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Research Article

The total mutual visibility number in bicyclic graphs

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Abstract: Let G be a graph. A subset of vertices $X \subseteq V(G)$ is termed a total mutual-visibility set if, for any pair of vertices $u, v \in V(G)$, one can find a shortest path connecting them such that none of its internal vertices belong to X. The cardinality of the largest such set is known as the total mutual-visibility number, denoted by $\mu_t(G)$. In this study, we determine this number for the class of connected bicyclic graphs. Our analysis categorizes these graphs into three structural types based on the connection between their two cycles: disjoint cycles linked by a path, cycles intersecting at a single vertex, and cycles sharing a common path (Theta-graphs). For each category, we establish explicit formulas to calculate the exact value of $\mu_t(G)$. These results cover all structural configurations, including variations in cycle lengths, leaf attachments, and internal vertex arrangements. Ultimately, this work provides a complete characterization of the total mutual-visibility number for bicyclic graphs and sets a foundation for future research.

Keywords: bicyclic graphs, mutual-visibility set, total mutual-visibility set, total mutual-visibility number.

AMS Subject Classification: 05C38, 05C12, 05C75, 05C99

1. Introduction

Mutual visibility is a relatively recent graph-theoretic concept that captures the possibility of communication between vertices along shortest paths while avoiding other selected vertices. Basic definitions and foundational properties of graphs in our terminology follow standard references such as Bondy and Murty [3].

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More precisely, let G be a connected graph and let $X \subseteq V(G)$ be a subset of vertices. Two vertices $x, y \in V(G)$ are said to be X-visible if there exists a shortest x, y-path whose internal vertices do not belong to X. A set $X \subseteq V(G)$ is called a mutual-visibility set if every two vertices of X are X-visible. The maximum cardinality of a mutual-visibility set is called the mutual-visibility number of G and is denoted by $\mu(G)$ [9]. This concept was introduced in graph theory by Di Stefano [9] and has since attracted considerable attention in both theoretical and applied settings. In particular, mutual visibility is related to communication and navigation problems in networks, including applications involving mobile robots and distributed systems [1, 8, 15].

A natural strengthening of mutual visibility is obtained by requiring the visibility condition to hold for every pair of vertices of the graph. More precisely, a set $X \subseteq V(G)$ is called a total mutual-visibility set if every two vertices of G are X-visible. The maximum cardinality of a total mutual-visibility set is called the total mutual-visibility number of G and is denoted by $\mu_t(G)$ [16]. A maximum total mutual-visibility set is called a μ_t-set. Since the total condition is stronger than the ordinary mutual-visibility condition, we always have

$$\mu_t(G) \leq \mu(G).$$

The total mutual-visibility problem is computationally difficult in general, and designing optimal or efficient approaches for specific graph classes is a central theme in this area (see, for instance, [2]). Its study has led to several exact results and structural characterizations for particular graph classes [16].

Mutual-visibility problems have been investigated for a variety of graph families. For example, Cicerone et al. studied the geodesic mutual-visibility problem in trees [4] and mutual visibility in distance-hereditary graphs, where they developed a linear-time algorithm [6]. Mutual visibility has also been investigated in Cartesian products and triangle-free graphs [7]. A closely related concept is the general position problem, which has also been extensively studied in graph theory [14].

More recent developments have substantially expanded the range of graph classes and visibility parameters considered in the literature. Korže and Vesel [12] studied mutual-visibility sets in Cartesian products of paths and cycles. They considered both Cartesian products of a cycle and a path and Cartesian products of two cycles. For the former family, they obtained optimal solutions for the majority of the considered graphs, while for Cartesian products of two cycles they completely determined the mutual-visibility number. Their results demonstrate that the structure of Cartesian products can be exploited to obtain precise information about mutual-visibility sets [12].

The study of total mutual visibility has also recently been extended to highly structured network classes. Cicerone et al. [5] investigated mutual and total mutual visibility in several hypercube-like networks, namely hypercubes, Fibonacci cubes, cube-connected cycles, and butterflies. For mutual visibility, they established bounds and approximation results for several of these families and obtained an exact formula for butterflies. For total mutual visibility, they obtained exact formulas for cube-connected cycles and butterflies. These results show that total mutual visibility can exhibit substantially different behavior on structured network families and can admit exact characterizations even when the corresponding optimization problem is computationally difficult in general [5].

Korže and Vesel [13] subsequently investigated a variety of mutual-visibility parameters in hypercubes, including mutual, total, outer, and dual mutual visibility. In particular, they obtained bounds and exact values for several dimensions and established an important connection between total mutual visibility in the hypercube and binary coding theory. More precisely, finding a largest total mutual-visibility set in an h-cube is equivalent to determining the maximum size of a binary code of length h with minimum Hamming distance four. This result highlights a connection between total mutual visibility, coding theory, and combinatorial optimization [13].

More recently, Hamed-Labbafian et al. [10] studied mutual visibility in hierarchical products of graphs. They established results and bounds for the mutual-visibility number of hierarchical products and also considered computational approaches based on integer linear programming and genetic algorithms. In addition, they discussed applications of mutual visibility to the integration of power and communication networks. These recent developments illustrate the growing scope of mutual-visibility problems from both structural and computational perspectives [10].

