

On linear ternary self-dual codes arising from strongly regular signed graphs

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Received: 10 July 2026; Accepted: 22 September 2026

Published Online: 30 September 2026

Abstract: Linear ternary self-dual codes generated by matrices of the form $[I \ A+xI]$ are considered, where A is the adjacency matrix of a strongly regular signed graph and $x \in \mathbb{F}_3$. It is shown that, in the connected case, every such code has distance at least six, apart from a trivial exception in dimension four. Existence is established in the case where A has exactly two distinct eigenvalues λ and μ which must be irrational, and satisfy $\lambda = -\mu$ and $\lambda^2 \equiv -1 \pmod{3}$. Moreover, among complete and net-regular signed graphs, only the complete case may occur, and exclusively within the preceding two-eigenvalue setting, while the net-regular case is excluded entirely. In the connected case, the analysis reduces the problem to exactly two unresolved exceptional cases. Examples of the obtained codes are provided.

Keywords: Linear ternary code, strong regularity, self-orthogonality, code distance, eigenvalues.

AMS Subject Classification: 05C22, 05C50, 94B05

1. Introduction

The central theme of this paper is the relationship between two structures, strongly regular signed graphs and linear ternary self-dual codes, together with the question of whether a strongly regular signed graph gives rise to (more precisely, generates) such a code. This connection naturally links combinatorial properties of signed graphs with algebraic properties of the associated codes, particularly dimension and distance. In this section, we introduce the necessary definitions and basic properties of both structures, establish the notation used throughout the paper, and conclude by outlining the main results obtained in the subsequent sections.

A *signed graph* $\dot{G}$ is an ordinary graph G in which every edge is declared either positive or negative; equivalently, $\dot{G} = (G, \sigma)$, where G is the underlying graph and σ is the

signature mapping the edge set in $\{+1, -1\}$. A signed graph is *homogeneous* if all its edges share the same sign; this includes, for instance, the case of an empty edge set. A signed graph is connected, bipartite or regular if the same holds for its underlying graph. The *net degree* of a vertex is the difference between the number of positive and the number of negative edges incident with it. A signed graph is *net-regular* if the net degree considered as a function on the vertex set is a constant.

If the vertices i and j are adjacent in $\dot{G}$, we write $i \sim j$. More specifically, the presence of a positive (resp. negative) edge between i and j is denoted by $i \overset{+}{\sim} j$ ($i \overset{-}{\sim} j$). The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0, 1)$ -adjacency matrix of G by reversing the sign of all entries which correspond to negative edges. The *spectrum* of $\dot{G}$ is the multiset of eigenvalues of $A_{\dot{G}}$.

A *walk* in a signed graph is a sequence of alternate vertices and edges, such that consecutive vertices are the endpoints of the corresponding edge. A walk is *positive* if the number of its negative edges (counted with their multiplicity if there are repeated edges) is even. Otherwise, it is *negative*. The difference between the numbers of positive and negative walks of length 2 between the vertices i and j is denoted by $w_2(i, j)$.

Strongly regular signed graphs were introduced in 2019 [8], and have since been extensively studied and classified in the literature. Their role has been recognized in constructing related structures requiring a high level of symmetry and a large automorphism group [9, 11]. We say that a signed graph $\dot{G}$ is *strongly regular* (abbreviated *SRS*) whenever it is regular and satisfies the following conditions:

- (S1) $\dot{G}$ is neither homogeneous complete nor edgeless;
- (S2) there exists $a \in \mathbb{Z}$ such that $w_2(i, j) = a$, for all $i \overset{+}{\sim} j$;
- (S3) there exists $b \in \mathbb{Z}$ such that $w_2(i, j) = b$, for all $i \overset{-}{\sim} j$;
- (S4) there exists $c \in \mathbb{Z}$ such that $w_2(i, j) = c$, for all $i \not\sim j$, $i \neq j$.

This definition extends the classical notion of a strongly regular graph. In matrix form, it can be expressed as

$$A^2 = \frac{a}{2}(A + A_G) - \frac{b}{2}(A - A_G) + cA_{\overline{G}} + rI, \quad (1.1)$$

where r denotes the vertex degree of $\dot{G}$, $\overline{G}$ is the complement of the underlying graph G , and A_H denotes the adjacency matrix of an unsigned graph H .

