = \omega(f|_{T'}) - 1 \geq \frac{-(n-3)+8}{2} - 1 > \frac{-n+8}{2}$. Thus we can assume that $s = 2$ and so $\deg(v_2) = 3$. Consider the tree $T' = T - \{v_2, u_1, u_2, z_1\}$ of order $n - 4$. Then the function $f|_{T'}$ is a WSTDRDF of T'. If T' has order three, then clearly $n = 7$ and $\omega(f') = 3$, and we have $\gamma_{wstdR}(T) \geq 1 > \frac{-n+8}{2}$. Hence we can assume that T' has order at least four. Using the induction hypothesis on T', we have

$$\gamma_{wstdR}(T) \geq \omega(f|_{T'}) - 2 \geq \frac{-(n-4)+8}{2} - 2 = \frac{-n+8}{2}. \tag{8.1}$$

Assume that $\gamma_{wstdR}(T) = \frac{-n+8}{2}$. Then the inequality occurring in (8.1) becomes equality, that is $\omega(f|_{T'}) = \frac{-n(T')+8}{2}$. It follows from the induction hypothesis that $T' \in \mathcal{T}_1$ with head v_3. It follows that $T \in \mathcal{T}_1$.
Conversely, let $T \in \mathcal{T}_1$. If $T \in \{K_{1,3}, P_4, H\}$, then clearly $\gamma_{wstdR}(T) = \frac{-n(T)+8}{2}$. Assume that $T = T_r \in \mathcal{T}_1$. Clearly, the function g with $g(y) = -1$ for each leaf y, $g(x) = 3$ and $g(x_i) = 1$ for $1 \leq i \leq r$ is a WSTDRDF on T_r of weight $-2r + 4 = \frac{-n(T_r)+8}{2}$ and therefore $\gamma_{wstdR}(T_r) \leq \frac{-n(T_r)+8}{2}$. This demonstrates that $\gamma_{wstdR}(T) = \frac{-n(T)+8}{2}$ for each $T \in \mathcal{T}_1$. $\square$

8.2. $k = 2$

In this subsection, we show that every nontrivial tree T satisfies the lower bound $\gamma^2_{wstdR}(T) \geq \frac{-n(T)+19}{4}$. To characterize the trees that attain equality, we introduce the following family. For any positive integer $r \geq 2$, let F_r be the trees obtained from r disjoint copies of P_2, that is, $x_1y_1, x_2y_2, \ldots, x_ry_r$ by adding a star $K_{1,2r-2}$ centered at x and the edges $xx_1, xx_2, \ldots, xx_r$ (see Figure 2). Let $\mathcal{T}_2 = \{F_r \mid r \geq 2\} \cup \{P_3\}$.

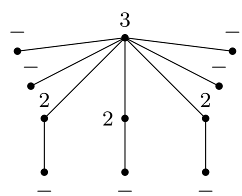

Figure 2. **The tree F_3 in family $\mathcal{T}_2$**

Theorem 5. *Let T be a tree of order $n \geq 3$. Then $\gamma^2_{wstdR}(T) \geq \frac{-n+19}{4}$, with equality if and only if $T \in \mathcal{T}_2$.*

Proof. The proof is by induction on the order n. Note that the base step deals with trees with few vertices or small diameter. If $\mathrm{diam}(T) = 2$, then T is a star and we have $\gamma^2_{wstdR}(T) = 4 \geq \frac{-n+19}{4}$ with equality if and only if $T = P_3 \in \mathcal{T}_2$. Similarly, if $\mathrm{diam}(T) = 3$, then T is a double star $S_{r,s}$ for some positive integers $r \geq s \geq 1$ centered at u, v, and for any $\gamma^2_{wstdR}(T)$-function f we have $\gamma^2_{wstdR}(T) = \omega(f) = f(N(u)) + f(N(v)) \geq 4 > \frac{-n+19}{4}$. This proves the base case. Let the statements hold for any tree T' of order $3 \leq n' < n$. Let T be a tree of order n and let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^2_{wstdR}(T)$-function. By the observations above, we can assume that $\mathrm{diam}(T) \geq 4$. If there exists a non-pendant edge uv, which satisfies that both endpoints u and v belong to V_{-1}, then as in the proof of Theorem 4 one can see that $\gamma^2_{wstdR}(T) > \frac{-n+19}{4}$. Hence, we can consider V_{-1} as an independent set. We distinguish the following claims.