Despite these developments, the total mutual-visibility number has not yet been systematically determined for connected bicyclic graphs. In particular, the existing results mainly concern structured graph families such as Cartesian products, hypercube-like graphs, and hierarchical products, whereas a general structural characterization of $\mu_t(G)$ for connected bicyclic graphs is still lacking. This motivates the present study.

A connected graph G with order n and size m has cyclomatic number

$$c(G) = m - n + 1.$$

Graphs with cyclomatic number 1 are called unicyclic, while graphs with cyclomatic number 2 are called bicyclic. Bicyclic graphs form a natural next class beyond trees and unicyclic graphs: they contain exactly two independent cycles and consequently exhibit a richer structural behavior while still admitting a meaningful classification according to the interaction between their cycles.

Although bicyclic graphs have been studied in connection with other shortest-path and position parameters, including the general position number [11], a systematic determination of $\mu_t(G)$ for all connected bicyclic graphs is still lacking. The presence and interaction of two cycles, connecting paths, cut-vertices, and possible pendant structures make the analysis of total mutual visibility substantially more involved than in the unicyclic case.

In this paper, we fill this gap by determining the total mutual-visibility number of connected bicyclic graphs. We classify these graphs according to the structural relationship between their two cycles and derive an exact formula for $\mu_t(G)$ for each resulting class. In particular, our analysis covers the three fundamental structural types of connected bicyclic graphs: graphs in which the two cycles are connected by a path, graphs in which the two cycles share a single vertex, and graphs in which the two cycles share a path. Consequently, our results provide a systematic characterization of the total mutual-visibility number for connected bicyclic graphs.

2. Total mutual-visibility number

In this section, we review key results from the literature and highlight fundamental properties of cut-vertices and leaves that are essential for our analysis. We then present the structural classification of connected bicyclic graphs, which provides the foundation for determining the total mutual-visibility number in the following sections.

2.1. Preliminary results

The presence of cut-vertices and leaves plays a crucial role in determining the total mutual-visibility number $\mu_t(G)$. We base our discussion on the following foundational results, originally established for standard mutual visibility, which naturally extend to the total mutual-visibility case.

Theorem 1 ([9]). *Let T be a tree, and let $L \subseteq V(T)$ denote the set of its leaves. Then:*

1. *The set L is a mutual-visibility set for T.*
2. *The mutual-visibility number of the tree is equal to the number of its leaves, i.e., $\mu(T) = |L|$.*

Theorem 2 ([9]). *Let G be a connected block graph, and let $C \subseteq V(G)$ denote the set of its cut-vertices. Then:*

1. *The set $V(G) \setminus C$ is a mutual-visibility set for G.*
2. *The mutual-visibility number is given by $\mu(G) = |V(G) \setminus C|$.*

Building on these observations, we establish two theorems specific to total mutual visibility, which are fundamental to our proofs.

Theorem 3. *Let G be a connected graph and let $L \subseteq V(G)$ denote the set of its leaves. Then any maximum total mutual-visibility set X must contain all leaves. That is, $L \subseteq X$.*

Proof. Let X be a total mutual-visibility set of maximum cardinality in G, that is, $|X| = \mu_t(G)$. Suppose, to obtain a contradiction, that there exists a leaf $v \in L$ such that $v \notin X$. Since $\deg(v) = 1$, the vertex v cannot appear as an internal vertex of any path between two distinct vertices of $V(G) \setminus \{v\}$.

We claim that $X' := X \cup \{v\}$ is also a total mutual-visibility set, which contradicts the maximality of $|X|$.
Indeed, let $x, y \in V(G)$ be arbitrary. If neither x nor y equals v, then by the assumption that X is a total mutual-visibility set, there exists a shortest path $P_{x,y}$ whose internal vertices lie outside X. Since v cannot be an internal vertex of such a path (because $\deg(v) = 1$), the same path is also internally disjoint from X'.
If one of the vertices equals v, say $x = v$ and $y \in V(G)$, let u be the unique neighbor of v. Choose a shortest path $P_{u,y}$ from u to y whose internal vertices lie outside X (such a path exists because X is a total mutual-visibility set and u and y are mutually visible with respect to X). Then the path $v, u, \ldots, y$, formed by the edge vu followed by $P_{u,y}$, is a shortest path between v and y. Its internal vertices consist of u together with the internal vertices of $P_{u,y}$. Since u is an internal vertex of the path from v to y and X is a valid set, we must have $u \notin X$; otherwise, v and y would not be visible with respect to X. Consequently, none of the internal vertices belong to X'.
Thus, every pair of vertices in G admits a shortest path whose internal vertices are disjoint from X', and hence X' is a total mutual-visibility set. This contradicts the maximality of X. Therefore, no such leaf v exists, and every maximum total mutual-visibility set contains all leaves, that is, $L \subseteq X$. □