Linear codes have been studied extensively for many decades, with particular emphasis on binary codes. Although less numerous, the literature on ternary codes is also substantial; representative monographs include [2, 6, 13]. In contrast, linear ternary codes associated with signed graphs have received considerably less attention, with some related results appearing in [5, 7, 11].

A *linear ternary code* $\mathcal{C}$ of *length* n and *dimension* $|\mathcal{C}|$ is a $|\mathcal{C}|$ -dimensional linear subspace $\mathcal{C}$ of the vector space $\mathbb{F}_3^n$, where $\mathbb{F}_3$ is the field of order 3. The elements of $\mathcal{C}$ are called *codewords*. The size of $\mathcal{C}$ is the number of codewords and equals $3^{|\mathcal{C}|}$.

The *weight* $\text{wt}(\mathbf{c})$ of a codeword $\mathbf{c}$ is the number of its non-zero entries, and the *distance* between two codewords is the Hamming distance between them, that is the number of entries in which they differ. The *distance* d of $\mathcal{C}$ is the minimum distance between distinct codewords and it equals the minimum weight of its non-zero codewords [2, 6, 13].

Every codeword of $\mathcal{C}$ can be expressed as a linear combination of a basis of $\mathcal{C}$. The vectors of such a basis are usually arranged as the rows of a matrix C , called a *generator matrix*. The *dual code* $\mathcal{C}^\perp$ is defined by

$$\mathcal{C}^\perp = \{\mathbf{c} \in \mathbb{F}_3^n : \mathbf{x} \cdot \mathbf{c} \equiv 0 \pmod{3}, \text{ for all } \mathbf{x} \in \mathcal{C}\},$$

where $\cdot$ denotes the Euclidean inner product. It is well known that $|\mathcal{C}| + |\mathcal{C}^\perp| = n$ [6, 13]. The code $\mathcal{C}$ is called *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^\perp$, and *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$.

For every codeword $\mathbf{c}$ of a ternary self-orthogonal code, the weight $\text{wt}(\mathbf{c})$ is divisible by 3. Indeed, $0 = \mathbf{c} \cdot \mathbf{c} \equiv \text{wt}(\mathbf{c}) \pmod{3}$, where the first equality follows from self-orthogonality. The code $\mathcal{C}$ is self-orthogonal if and only if $CC^\top = O$. In particular, if M is an $n \times n$ matrix over $\mathbb{F}_3$ and $C = [I \ M]$, then the dimension is n , and self-orthogonality is equivalent to self-duality.

Following [2, 6, 13], we recall that ternary self-dual codes of length n exist if and only if n is divisible by 4. Their distance satisfies the Mallows–Sloane bound $d \leq 3 \lfloor \frac{n}{12} \rfloor + 3$. Codes attaining this upper bound are called extremal. Classical examples include the extended ternary Golay code, together with wider families such as the Pless symmetry codes and extended quadratic residue codes. Complete classifications of ternary self-dual codes are known for all lengths up to and including 24 [3].

In this paper, linear ternary self-dual codes generated by matrices of the form $[I \ A + xI]$ are studied, where A is the adjacency matrix of a strongly regular signed graph and $x \in \mathbb{F}_3$. To avoid ambiguity, the field $\mathbb{F}_3$ is identified with $\{1, 0, -1\}$ throughout, in accordance with the notation used for signed graphs.

Here is the outline of the obtained results. Necessary and sufficient conditions for self-duality are established in terms of the parameters of the corresponding strongly regular signed graph. It is further shown that, for $r > 2$, every code obtained in this way has distance at least six. In the case where A has exactly two distinct eigenvalues λ and μ , the corresponding code is proved to be self-dual if and only if $x = 0$, the eigenvalues are irrational, $\lambda = -\mu$, and $\lambda^2 \equiv -1 \pmod{3}$.

Several additional restrictions are derived, leading to the elimination of many classes of strongly regular signed graphs. Ultimately, for connected SRSGs, the problem is reduced to two unresolved exceptional cases. In the first case, G is non-complete, $\overline{G}$ is disconnected, $a = -b$ and $c \neq 0$. In the second case, G is non-complete, $a \neq -b$, $c \notin \{0, \frac{a+b}{2}\}$ and $\frac{2c}{a+b-2c} \notin \mathbb{R} \setminus (-1, 0)$.