Claim 1. If there exists a vertex v with $\deg_T(v) \geq 2$ such that $f(v) = -1$, then $\gamma^2_{wstdR}(T) > \frac{-n+19}{4}$.

Proof of Claim 1. By the definition of a WSTDR2DF, v cannot be a support vertex. Let t be the number of K_2-components of $T - v$ (if any) and $T_1, T_2, \ldots, T_r$ be the components of $T - v$ of order at least three (if there exists). Obviously, for any K_2-component xy of $T - v$ we have $f(x) + f(y) \geq 5$ and also the function $f|_{T_i}$ is a WSTDR2DF of T_i for each $i \in \{1, 2, \ldots, r\}$. Using the induction hypothesis on every T_i of order at least three and the fact that $t + r \geq 2$, we obtain

$$\begin{aligned}\gamma^2_{wstdR}(T) &= 5t + \sum_{i=1}^{r} \omega(f_i) + f(v) \geq 5t + \sum_{i=1}^{r} \frac{-|V(T_i)| + 19}{4} - 1 \\ &\geq \frac{-(n-1) + 22t + 19r - 4}{4} > \frac{-n+19}{4}.\end{aligned}$$

According to Claim 1, we can now suppose that each vertex belonging to V_{-1} is a leaf. Thus any vertex of T with degree at least two is assigned a positive value under f. If there exists a support vertex v with two leaf neighbors u, w such that $f(u) = -1$ and $f(w) \geq 1$, then as in the proof of Theorem 4 we can see that $\gamma^2_{wstdR}(T) > \frac{-n+19}{4}$. Consequently, we can suppose that each leaf of T belongs to V_{-1}. Let $v_1 v_2 \ldots v_d$ be a diametral path in T, and root the tree at v_d. Since $\mathrm{diam}(T) \geq 4$, we have $d \geq 5$. Let $v_1 = u_1, u_2, \ldots, u_s$ denote the leaves adjacent to v_2. By the definition we have $f(v_2) = f(N(v_1)) \geq 2$ and $f(v_3) + \sum_{i=1}^{s} f(u_i) = f(N(v_2)) \geq 2$. Since $f(u_i) = -1$, the last inequality leads to $f(v_3) = 3$ and $s = 1$. If v_3 is not a support vertex, then consider the tree $T' = T - u_1$. Now reassigning v_2 the value $f(v_2) - 1$ provides a WSTDR2DF of T' with weight $\omega(f)$. Using the induction hypothesis on T', we have

$$\gamma^2_{wstdR}(T) \geq \frac{-(n-1) + 19}{4} > \frac{-n+19}{4}.$$

Now let v_3 be a support vertex and $z_1, z_2, \ldots, z_m$ be the leaf-neighbors of v_3. By our earlier assumption, $f(z_1) = f(z_2) = \cdots = f(z_m) = -1$, $s = 1$ and $f(v_2) \geq 2$. If $f(v_2) = 3$, then consider the tree $T' = T - z_1$. Now by reassigning v_2 the value 2, we obtain a WSTDR2DF of T' with weight $\omega(f)$, and as above we can see that $\gamma^2_{wstdR}(T) \geq \frac{-(n-1)+19}{4} > \frac{-n+19}{4}$. Hence we can assume that $f(v_2) = 2$. First let $m \geq 2$ and consider the tree $T' = T - \{v_2, u_1, z_1, z_2\}$ of order $n-4$. Clearly, the function $f|_{T'}$ is a WSTDR2DF of T'. Using the induction hypothesis on T', we have

$$\gamma^2_{wstdR}(T) = \omega(f|_{T'}) - 1 \geq \frac{-(n-4)+19}{4} - 1 \geq \frac{-n+19}{4}. \tag{8.2}$$

Assume that $\gamma^2_{wstdR}(T) = \frac{-n+19}{4}$. Then the inequality occurring in (8.2) become equality, that is $\omega(f|_{T'}) = \frac{-n(T')+19}{4}$. It follows from the induction hypothesis that $T' \in \mathcal{T}_2$ with head v_3. It follows that $T \in \mathcal{T}_2$.