Theorem 4. *If G is a connected graph and v is a cut-vertex of G, then v cannot belong to any total mutual visibility set. Consequently, if S_{cut} is the set of all cut-vertices in G, then*

$$X \cap S_{\text{cut}} = \emptyset.$$

Proof. Let v be a cut-vertex of G. Removal of v disconnects G into at least two components; let G_1 and G_2 be two distinct components of $G \setminus \{v\}$. Choose $u \in V(G_1)$ and $w \in V(G_2)$. Every path connecting u and w must pass through v; hence v belongs to every u-w path, and in particular to every shortest u-w path.
Suppose, for contradiction, that $v \in X$ for some total mutual-visibility set X. Then for the pair (u, w) there cannot exist a shortest path whose internal vertices lie outside X, because any shortest u-w path has v as an internal vertex and $v \in X$. This contradicts the assumption that X is a total mutual-visibility set. Thus, no cut-vertex can belong to any total mutual-visibility set; equivalently, $X \cap S_{\text{cut}} = \emptyset$. □

A connected graph G with n vertices and m edges is called *bicyclic* if $m = n+1$. Such graphs contain exactly two linearly independent cycles. The geometric structure of a bicyclic graph is determined by the interaction between these two cycles. Based on the connectivity and intersection of the cycles, we classify connected bicyclic graphs into three distinct types.

Type I: Disjoint cycles connected by a path

In this configuration, the two cycles, denoted by C_k and C_l, are vertex-disjoint ($V(C_k) \cap V(C_l) = \emptyset$). They are connected by a unique simple path P of length at least 1. The endpoints of P, say $u \in V(C_k)$ and $v \in V(C_l)$, act as cut-vertices, as do any internal vertices of P. The graph is represented as $G = C_k \cup P \cup C_l$.

Type II: Cycles sharing a single vertex

Here, the two cycles C_k and C_l share exactly one common vertex v ($V(C_k) \cap V(C_l) = \{v\}$). This vertex v is a cut-vertex that connects the two cycles, forming a structure resembling a figure-eight. The graph is denoted as $G(C_k, C_l, v)$.

Type III: Cycles with a shared path (Theta-graphs)

In this type, the two cycles share two or more vertices, forming a structure known as a Theta graph (Θ-graph). Specifically, the intersection of the two cycles is a path P of length at least 1 connecting two vertices u and v. Unlike Types I and II, removing the internal vertices of the common path P does not disconnect the graph; however, these internal vertices lie on shortest paths between the non-overlapping parts of the cycles.

Remark. Each of these three structural types may also possess pendant trees (leaves) attached to any vertex of the structure. However, by Theorems 3 and 4, the presence of leaves converts their attachment vertices into cut-vertices, thereby excluding them from the total mutual-visibility set. Therefore, our analysis primarily focuses on the contribution of the cycles and the impact of the cut-vertices induced by the structure or the leaves.

3. Main results

In this section, we determine the total mutual-visibility number $\mu_t(G)$ for Type-I bicyclic graphs. We do so by decomposing the graph into its structural components: the two cycles, the connecting path between them, and any attached pendant trees. We then analyze how each cycle contributes to a maximum total mutual-visibility set. To establish a general formula, we introduce the notions of the free set and cycle contribution.

3.1. Type-I bicyclic graphs: disjoint cycles connected by a path

A Type-I bicyclic graph consists of two vertex-disjoint cycles, C_k and C_l, connected by a path P of length at least 1. To derive a general formula for $\mu_t(G)$, we first introduce the concepts of the free set and cycle contribution.

3.1.1. Free set and cycle contribution

Definition 1. Let G be a graph and let C be a cycle in G. The *free set* of C, denoted by $S(C)$, is the set of vertices in the cycle having degree 2 in G:

$$S(C) := \{v \in V(C) : \deg_G(v) = 2\}.$$

Remark. Vertices in $S(C)$ are not cut-vertices, nor are they incident to leaves or the connecting path P. Their open neighborhood lies entirely within the cycle C.

Definition 2 (Cycle contribution). The cycle contribution of C, denoted by $\alpha(C)$, is the maximum number of vertices from $S(C)$ that can belong to a maximum total mutual-visibility set of G. Formally,

$$\alpha(C) = \max\{|Y| : Y \subseteq S(C) \text{ and } Y \subseteq X \text{ for some } \mu_t\text{-set } X \text{ of } G\}.$$

3.1.2. Analysis of cycle contributions

We evaluate $\alpha(C)$ based on the cycle length and the structure of its free set.