The next section introduces several additional definitions together with a simple result showing that the code generated by $[I \ A + xI]$ depends only on the switching class of the corresponding signed graph, without assuming strong regularity. The main results are presented in the subsequent two sections. Section 3 contains the theoretical

analysis, while Section 4 is devoted to existence questions for particular classes of SRSGs. Finally, examples of the obtained codes are collected in Section 5.

2. Switching equivalence and the corresponding codes

We believe that the result of this section, stating that switching equivalent signed graphs generate equivalent codes, is already implicit in the existing literature, or can at least be derived from known results. Nevertheless, for the sake of completeness and to introduce the corresponding notions, we provide a short proof.

The signed graphs $\dot{G}$ and $\dot{H}$, with adjacency matrices $A_{\dot{G}}$ and $A_{\dot{H}}$, are *switching equivalent* if there is a diagonal matrix S with ± 1 on the main diagonal such that $A_{\dot{H}} = S^{-1}A_{\dot{G}}S$. Clearly, they share the same spectrum.

Recall that codes are equivalent if their generating matrices can be obtained from one another by invertible elementary row operations, column permutations, and multiplication of individual columns by non-zero field elements; for a formal definition, we refer to [6, 13].

In what follows, we show that the codes $[I \ A_{\dot{G}} + xI]$ and $[I \ A_{\dot{H}} + xI]$ are equivalent. Since $S^{-1} = S$, from $A_{\dot{H}} + xI = SA_{\dot{G}}S + xI = S(A_{\dot{G}} + xI)S$, we obtain $[I \ A_{\dot{H}} + xI] = [I \ S(A_{\dot{G}} + xI)S]$. Now multiply the rows of this matrix by S ; since S is invertible, this row operation does not change the generated code. We obtain $[S \ (A_{\dot{G}} + xI)S]$. Finally,

$$[S \ (A_{\dot{G}} + xI)S] \begin{bmatrix} S & O \\ O & S \end{bmatrix} = [SS \ (A_{\dot{G}} + xI)SS] = [I \ A_{\dot{G}} + xI].$$

Hence, the two generator matrices are related by invertible row operations and a monomial transformation of coordinates. Thus, the codes are equivalent. In other words, the code generated by $[I \ A_{\dot{G}} + xI]$ depends only on the switching class of $\dot{G}$.

3. Self-duality

In this section $\dot{G}$ is a SRSG with n vertices whose parameters r, a, b, c are given in the defining conditions (S1)–(S4). We start with necessary and sufficient conditions for self-duality of the corresponding code.

Theorem 1. *Let $\dot{G}$ be an inhomogeneous non-complete SRSG with the adjacency matrix A . For $x \in \mathbb{F}_3$, let $[I \ A + xI]$ be the generator matrix of a ternary code. The code is self-dual if and only if the parameters of $\dot{G}$ satisfy*

$$(r, a, b, c) = (-x^2 - 1, x, -x, 0) \tag{3.1}$$

in $\mathbb{F}_3$.

Proof. Denote the generator matrix by C . The corresponding ternary code is self-dual if and only if $CC^\top = O$.

Since A is symmetric, we obtain $CC^\top = I + (A + xI)(A + xI)^\top = I + (A + xI)^2$. Hence,

$$CC^\top = A^2 + 2xA + (x^2 + 1)I.$$

Substituting the quadratic relation (1.1), we get $CC^\top = \frac{a}{2}(A + A_G) - \frac{b}{2}(A - A_G) + cA_{\overline{G}} + 2xA + (r + x^2 + 1)I$. Collecting coefficients yields

$$CC^\top = \left(\frac{a-b}{2} + 2x \right) A + \frac{a+b}{2} A_G + cA_{\overline{G}} + (r + x^2 + 1)I.$$

Denote by P and N the matrices of underlying graphs induced by the positive and the negative edges of $\dot{G}$, respectively. Then $A = P - N$, $A_G = P + N$ and $A_{\overline{G}} = J - I - (P + N)$. Consequently, $CC^\top = aP + bN + c(J - I - (P + N)) + rI + 2x(P - N) + (x^2 + 1)I$, giving

$$CC^\top = (a - c + 2x)P + (b - c - 2x)N + c(J - I) + (r + x^2 + 1)I.$$

Since $\dot{G}$ is inhomogeneous and non-complete, the matrices $P, N, J - I, I$ are linearly independent over $\mathbb{F}_3$. Therefore, the equality $CC^\top = O$ holds if and only if $a - c + 2x = 0$, $b - c - 2x = 0$, $c = 0$ and $r + x^2 + 1 = 0$ in $\mathbb{F}_3$. Over the same field, these equalities are equivalent to