Now assume that $m = 1$. Clearly, $|V(T)| \geq 6$. If $f(v_4) \geq 2$ or $\deg(v_3) \geq 4$, then consider the tree $T' = T - \{v_2, v_1, z_1\}$ of order $n-3$. Clearly, the function $f|_{T'}$ is a WSTDR2DF with weight $\omega(f)$. Using the induction hypothesis on T', we have $\gamma^2_{wstdR}(T) = \omega(f|_{T'}) \geq \frac{-(n-3)+19}{4} > \frac{-n+19}{4}$. Hence we can assume that $f(v_4) = 1$ and $\deg(v_3) = 3$. It follows that v_4 is not a support vertex. Consider the tree $T' = T - \{v_1, z_1\}$ of order $n-2$. Now reassigning v_3 the value 2 and v_2 the value 1, provides a WSTDR2DF of T' with weight $\omega(f)$. Using the induction hypothesis on T', we obtain $\gamma^2_{wstdR}(T) \geq \frac{-(n-2)+19}{4} > \frac{-n+19}{4}$.

Conversely, let $T \in \mathcal{T}_2$. If $T = P_3$, then clearly $\gamma^2_{wstdR}(P_3) = 4 = \frac{-n(T)+19}{4}$. Let $T \in \mathcal{T}_2 - \{P_3\}$. Clearly, the function g with $g(y) = -1$ for each leaf y, $g(x) = 3$ and $g(x_i) = 2$ for $1 \leq i \leq r$ is a WSTDR2DF on T of weight $-r + 5 = \frac{-n(T)+19}{4}$ and therefore $\gamma^2_{wstdR}(T) \leq \frac{-n(T)+19}{4}$. This demonstrates that $\gamma^2_{wstdR}(T) = \frac{-n(T)+19}{4}$ for each $T \in \mathcal{T}_2$. This completes the proof. $\square$

8.3. k=3

In this subsection, we establish that every nontrivial tree T satisfies the lower bound $\gamma^3_{wstdR}(T) \geq 6$. To characterize the trees that attain equality, we introduce the following family. Let T be a nontrivial tree. We define T' to be any tree obtained from T by attaching to each vertex v either $3\deg(v) - 3$ or at least $3\deg(v) - 1$ new leaves (see Figure 3). Let $\mathcal{T}_3 = \{T' \mid T \text{ is a nontrivial tree}\}$. It is not hard to see that $\gamma^3_{wstdR}(T') \leq 6$ for any tree $T' \in T$.

Theorem 6. *Let T be a tree of order $n \geq 2$. Then $\gamma^3_{wstdR}(T) \geq 6$, with equality if and only if $T \in \mathcal{T}_3$.*

Proof. We use induction on the order n. Observe that the base step addresses trees characterized by a small number of vertices or a small diameter. If $\mathrm{diam}(T) = 2$, then T is a star and we have $\gamma^3_{wstdR}(T) = 6$ and $T \in \mathcal{T}_3$. Similarly, if $\mathrm{diam}(T) = 3$, then T is a double star $S_{r,s}$ for some positive integers $r \geq s \geq 1$ centered at u, v, and

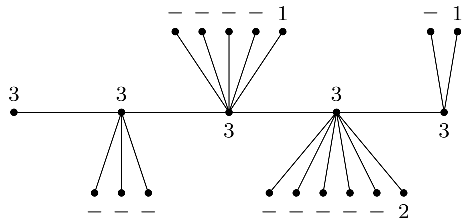

Figure 3. **A tree in family $\mathcal{T}_3$ obtained from P_5**

for any $\gamma^3_{wstdR}(T)$-function f we have $\gamma^3_{wstdR}(T) = \omega(f) = f(N(u)) + f(N(v)) \geq 6$ with equality if and only if $s \geq 2$ and so $T \in \mathcal{T}_3$. This proves the base case. Let the statements hold for any tree T' of order $2 \leq n' < n$. Let T be a tree of order n and let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^3_{wstdR}(T)$-function. By observations above, we can assume that $\mathrm{diam}(T) \geq 4$.