Lemma 1. *Let C_k be a cycle in a Type-I bicyclic graph with $k \geq 5$. Then*

$$\alpha(C_k) = 0.$$

Proof. Suppose, to the contrary, that there exists a vertex $u \in S(C_k) \cap X$, where X is a μ_t-set of G. Let x and y be the neighbors of u on the cycle. Since $k \geq 5$, the path $P : x, u, y$ has length 2, while the other path between x and y on the cycle has length $k - 2 \geq 3$. Thus, P is the unique shortest path between x and y. However, since u lies on this unique shortest path and $u \in X$, the vertices x and y are not mutually visible. This contradicts the fact that X is a total mutual-visibility set. Therefore, $S(C_k) \cap X = \emptyset$, implying $\alpha(C_k) = 0$. □

Lemma 2. *Let C_4 be a cycle with vertices v_1, v_2, v_3, v_4 in cyclic order. The cycle contribution $\alpha(C_4)$ depends on the structure of the free set $S(C_4)$ as follows:*

$$\alpha(C_4) = \begin{cases} 2, & \textit{if } S(C_4) \textit{ contains adjacent vertices}, \\ 1, & \textit{if } S(C_4) \neq \emptyset \textit{ and contains no adjacent vertices}, \\ 0, & \textit{if } S(C_4) = \emptyset. \end{cases}$$

Proof. Since G is a Type-I bicyclic graph, the cycle is attached to the path P at some vertex. Without loss of generality, let v_1 be this attachment vertex. By Theorem 4, v_1 is a cut-vertex, so $v_1 \notin X$ for any total mutual-visibility set X. Thus, any contribution must come from $\{v_2, v_3, v_4\} \cap S(C_4)$.

Case 1: $S(C_4) = \emptyset$.

Since no vertices are free, all available vertices are excluded (being cut-vertices or neighbors of leaves). Thus, $\alpha(C_4) = 0$.

Case 2: $S(C_4)$ contains vertices but no adjacent pairs.

This implies $S(C_4) \subseteq \{v_2, v_4\}$ (since including v_3 would create an adjacency if $|S(C_4)| \geq 2$). The vertices v_1 and v_3 are connected by two paths of length 2: v_1, v_2, v_3 and v_1, v_4, v_3. If both v_2 and v_4 were in X, both shortest paths between v_1 and v_3 would be blocked. Hence, we can select at most one vertex (e.g., v_2). Thus, $\alpha(C_4) = 1$.

Case 3: $S(C_4)$ contains adjacent vertices.

Let $\{v_2, v_3\} \subseteq S(C_4)$ be an adjacent pair. (Note that if $|S(C_4)| = 3$, such a pair necessarily exists.) We claim $Y = \{v_2, v_3\}$ is a valid subset. Since $v_2 \sim v_3$, they are mutually visible. For the non-adjacent pair v_2, v_4, the shortest path is v_2, v_1, v_4. Since $v_1 \notin X$, this visibility is preserved. Furthermore, the path to any external vertex passes through v_1, which is clear of X. Thus, we can include 2 vertices, so $\alpha(C_4) = 2$. $\square$

Lemma 3. *For a cycle C_3 in a Type-I bicyclic graph, we have*

$$\alpha(C_3) = |S(C_3)|.$$

Proof. First, observe that since the cycle is attached to the connecting path P, at least one vertex of C_3 is a cut-vertex. Consequently, $|S(C_3)| \leq 2$.
Since C_3 is a complete graph (K_3), any two vertices in the cycle are adjacent. Adjacent vertices are mutually visible regardless of the composition of the rest of the set X, as the edge connecting them contains no internal vertices.
Furthermore, any shortest path connecting a vertex in $S(C_3)$ to a vertex outside the cycle must pass through the attachment cut-vertex. By Theorem 4, this cut-vertex does not belong to X. Therefore, including vertices from $S(C_3)$ in X does not obstruct visibility with the rest of the graph.
Thus, every vertex in the free set can be included in X, yielding $\alpha(C_3) = |S(C_3)|$. $\square$

Theorem 5. *Let G be a Type-I bicyclic graph composed of two disjoint cycles $C^{(1)}$ and $C^{(2)}$ connected by a path P, and let L be the set of leaves. Then*

$$\mu_t(G) = |L| + \alpha(C^{(1)}) + \alpha(C^{(2)}).$$

Proof. Let X be a maximum total mutual-visibility set of G. By Theorem 3, we have $L \subseteq X$. Conversely, Theorem 4 implies that no cut-vertex can belong to X.
In a Type-I graph, the internal vertices of the path P, as well as the attachment vertices of the cycles, are all cut-vertices. Therefore, $X \cap V(P) = \emptyset$. Consequently, the only remaining candidates for inclusion in X (other than leaves) are the vertices in the free sets $S(C^{(1)})$ and $S(C^{(2)})$.
Since $V(P) \cap X = \emptyset$, the path P remains clear and preserves visibility between any pair of vertices residing in different cycles. This implies that the choice of vertices in $C^{(1)}$ is independent of the choice in $C^{(2)}$.
Thus, the maximization problem decouples, and we obtain $\mu_t(G) = |L| + \alpha(C^{(1)}) + \alpha(C^{(2)})$. $\square$

As shown in Table 1, the cycle contributions $\alpha(C)$ vary according to the length and structure of the cycle.

Cycle Type	**Structure of Free Set** $S(C)$	**Contribution** $\alpha(C)$	**Reference**
Cycle C_k $(k \geq 5)$	Any configuration	0	Lemma 1
Cycle C_4	Contains an adjacent pair	2	Lemma 2
Cycle C_4	Non-empty independent set	1	Lemma 2
Cycle C_4	$S(C) = \emptyset$	0	Lemma 2
Cycle C_3	Any configuration	$\lvert S(C)\rvert$	Lemma 3

Table 1. Values of the cycle contribution $\alpha(C)$ used in Theorem 5 to determine $\mu_t(G)$ for Type-I bicyclic graphs.