$$a = x, b = -x, c = 0, r + x^2 + 1 = 0,$$

as desired. $\square$

Remark 1. Observe that if $\dot{G}$ is complete, then $A_{\overline{G}} = O$, and therefore the parameter c is arbitrary, while the conditions on the remaining parameters remain unchanged. Similarly, if $\dot{G}$ is homogeneous, then one of the matrices P and N vanishes, and the corresponding condition on the associated parameter becomes redundant. In conclusion, Theorem 1 essentially covers these particular cases upon omitting the corresponding parameter.

We now proceed to consider the distance of the code. Throughout, we say that two edges *agree* at a common vertex if they have the same sign. Similarly, two vertices *agree* at a common neighbour if the corresponding incident edges have the same sign. We first set $x = 0$ in the generator matrix.

Theorem 2. *Let $C = [I \ A]$ be the generator matrix of a ternary self-dual code where A is the adjacency matrix of a SRSg with degree $r > 2$. Then the distance of this code is at least six.*

Proof. Every codeword of weight 3 is a linear combination of at most three rows of the generator matrix C . Let $\mathbf{c}$ be an arbitrary codeword of weight 3. Since the code is self-dual, $CC^\top = I + A^2 = O$ over $\mathbb{F}_3$. Every diagonal entry of A^2 equals r , so

$$r \equiv 2 \pmod{3}. \quad (3.2)$$

Together with $r > 2$, this gives $r \geq 5$. In particular, a codeword of weight 3 cannot be a single row of C .

Assume now that $\mathbf{c}$ is a linear combination of two rows corresponding to vertices u and v . Then u and v must be non-adjacent in $\dot{G}$; otherwise, $\text{wt}(\mathbf{c}) > 3$. Moreover, one of the following holds:

- (a) u and v agree at $r - 1$ common neighbours and disagree at exactly one;
- (b) u and v disagree at $r - 1$ common neighbours and agree at exactly one.

In either case, let w index the unique non-zero entry, say γ , in the second block of $\mathbf{c}$. Since $r \geq 5$, the vertex w has a neighbour $z \notin \{u, v\}$. Let $\mathbf{r}_z$ be the generator row corresponding to z . The first block of $\mathbf{c}$ is supported on $\{u, v\}$, and hence

$$\mathbf{c} \cdot \mathbf{r}_z = \gamma A_{wz} \neq 0$$

over $\mathbb{F}_3$. This contradicts self-orthogonality.

Finally, assume that $\mathbf{c}$ is a linear combination of three rows corresponding to vertices u, v, w . Then u, v, w are pairwise non-adjacent; otherwise, $\mathbf{c}$ would contain an additional non-zero entry in the part corresponding to these vertices.

Let s denote the number of common neighbours of u, v, w , and let t be the number of vertices adjacent to exactly two of them. Then each pair among u, v, w shares exactly $t/3$ additional common neighbours, and clearly $t \equiv 0 \pmod{3}$, so we may write $t = 3k$. Two configurations are possible:

- (c1) the vertices u, v, w agree at the s common neighbours and pairwise disagree at the additional neighbours;
- (c2) the vertices u and v agree at the s common neighbours and disagree at the remaining common neighbours, while w disagrees with both on s vertices and agrees with both at the remaining neighbours.

In the case (c1), we obtain

$$r = s + \frac{2t}{3} \equiv 2 \pmod{3}, \quad c = s - \frac{t}{3} \equiv 0 \pmod{3}.$$

Substituting $t = 3k$, the system becomes

$$s + 2k \equiv 2 \pmod{3}, \quad s - k \equiv 0 \pmod{3}.$$

From the second congruence, $s \equiv k \pmod{3}$. Substituting this into the first congruence yields $3k \equiv 2 \pmod{3}$, a contradiction.

In the case (c2), considering the pairs (u, v) and (u, w) yields $c = s - \frac{t}{3}$ and $c = -s + \frac{t}{3}$. Hence, $s = \frac{t}{3}$, and consequently

$$r = s + \frac{2t}{3} = t = 3k,$$

contrary to (3.2).