If there is a non-pendant edge uv such that $u, v \in V_{-1}$, then as in the proof of Theorem 4 one can see that $\gamma^3_{wstdR}(T) > 6$. Hence we may assume that the set V_{-1} is independent. We consider the following cases. If there exists a non-leaf vertex v in V_{-1}, then again as in the proof of Theorem 4 one can see that $\gamma^3_{wstdR}(T) > 6$. Hence, we may assume that every vertex in V_{-1} is a leaf. We proceed with three cases.

Case 1. There is a non-leaf vertex $v \in V_1$. By definition, v is not a support vertex. Let $T_1, T_2, \ldots, T_k$ be the components of $T-v$ and let $v_i \in V(T_i)$ be the vertex adjacent to v for each $i \in \{1, 2, \ldots, k\}$. By our earlier assumption, we have $f(v_i) \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. Clearly, $k \geq 2$. Let T' be the tree obtained from $T - v$ by adding the edges $v_i v_{i+1}$ for $i \in \{1, 2, \ldots, k-1\}$. It is not hard to see that the function f restricted to $T - v$ is a WSTDR3DF of T' and applying the induction hypothesis on T' it follows that

$$\gamma^3_{wstdR}(T) = \omega(f|_{V(T-v)}) + 1 \geq 6 + 1 > 6.$$

Case 2. There is a non-leaf vertex v with $f(v) = 2$.

By definition v is not a support vertex. Let $T_1, T_2, \ldots, T_k$ be the nontrivial components of $T - v$ and let $v_i \in V(T_i)$ be the vertex adjacent to v for each $i \in \{1, 2, \ldots, k\}$. By Case 1 and our earlier assumption, we have $f(v_i) \geq 2$ for each $i \in \{1, 2, \ldots, k\}$. Using the above arguments, we can see that it is $\gamma^3_{wstdR}(G) = \omega(f) > 6$.

Regarding the above cases, we can assume that all non-leaf vertices of T are assigned 3 under f.

Case 3. There is a non-leaf vertex v such that v is not a support vertex.

Let $N(v) = \{v_1, v_2, \ldots, v_s\}$. By assumption we have $N[v] \subseteq V_3$. Let T^1 be the tree obtained from $T - v$ by adding the edges $v_1v_2, v_2v_3, \ldots, v_{s-1}v_s$. Clearly, T^1 is a tree of order $n-1$ and the function $f|_{T^1}$ is a WSTDR3DF on T^1 of weight $\omega(f) - 3$. We conclude from the induction hypothesis that $\gamma^3_{wstdR}(T) = \omega(f|_{T^1}) + 3 \geq 6 + 3 > 6$.

In light of the above cases, we conclude that each vertex of T is either a leaf or a support vertex and any support vertex is assigned 3 under f. Since $\mathrm{diam}(T) \geq 4$,

for any support vertex, f assigns a -1 to at least one leaf adjacent to it. Let v be an arbitrary support vertex and let $u_1, u_2, \ldots, u_t$ be the leaf neighbors of v. Since $\mathrm{diam}(T) \geq 4$, v is adjacent to at least one support vertex.
If $\sum_{i=1}^{t} f(u_i) \geq 1$, then consider the tree $T' = T - \{u_1, \ldots, u_t\}$. Clearly, the function f, restricted to T' is a WSTDR3DF of T' of weight at most $\omega(f) - 1$, and using the induction hypothesis on T' it follows that $\gamma^3_{wstdR}(T) > \omega(f|_{T'}) \geq 6$. Hence we can assume that $\sum_{i=1}^{t} f(u_i) \leq 0$. This implies that if $f(u_i) = m \in \{1, 2, 3\}$ for some i, say $i = 1$, then there are m leaves adjacent to v assigned -1 under f, say $u_2, \ldots, u_{m+1}$. Consider the tree $T' = T - \{u_1, u_2, \ldots, u_{m+1}\}$. Clearly, the function f, restricted to T' is a WSTDR3DF of T' of weight $\omega(f)$, and by the induction hypothesis we have $\gamma^3_{wstdR}(T) = \omega(f|_{T'}) \geq 6$. By repeating this process, consider a tree T' obtained from T such that all leaves of T' belong to V_{-1}. Recall that all support vertices belong to V_3. For every support vertex v, let ℓ_v and ℓ'_v be the number of leaves adjacent to v in T and T', respectively. Clearly, $\ell_v = \ell'_v$ or $\ell_v \geq \ell'_v + 2$. Suppose that T'' is the tree obtained from T' by removing all leaves of T'. Since $f(v) = 3$ for every support vertex, we must have $\ell'_v \leq 3(\deg_{T''}(v) - 1)$. Hence

$$\sum_{v \in V(T'')} \ell'_v \leq \sum_{v \in V(T'')} (3 \deg_{T''}(v) - 3) = 3n(T'') - 6. \tag{8.3}$$