3.2. Type-II bicyclic graphs

3.2.1. Definition and structure

Let $G = G(C_k, C_\ell, v)$ be a Type-II bicyclic graph composed of two simple cycles C_k and C_ℓ sharing exactly one common vertex v. The vertex v serves as a cut-vertex connecting the two cycles. To determine $\mu_t(G)$, we apply the notions of the free set $S(C)$ and the cycle contribution $\alpha(C)$, as introduced in Definition 1.

Observe that the structural analysis of cycles in Type-II graphs mirrors that of Type-I graphs. In both configurations, the connecting vertices (whether the shared vertex v or the attachment points of a path) serve as cut-vertices and are consequently excluded from any total mutual-visibility set (Theorem 4). Therefore, the cycle contribution $\alpha(C)$ follows the same derivation rules established in Section 3.1. Since $v \notin X$, the choices of vertices in the two cycles are independent with respect to mutual visibility. Thus, the problem reduces to calculating $\alpha(C)$ for each cycle separately, treating v as a restricted vertex similar to the attachment vertices in Type-I graphs.

3.2.2. Leafless Type-II bicyclic graphs

Theorem 6. *Let $G = G(C_k, C_l, v)$ be a leafless Type-II bicyclic graph. The total mutual-visibility number of G is given by*

$$\mu_t(G) = \alpha(C_k) + \alpha(C_l),$$

where the cycle contribution $\alpha(C)$ is determined according to Lemmas 1, 2, and 3.

Proof. Since G is leafless, the vertex v is the unique cut-vertex in the graph. By Theorem 4, we have $v \notin X$ for any total mutual-visibility set X. This exclusion implies that the two cycles share no candidate vertices for the set X, and the visibility constraints within one cycle do not affect the other. Thus, the problem decomposes into two independent sub-problems.

For each cycle $C \in \{C_k, C_l\}$, the free set consists of all vertices except the shared vertex v, that is,

$$|S(C)| = |V(C)| - 1.$$

We evaluate the contribution $\alpha(C)$ based on the cycle length, referencing the results established in Section 3.1:

Table 2. **The total mutual-visibility number $\mu_t(G)$ for leafless Type-II bicyclic graphs.**

Case	Cycle Lengths (k, l)	Cycle Contributions $(\alpha(C_k), \alpha(C_l))$	$\mu_t(G)$	Reference (Section 3.1)
1	$k \geq 5$, $l \geq 5$	$(0, 0)$	0	Lemma 1
2	$k \geq 5$, $l = 4$	$(0, 2)$	2	Lemma 1 & Lemma 2
3	$k \geq 5$, $l = 3$	$(0, 2)$	2	Lemma 1 & Lemma 3
4	$k = 4$, $l = 4$	$(2, 2)$	4	Lemma 2 (Case 3)
5	$k = 4$, $l = 3$	$(2, 2)$	4	Lemma 2 & Lemma 3
6	$k = 3$, $l = 3$	$(2, 2)$	4	Lemma 3

- **Cycle length $k \geq 5$:** By Lemma 1, the contribution is $\alpha(C_k) = 0$.
- **Cycle length 4 (C_4):** The free set $S(C_4)$ contains exactly 3 vertices. Since any set of three vertices in a 4-cycle must contain at least one pair of adjacent vertices, the condition of "containing adjacent vertices" (Case 3 of Lemma 2) is automatically satisfied. Thus, $\alpha(C_4) = 2$.
- **Cycle length 3 (C_3):** The free set $S(C_3)$ contains 2 vertices. By Lemma 3, all free vertices can be included, yielding $\alpha(C_3) = 2$.

Summing these independent contributions completes the proof. □

Using the formula established in Theorem 6, we compute the exact values of $\mu_t(G)$ for various combinations of cycle lengths in leafless graphs. These results are summarized in Table 2.

3.2.3. Type-II bicyclic graphs with leaves

Theorem 7. *Let $G = G(C_k, C_l, v)$ be a connected Type-II bicyclic graph and let L denote the set of its leaves. The total mutual-visibility number of G is given by*

$$\mu_t(G) = |L| + \alpha(C_k) + \alpha(C_l),$$

where the cycle contribution $\alpha(C)$ is calculated based on the free set $S(C)$ as defined in Section 3.1.

Proof. Let X be a maximum total mutual-visibility set. By Theorem 3, all leaves must be included in X (i.e., $L \subseteq X$). Conversely, Theorem 4 states that no cut-vertex can belong to X.

In a Type-II graph with leaves, the set of cut-vertices consists of two types: the shared vertex v connecting the cycles, and any vertex on the cycles to which a pendant tree (or leaf) is attached. Since the definition of the free set is

$$S(C) = \{u \in V(C) : \deg_G(u) = 2\},$$

any vertex on the cycle that is incident to a leaf or is the shared vertex v has degree at least 3, and is thus automatically excluded from $S(C)$.