Therefore, the code does not contain a codeword of weight 3, completing the proof. $\square$

We remark that the previous theorem does not hold for $r = 2$, as demonstrated by the quadrangle with exactly one negative edge in the role of $\dot{G}$. We proceed with $x \neq 0$.

Theorem 3. *Let $C = [I \ A + xI]$, $x \in \{1, -1\}$, be the generator matrix of a ternary self-dual code where A is the adjacency matrix of a connected SRSG. Then the distance of this code is at least six.*

Proof. Only the case $x = 1$ is proved, as the other one is analogous. We continue to use the notation of the previous proof and show that the code does not contain a non-zero codeword of weight 3.

Since $x = 1$ and the code is self-dual, $CC^\top = I + (A + I)^2 = O$ over $\mathbb{F}_3$. Every diagonal entry of this matrix equals $r + 2$, and hence $r \equiv 1 \pmod{3}$. If $r = 1$, connectedness would force $\dot{G}$ to be a signed K_2 , contrary to (S1). Thus, $r \geq 4$ and a codeword $\mathbf{c}$ of weight 3 cannot be a single row of C .

Assume now that $\mathbf{c}$ is a linear combination of two rows corresponding to vertices u and v . We claim that these vertices must be adjacent. To prove it, assume that $\mathbf{c} = \alpha \mathbf{r}_u + \beta \mathbf{r}_v$, where $\alpha, \beta \in \{\pm 1\}$ and $\mathbf{r}_u$ is the row of the generator matrix corresponding to u . Since the identity block contributes two non-zero coordinates, the part corresponding to $A + I$ must have weight one. If $u \not\sim v$, then the coordinates u and v in this part are both non-zero, yielding $\text{wt}(\mathbf{c}) \geq 4$, a contradiction. Thus, $u \sim v$. Moreover, the coordinates at u and v in the second block differ by the factor A_{uv} , and hence both vanish. If u and v had different closed neighbourhoods, regularity would give at least two vertices outside $\{u, v\}$ adjacent to exactly one of them, producing at least two non-zero coordinates. Hence, u and v have the same closed neighbourhoods. Similarly to the proof of Theorem 2, one of the following holds:

- (a) $u \overset{+}{\sim} v$, while u and v agree at $r - 2$ common neighbours and disagree at exactly one;
- (b) $u \overset{-}{\sim} v$, while u and v disagree at $r - 2$ common neighbours and agree at exactly one.

In either case, let w index the unique non-zero entry, say γ , in the second block of $\mathbf{c}$. Since $r \geq 4$, the vertex w has a neighbour $z \notin \{u, v\}$. As $z \neq w$, the generator row $\mathbf{r}_z$ satisfies

$$\mathbf{c} \cdot \mathbf{r}_z = \gamma(A + I)_{zw} = \gamma A_{zw} \neq 0,$$

contrary to self-orthogonality.

Finally, assume that $\mathbf{c}$ is a linear combination of three rows corresponding to vertices u, v and w . The vanishing second block forces u, v, w to form a triangle with edge signs $\alpha_i \alpha_j$, where the α_i are the corresponding coefficients. Assume that $\alpha_u = \alpha_v$, so $u \stackrel{+}{\sim} v$. Let s count their common neighbours. By regularity, each pair has $k = (r-s-2)/2$ additional common neighbours outside $\{u, v, w\}$. Cancellation implies that u, v agree at the s common neighbours and disagree at their k additional ones, while w contributes 1 to $w_2(u, v)$. Using $CC^\top = O$, we obtain $r = s + 2 + 2k \equiv 1 \pmod{3}$ and $a = s + 1 - k \equiv 1 \pmod{3}$. Consequently, $3k \equiv 2 \pmod{3}$, a contradiction.

Therefore, the distance is at least six. $\square$

4. Existence

As indicated by the title, in this section we investigate the existence of the codes considered in the previous section.

Theorem 4. *Let $\dot{G}$ be a SRSG with adjacency matrix A having exactly two distinct eigenvalues λ and μ . The ternary code $[I \ A + xI]$ is self-dual if and only if $x = 0$, $\lambda = -\mu$ and $\lambda^2 \equiv 2 \pmod{3}$. In this case, $\lambda \notin \mathbb{Z}$.*

Proof. It holds

$$A^2 - (\lambda + \mu)A + \lambda\mu I = 0.$$

Comparing this identity with the defining identity of a SRSG, and choosing any unused parameters accordingly, we obtain

$$(r, a, b, c) = (-\lambda\mu, \lambda + \mu, -\lambda - \mu, 0), \quad (4.1)$$

In addition, λ and μ are either integers or irrational algebraic conjugates. In the latter case, we necessarily have $\lambda^2 = r$ and $\mu = -\lambda$.