On the other hand, since $n \geq n(T') = n(T'') + \sum_{v \in V(T'')} \ell'_v$ and $\gamma^3_{wstdR}(T) = \gamma^3_{wstdR}(T') = 3n(T'') - \sum_{v \in V(T'')} \ell'_v$, it follows from (8.3) that

$$\gamma^3_{wstdR}(T) = 3n(T'') - \sum_{v \in V(T'')} \ell'_v \geq 3n(T'') - 3n(T'') + 6 = 6. \tag{8.4}$$

If moreover $\gamma^3_{wstdR}(T) = 6$, then all inequalities occurring in (8.3) and (8.4) become equality. In particular, we must have $\ell'_v = 3(\deg_{T''}(v) - 1)$ for every $v \in V(T'')$, and therefore $T \in \mathcal{T}_3$ because $\ell_v = \ell'_v$ or $\ell_v \geq \ell'_v + 2$ for each support vertex .
Conversely, if $T \in \mathcal{T}_3$, then clearly, $\gamma^3_{wstdR}(T) = 6$ and the proof is complete. □

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] H. Abdollahzadeh Ahangar, M.A. Henning, Y. Zhao, C. Löwenstein, and V. Samodivkin, *Signed Roman domination in graphs*, J. Comb. Optim. **27** (2014), 241–255.
https://doi.org/10.1007/s10878-012-9500-0.

[2] J. Amjadi, S. Nazari-Moghaddam, S.M. Sheikholeslami, and L. Volkmann, *On the signed Roman k-domination in graphs*, Quaest. Math. **43** (2020), no. 8, 1065–1082.
https://doi.org/10.2989/16073606.2019.1600068.

[3] L. Asgharsharghi, R. Khoeilar, and S.M. Sheikholeslami, *Signed strong Roman domination in graphs*, Tamkang J. Math. **48** (2017), no. 2, 135–147.
http://dx.doi.org/10.5556/j.tkjm.48.2017.2240.

[4] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Roman domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2020, pp. 365–409.

[5] ______, *Varieties of Roman domination in graphs II*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 966–984.
https://doi.org/10.1016/j.akcej.2019.12.001.

[6] ______, *Varieties of Roman domination in graphs*, Structures of Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2021, pp. 273–307.

[7] T.W. Haynes, S.T. Hedetniemi, and M.A. Henning (eds.), *Topics in Domination in Graphs*, Developments in Mathematics, vol. 64, Springer, Cham, 2020.

[8] T.W. Haynes, S.T. Hedetniemi, and M.A. Henning (eds.), *Structures of Domination in Graphs*, Developments in Mathematics, vol. 66, Springer, Cham, 2021.

[9] L. Shahbazi, H. Abdollahzadeh Ahangar, R. Khoeilar, and S.M. Sheikholeslami, *Bounds on signed total double Roman domination*, Commun. Comb. Optim. **5** (2020), no. 2, 191–206.
https://doi.org/10.22049/CCO.2020.26761.1140.

[10] ______, *Signed total double Roman k-domination in graphs*, Discrete Math. Algorithms Appl. **12** (2020), no. 1, 2050009.
https://doi.org/10.1142/S1793830920500093.

[11] L. Shahbazi, H. Abdollahzadeh Ahangar, and S.M. Sheikholeslami, *Signed total double Roman dominating functions in graphs*, AKCE Int. J. Graphs Comb. **21** (2024), no. 3, 294–301.
https://doi.org/10.1080/09728600.2024.2357561.

[12] S.M. Sheikholeslami, L. Volkmann, Y. Shang, and M. Esmaeili, *More results on the signed total double Roman k-domination of graphs*, submitted.