Since the shared vertex v acts as a cut-vertex, we have $v \notin X$. This effectively separates the two cycles, meaning the visibility constraints within C_k are independent of those in C_l. Therefore, we can maximize the contribution of each cycle separately using the results from Lemmas 1, 2, and 3. The total number is the sum of the leaves and the independent contributions from the cycles. □

Remark. It is worth noting that the determination of $\alpha(C)$ depends exclusively on the structure of the free set $S(C)$. The presence of leaves simply restricts $S(C)$ by removing the attachment vertices. Once the reduced free set is identified, the computation of $\alpha(C)$ follows the exact same structural rules as analyzed for Type-I graphs. Therefore, the summary of contribution values provided in Table 1 (Section 3.1) remains fully applicable to this case.

3.3. Type-III bicyclic graphs (Theta-graphs)

In this section, we determine the exact value of the total mutual-visibility number for Type-III bicyclic graphs, commonly referred to as Theta-graphs. A key distinction of this class is its structural connectivity; unlike Type-I and Type-II graphs, the underlying Theta-structure is 2-connected, meaning that the removal of internal vertices from the paths does not disconnect the cycles. However, the presence of leaves fundamentally alters this property by introducing cut-vertices at their attachment points. To address these structural variations effectively, we divide our analysis into two subsections: Theta-graphs with leaves and leafless Theta-graphs.

3.3.1. Theta-graphs with leaves

We begin by focusing on Theta-graphs with leaves. Leveraging the fundamental properties established in Section 2—specifically, the mandatory inclusion of leaves in any maximum total mutual-visibility set and the exclusion of their neighbors—the problem reduces to analyzing the visibility behavior of the core Theta-structure under these constraints.

A Theta-graph $G = \Theta(l_1, l_2, l_3)$ consists of two vertices u and v of degree 3, connected by three internally disjoint paths P_1, P_2, and P_3 of lengths l_1, l_2, and l_3, respectively. From a structural point of view, G can be regarded as the union of two cycles, $C^{(1)}$ and $C^{(2)}$, sharing a common path P_{share}. To obtain a unified formula, the geodesic interaction between these two cycles must be taken into account. Our analysis reveals a general additive behavior, subject to specific topological conditions based on the cycle types.

Theorem 8. *Let $G = \Theta(l_1, l_2, l_3)$ be a connected Type-III bicyclic graph with a set of leaves L. Without loss of generality, assume l_2 and l_3 are the lengths of the non-shared paths and l_1 is the length of the shared path (P_{share}). The total mutual-visibility number $\mu_t(G)$ is determined as follows:*

***Case 1: The Twin-Short-Path Configuration** ($l_2 = l_3 = 2$).*

When the two non-shared paths both have length 2, the visibility depends on the length of the shared path:

- ***Subcase 1.1: Compact Structure** ($l_1 \leq 2$).*
If the shared path is also short ($l_1 = 1$ or 2), the graph is structurally tight. In this scenario, one of the branching vertices (provided it is not a cut-vertex due to leaf attachment) can be included in the set alongside all internal vertices of degree 2. Thus:

$$\mu_t(G) = |L| + |D_2(G)| + 1,$$

where $|D_2(G)|$ is the total count of degree-2 vertices in the Theta-subgraph.

- ***Subcase 1.2: Elongated Structure** ($l_1 \geq 3$).*
If the shared path is long, the visibility is restricted to the branches. The branching vertices cannot be included, and the shared path contributes nothing. Thus:

$$\mu_t(G) = |L| + |S_{\text{ext}}|,$$

where $|S_{\text{ext}}|$ denotes the number of degree-2 vertices lying strictly on the non-shared paths (i.e., the external arcs).

Case 2: General Configuration.
For all other configurations (i.e., when at least one of the non-shared paths has length $l_i \geq 3$, or the structure does not match Case 1), the total mutual-visibility number follows the additive cycle contribution formula:

$$\mu_t(G) = |L| + \alpha(C^{(1)}) + \alpha(C^{(2)}).$$

Here, $\alpha(C^{(i)})$ is the cycle contribution calculated on the full length of each cycle (including the vertices of the shared path) as defined in Section 3.1.

Proof. Let X be a maximum total mutual-visibility set in G. We begin by applying the fundamental reduction properties. By Theorem 3, the set of leaves L must be included in X ($L \subseteq X$), and Theorem 4 implies that any vertex incident to a leaf serves as a cut-vertex and must be excluded from X. Let S_{cut} denote this set of excluded vertices. The problem thus reduces to selecting the maximum number of remaining vertices from the core Theta-structure $G' = G \setminus (L \cup S_{cut})$. Figure 1 provides a visual overview of the structural configurations analyzed below.

Case 1: The Twin-Short-Path Configuration ($l_2 = l_3 = 2$).
In this scenario, the two non-shared paths, say P_2 and P_3, consist of exactly one internal vertex each. Let w_2 and w_3 be these vertices. The visibility between w_2 and w_3 is the controlling factor.