By Theorem 1 and Remark 1, the code is self-dual if and only if $(r, a, b, c) = (-x^2 - 1, x, -x, 0)$ in $\mathbb{F}_3$. Substituting in (4.1), we obtain

$$\lambda + \mu = x, \lambda\mu = x^2 + 1. \quad (4.2)$$

We now solve the system (4.2) in $\mathbb{F}_3$. Suppose first that λ, μ are integers. By substituting μ from the first equality in the second one, we arrive at

$$\lambda^2 - x\lambda + x^2 + 1 = 0. \quad (4.3)$$

For $x = 0$, we obtain $\lambda^2 = -1$, which is impossible. Likewise, the choices $x = 1$ and $x = -1$ lead to the impossible equations $\lambda^2 \mp \lambda + 2 = 0$, respectively, since none of the three possible values of λ satisfies them. Therefore, if the eigenvalues are integers, the code is not self-dual.

Suppose now that the eigenvalues are non-integral. Since $\lambda = -\mu$, the first equation in (4.2) yields $x = 0$, while the second gives $r = \lambda^2 \equiv -1 \pmod{3}$, proving the assertion. $\square$

Every signed graph with exactly two distinct irrational eigenvalues is strongly regular, and every switching of such a graph is again a SRSG. For general SRSGs, switching need not preserve strong regularity [12, Chapter 3]. In view of Section 2, switching equivalent SRSGs arising in the previous theorem generate equivalent self-dual codes. We proceed to consider net-regular SRSGs. The following lemma is needed.

Lemma 1. *The net degree ϱ of a net-regular SRSG satisfies*

$$\varrho^2 - \frac{a-b}{2}\varrho = \left(\frac{a+b}{2} - c\right)r + cn + r - c.$$

Proof. It holds $A\mathbf{j} = \varrho\mathbf{j}$, where $\mathbf{j}$ denotes the all-ones vector, and we consider the quadratic relation (1.1). Since $A_{\overline{G}} = J - I - A_G$, it reads

$$A^2 = \frac{a-b}{2}A + \frac{a+b}{2}A_G + c(J - I - A_G) + rI,$$

that is,

$$A^2 = \frac{a-b}{2}A + \left(\frac{a+b}{2} - c\right)A_G + cJ + (r-c)I.$$

Multiplying by $\mathbf{j}$, and using $A\mathbf{j} = \varrho\mathbf{j}$, $A_G\mathbf{j} = r\mathbf{j}$, and $J\mathbf{j} = n\mathbf{j}$, we obtain

$$\varrho^2\mathbf{j} = \frac{a-b}{2}\varrho\mathbf{j} + \left(\frac{a+b}{2} - c\right)r\mathbf{j} + cn\mathbf{j} + (r-c)\mathbf{j}.$$

Hence, ϱ satisfies the desired quadratic equality. $\square$

The next result excludes all net-regular SRSGs.

Theorem 5. *For a net-regular SRSG with the adjacency matrix A , the ternary code C_x generated by $[I \ A + xI]$ is not self-dual.*

Proof. In light of (3.1), the equality on the net degree ϱ established in Lemma 1 simplifies over $\mathbb{F}_3$ as follows. Since $a + b = 0$, $\frac{a-b}{2} = x$ and $c = 0$, we get $\varrho^2 - x\varrho = r$. As $r = -x^2 - 1$, it follows that

$$\varrho^2 - x\varrho + x^2 + 1 = 0.$$

This is the equation (4.3) which has no integer solutions as established in the corresponding proof, and we are done. $\square$