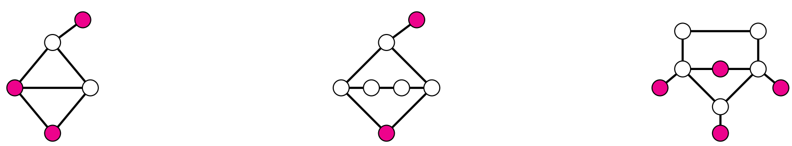
Subcase 1.1: Compact Structure ($l_1 \leq 2$).
Here, the shared path P_1 is short ($l_1 = 1$ or 2). As depicted in Figure 1(a), this compactness creates multiple shortest paths between the branching vertices u and v. The geometric obstruction normally caused by a unique shortest path is relaxed. Specifically, we can form a valid set including all degree-2 vertices on P_2 and P_3, plus one additional vertex from the shared structure (either an internal vertex of P_1 or one of the branching vertices u, v, provided it is not in S_{cut}). Thus, we gain one extra vertex beyond the degree-2 set, yielding $|L| + |D_2(G)| + 1$.

Subcase 1.2: Elongated Structure ($l_1 \geq 3$).
When the shared path is long, it acts as a "geodesic bridge" between the two short branches. This scenario is illustrated in Figure 1(b), where the long shared path isolates the two loops. Any path connecting w_2 and w_3 through P_1 has length $1 + l_1 + 1 \geq 5$, whereas the path through u (or v) has length 2. The crucial observation is that u and v (and any vertex on P_1) lie on the unique shortest paths connecting different parts of the graph. If any vertex from the shared path or branching points were added to X, it would block the visibility between w_2 and w_3. Therefore, X is strictly restricted to the leaves and the internal vertices of the non-shared paths (S_{ext}).

Case 2: General Configuration.
Assume the graph does not fit Case 1. This implies at least one non-shared path has length $k \geq 3$. An example of this configuration is shown in Figure 1(c). In this general setting, the visibility constraints within one cycle become largely independent of the other. The reasoning is as follows: if a cycle $C^{(i)}$ has length ≥ 5, its contribution is 0 (by Lemma 1), meaning no internal vertices are selected, avoiding any conflict with the other cycle. If a cycle is C_4 or C_3, the internal vertices can be selected according to $\alpha(C^{(i)})$ (by Lemmas 2 and 3), and since the branching vertices u, v are generally excluded (either as cut-vertices or to preserve path flow), the choices in $C^{(1)}$ do not obstruct the visibility in $C^{(2)}$. The additive formula

$$\mu_t(G) = |L| + \alpha(C^{(1)}) + \alpha(C^{(2)})$$

effectively captures this independence, summing the local maxima for each cycle. $\square$



(a) $\mu_t(G) = 3$ (b) $\mu_t(G) = 2$ (c) $\mu_t(G) = 4$

Figure 1. Visual illustration of Theorem 8 for the analyzed cases. Red vertices correspond to the total mutual-visibility set X.

3.3.2. Leafless Theta-Graphs

In this subsection, we address Type-III bicyclic graphs that contain no leaves ($L = \emptyset$). In the absence of leaves, the branching vertices u and v are not automatically forced to be cut-vertices. However, the geodesic structure of a Theta-graph still imposes strict restrictions on which vertices can be simultaneously included in a total mutual-visibility set.The following theorem determines the exact values of the total mutual-visibility number based on arbitrary selections from the admissible vertices.

Theorem 9 (Leafless Type-III Bicyclic Graphs). *Let $G = \Theta(l_1, l_2, l_3)$ be a connected leafless Type-III bicyclic graph. Assume that l_2 and l_3 are the lengths of the non-shared paths, and that l_1 is the length of the shared path (P_{share}). The total mutual-visibility number $\mu_t(G)$ is determined as follows:*

Case 1: The Twin-Short-Path Configuration ($l_2 = l_3 = 2$).

When both non-shared paths have length 2, the value of $\mu_t(G)$ depends on the length of the shared path:

Subcase 1.1: Compact Structure ($l_1 \leq 2$).

If the shared path is short, the maximum total mutual-visibility set consists of exactly three vertices: one branching vertex chosen arbitrarily, together with any two degree-2 vertices from the non-shared paths. As illustrated in Figure 2(a), this yields:

$$\mu_t(G) = 3.$$

(Note that even if the graph contains three degree-2 vertices, only two of them can be selected together with a branching vertex.)

Subcase 1.2: Elongated Structure ($l_1 \geq 3$).

If the shared path is long, the visibility is restricted to a single vertex (see Figure 2(b)). In this case, the maximum set consists of exactly one degree-2 vertex chosen arbitrarily from the non-shared paths. Thus:

$$\mu_t(G) = 1.$$

Case 2: General Configuration.

For all remaining configurations—namely, when at least one of the non-shared paths has length $l_i \geq 3$, or when the structure does not fall under Case 1—the branching vertices u and v cannot belong to a total mutual-visibility set. An example of this configuration is shown in Figure 2(c) . In this situation, the total mutual-visibility number is given by the sum of the contributions of the two cycles:

$$\mu_t(G) = \alpha(C^{(1)}) + \alpha(C^{(2)}),$$

where $\alpha(C^{(i)})$ denotes the cycle contribution defined in Section 3.1 (see also Lemmas 1, 2, and 3).

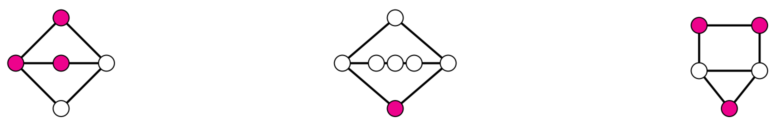
Proof. Since G is leafless, we have $L = \emptyset$. We examine the validity of vertex selections in each case.

Subcase 1.1 ($l_2 = l_3 = 2$ and $l_1 \leq 2$). The graph is structurally compact. By symmetry, one branching vertex, say u, may be chosen arbitrarily. As depicted in Figure 2(a) , with $u \in X$ (shown in red), the set can be extended by selecting any two degree-2 vertices, for instance, the internal vertices w_2 and w_3 of the non-shared paths. The set $X = \{u, w_2, w_3\}$ is valid, since the presence of multiple short shortest paths ensures that for every pair of vertices in X, there exists at least one shortest path whose internal vertices avoid X. Including any additional vertex would obstruct such paths, and therefore the maximum size is 3.

Subcase 1.2 ($l_1 \geq 3$). The elongation of the shared path imposes a strict geodesic constraint, as visualized in Figure 2(b). Any shortest path connecting internal vertices of the two non-shared paths must pass through one of the branching vertices u or

v. Consequently, neither u nor v can belong to X if an internal vertex is selected. Moreover, two internal vertices cannot be chosen simultaneously, as the shortest path between them would necessarily pass through u or v, which are excluded. Hence, the maximum total mutual-visibility set consists of exactly one degree-2 vertex chosen arbitrarily from the non-shared paths.

Case 2. General Configuration. The structural situation coincides with the general case analyzed in Theorem 8. Figure 2(c) illustrates a scenario where the path lengths force u and v (shown in white) to lie on unique shortest paths connecting the two cycles. Thus, the condition $u, v \notin X$ is necessary. Once these vertices are excluded, the visibility constraints of the two cycles become independent, and the additive formula follows directly. □



(a) $\mu_t(G) = 3$ (b) $\mu_t(G) = 1$ (c) $\mu_t(G) = 3$

Figure 2. Visual illustration of Theorem 9 for leafless Theta-graphs. Red vertices correspond to the total mutual-visibility set X.

Summary of Section 3

This concludes our comprehensive analysis of connected bicyclic graphs. The results established in Sections 3.1 through 3.3 provide a rigorous determination of the total mutual-visibility number for every structural variant— covering Type-I, Type-II, and Type-III configurations, along with all possible leaf arrangements. Consequently, this study offers a full characterization of the total mutual-visibility number for the entire class of connected bicyclic graphs.

4. Conclusion and future work

In this paper, we provided a complete characterization of the total mutual-visibility number, $\mu_t(G)$, for the class of connected bicyclic graphs. By decomposing this family into three canonical topological structures—Type-I graphs (disjoint cycles connected by a path), Type-II graphs (cycles sharing a single cut-vertex), and Type-III graphs (Theta-graphs)—we derived explicit formulas that cover all possible configurations.

Our analysis demonstrates that cut-vertices and leaves play a fundamental role in determining the structure of maximum total mutual-visibility sets. In particular, we showed that every leaf must be included in any such maximum set, whereas cut-vertices are necessarily excluded. Building on these structural observations, we introduced the notions of the *Free Set* and *Cycle Contribution*, which enable us to decouple the effect of each cycle from the surrounding graph structure. This framework allows the global visibility problem to be reduced to local computations for Type-I and Type-II graphs. In the case of Type-III graphs, where the underlying 2-connected structure introduces additional geodesic constraints, the exact values of $\mu_t(G)$ were obtained through a detailed analysis of path lengths and blocking vertices.

An important consequence of our results is that the derived formulas depend only on basic graph parameters, such as cycle lengths, vertex degrees, and the number of leaves. Since these parameters can be extracted efficiently, the total mutual-visibility number of a bicyclic graph can be computed in linear time, $O(n)$.

Several directions for future research naturally arise from this work. First, since Type-II bicyclic graphs form basic building blocks of cactus graphs—graphs in which each edge belongs to at most one cycle—it would be natural to extend the notion of Cycle Contribution to determine $\mu_t(G)$ for general cactus graphs. Second, the structural decomposition techniques developed here may be adaptable to graphs with higher cyclomatic number $c(G) \geq 3$, such as tricyclic or tetracyclic graphs, provided the interactions between cycles remain sufficiently constrained. From an algorithmic perspective, while this study establishes linear-time computability for bicyclic graphs, the computational complexity of determining $\mu_t(G)$ for general graphs remains open. Identifying polynomial-time algorithms or proving hardness results for broader graph classes would represent a significant advance. Finally, investigating alternative notions of visibility—such as outer mutual visibility or dual mutual visibility—within the setting of bicyclic graphs may reveal further structural phenomena and deepen the understanding of visibility parameters in graph theory.

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