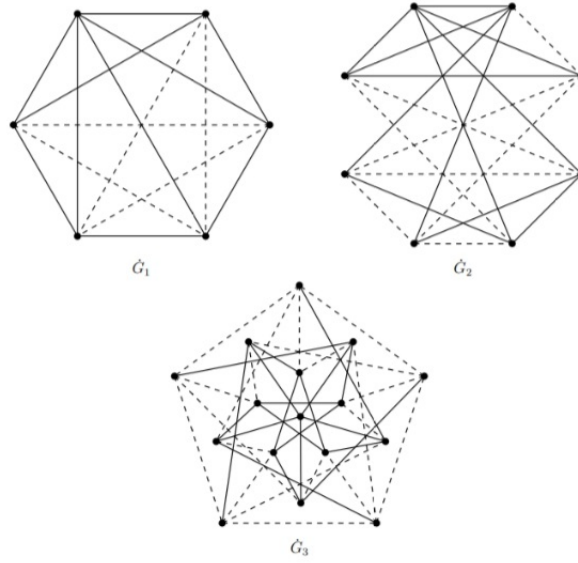


Figure 1. The three SRSs with eigenvalues $\pm\sqrt{5}$. Negative edges are dashed.

For the remainder of this section, assume that $\dot{G}$ is connected. The following discussion relies on the previous results and certain known findings. If, for a SRS $\dot{G}$, we have $a = -b$ and $\dot{G}$ is either complete or non-complete with $c = 0$, the identity (1.1) reduces to $A^2 + bA = rI$, meaning that $\dot{G}$ has exactly two distinct eigenvalues. This situation is resolved in Theorem 4. If $\dot{G}$ is non-complete with $c \neq 0$, then $A^2 + bA = cA_{\overline{G}} + rI$. We know from [4, Theorem 3.1] that, in this setting, $\dot{G}$ is net-regular whenever $\overline{G}$ is connected. Consequently, by Theorem 5 the corresponding code cannot be self-dual.

For $a \neq -b$, $\dot{G}$ is either (i) complete, or (ii) non-complete with $c \in \{0, \frac{a+b}{2}\}$, or (iii) non-complete with $c \notin \{0, \frac{a+b}{2}\}$. It has been proved in [4, Theorem 4.1] and [1, Proposition 3.1] that SRSs of (i) and (ii) are necessarily net-regular. By [1, Proposition 3.2], the same holds for SRSs of (iii) whenever either $a + b \equiv 1 \pmod{2}$ or $\frac{2c}{a+b-2c} \in \mathbb{R} \setminus (-1, 0)$.

Gathering the previous results, we arrive at the following conclusion.

Theorem 6. *Let $\dot{G}$ be a connected SRS with adjacency matrix A . Unless*

- (a) $\dot{G}$ is non-complete, $\overline{G}$ is disconnected, $a = -b$ and $c \neq 0$ or
- (b) G is non-complete, $a \neq -b$, $c \notin \{0, \frac{a+b}{2}\}$ and $\frac{2c}{a+b-2c} \notin \mathbb{R} \setminus (-1, 0)$,

the ternary code $[I \ A + xI]$ is self-dual if and only if $x = 0$ and $\dot{G}$ satisfies the spectral conditions of Theorem 4.

5. Examples

SRSGs with two eigenvalues $\pm\sqrt{5}$ are classified in [10]. In the bipartite case, there exist two infinite families together with five exceptional SRSGs. For such graphs, the code $[I \ A]$ decomposes into a combination of codes arising from the adjacencies between the two colour classes. We therefore focus on the non-bipartite case, which consists of infinite families and three exceptional SRSGs illustrated in Figure 1. The corresponding codes have dimensions 6, 8, and 16, respectively, while their lengths are twice the dimensions. The codeword weight distributions of these codes are given below.

Weights:	0	3	6	9	12	15	18	21	24	27	30
$\dot{G}_1$	1	0	264	440	24						
$\dot{G}_2$	1	0	224	2720	3360	256					
$\dot{G}_3$	1	0	64	1600	63424	1282048	8652160	18794752	12350080	1861632	40960

One may recognize the famous Golay code corresponding to $\dot{G}_1$ and attaining the Mallows–Sloane bound. The mentioned infinite families consist of SRSGs with $4t + 2$ and $4t + 4$ vertices, respectively, where $t \geq 2$. The corresponding codes are obtained by the same construction.

Another example is provided by the infinite family of r -dimensional signed cubes with negative quadrangles, where $r \equiv -1 \pmod{3}$. It is known that, for every r , there exists exactly one such cube up to switching isomorphism, regardless of the congruence class of r [12, Theorem 4.18]; further details on these SRSGs, including iterative constructions, are given in the same reference. In our setting, the first interesting case occurs for $r = 5$, yielding the code of dimension 32.

Acknowledgement: This research is supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia.

Conflict of Interest: The author declares that he has no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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