

On the eccentricity matrices of line graphs of trees and wheel graphs

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Abstract: The eccentricity matrix $\mathcal{E}(X)$ of a connected graph X is obtained from the distance matrix of X by keeping the largest nonzero entries in each row and each column, and leaving zeros in the remaining ones. In this paper, we investigate the spectral properties of the eccentricity matrices of line graphs of trees and wheel graphs. We prove that if X is a tree on n vertices of odd diameter $d \geq 5$, then the inertia of $\mathcal{E}(L(X))$ is $(2, 2, n - 5)$. Furthermore, for trees of even diameter $d \geq 4$, we show that the inertia is $(l, l, n - 2l - 1)$, where l denotes the number of diametrically distinguished vertices of X . Finally, we extend our analysis to determine the inertia, eigenvalues, and determinant of the eccentricity matrix for the line graphs of wheel graphs. We also relate its spectrum to the Dirichlet kernel.

Keywords: Tree, line graph, eccentricity matrix.

AMS Subject Classification: 05C50, 05C12

1. Introduction

Let $X = (V(X), E(X))$ be a simple undirected graph, where $V(X)$ and $E(X)$ denote the vertex set and the edge set. The *distance* $d(v_i, v_j)$ between the vertices v_i and v_j of X is the length of the shortest path from v_i to v_j , where $i \neq j$. The *eccentricity* of a vertex u in X , denoted by $e(u)$, is defined as $\max\{d(u, v) | v \in V(X)\}$. The *diameter* of a graph X , denoted by $d(X)$, is equal to $\max\{d(u, v) | u, v \in V(X)\}$. A *diametrical path* of a graph is the shortest path whose length equals the diameter of the graph. The adjacency matrix of a graph X , denoted by $A(X)$ is the matrix whose $(i, j)^{th}$ entry is 1 if v_i and v_j are adjacent and 0 otherwise. The distance matrix of a graph X , denoted by $D(X)$ is the matrix whose $(i, j)^{th}$ entry is equal to $d(v_i, v_j)$. Motivated by

Randić's [16] idea, Wang *et al.* [22] introduced the notion of the eccentricity matrix. The eccentricity matrix of a graph X , denoted by $\mathcal{E}(X)$ is defined as follows:

$$\mathcal{E}(X)_{ij} = \begin{cases} d(v_i, v_j) & \text{if } d(v_i, v_j) = \min(e(v_i), e(v_j)), \\ 0 & \text{otherwise.} \end{cases}$$

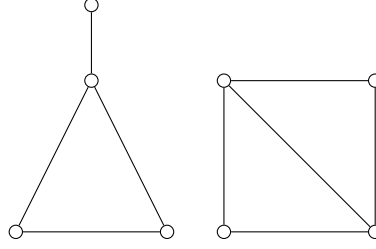
In a certain sense, the eccentricity matrix behaves oppositely to the adjacency matrix: it retains only the maximal distances in each row and column, while setting all other entries equal to zero. Therefore, it is also known as an anti-adjacency matrix. The eigenvalues of the eccentricity matrix are called eccentricity eigenvalues. The collection of the eigenvalues of a matrix together with their multiplicities is called the spectrum of the matrix. If $\lambda_1 > \lambda_2 > \dots > \lambda_k$ are the distinct eigenvalues of a matrix A , then the spectrum of A is denoted by

$$\left\{ \begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{array} \right\},$$

where m_i is the algebraic multiplicity of eigenvalue λ_i for $1 \leq i \leq k$. The spectrum of the eccentricity matrix is called the eccentricity spectrum. The inertia of a matrix is the triple of integers (n_+, n_-, n_0) , where n_+ , n_- , and n_0 denote the number of positive, negative, and zero eigenvalues of $\mathcal{E}(X)$, respectively.

Unlike the adjacency and distance matrices of a connected graph, the eccentricity matrix is not irreducible. For example, the eccentricity matrix of a complete bipartite graph is reducible. It was shown in [22, 24] that the eccentricity matrix of every tree, as well as that of certain other classes of graphs, is irreducible. The problem of characterizing the graphs for which the eccentricity matrix is irreducible remains open. One of the main applications of the eccentricity matrix is in the field of chemistry. Specifically, the eccentricity matrix is instrumental in modeling the boiling points of hydrocarbons [21] and developing various molecular descriptors [22]. Beyond its chemical applications, the spectral properties of the eccentricity matrix have been extensively characterized. In [20] and [25], the authors characterize all connected graphs with exactly one positive eccentricity eigenvalue and graphs whose second largest eccentricity eigenvalue is less than 1, respectively. Mahato *et al.* studied the eccentricity matrices of trees and of their complements in [12] and [13], respectively. The eccentricity matrices of wheel, block, and bi-block graphs have been studied in [5, 9, 11]. The behavior of the eccentricity matrix under various operations was examined in [14, 15], with further foundational literature provided in [10–12, 18, 19, 21, 23, 25]. Motivated by these developments, this paper investigates the eccentricity matrices of line graphs of trees and wheel graphs.

Among the various graph constructions, the line graph is perhaps the most intriguing. Its systematic study goes back to Whitney [26]. Given a graph X , its *line graph*, denoted by $L(X)$, is the graph whose vertex set is the edge set of X , and two vertices of $L(X)$ are adjacent if and only if the corresponding edges in X are adjacent. Figure

Figure 1: X and $L(X)$

1 shows a graph and its line graph. The relationship between the properties of X and those of $L(X)$ has been extensively studied; see, for example, [2, 6].

Recently, Balamoorthy et al. [1] determined the eccentricity spectra of line graphs of starlike trees, computed their eccentric energy, and established that their eccentricity matrices are irreducible, whereas Section 3 of this work establishes a uniform inertia theorem for the line graph of an arbitrary tree. On the common subclass of starlike trees, the inertia formulas obtained here are consistent with the spectra determined by Balamoorthy et al. Furthermore, there is no analogous overlap with Balamoorthy et al. in the wheel-graph part of the manuscript. This article is organized as follows. In Section 2, we collect some known results. In Section 3, we determine the inertia of the eccentricity matrices of line graphs of trees. We prove that if T is a tree of odd diameter $d \geq 5$, then the inertia of $L(T)$ is $(2, 2, n - 5)$. Furthermore, if T is a tree of even diameter $d \geq 4$, then the inertia of $\mathcal{E}(L(T))$ is $(l, l, n - 2l - 1)$, where l is the number of diametrically distinguished vertices of T , defined in the next section. Interestingly, the number of positive and negative eigenvalues of the tree and the line graph of a tree are identical (see [12]). Our proofs use some of the techniques developed in [12]. In Section 4, we determine the eigenvalues, determinant, inertia, and irreducibility of the eccentricity matrix of the line graph of a wheel graph.

2. Preliminaries

In this section, we provide the fundamental definitions, notations, and results used throughout the paper.

Definition 1. Let T be a tree with even diameter and u_0 be the center of T . A vertex v adjacent to u_0 is diametrically distinguished if there is a diametrical path that contains the vertex v . The set of diametrically distinguished vertices will be denoted by $D(T)$.

Let $T(k, s, t)$ denote the tree with $n = s + t + k + 1$ vertices, constructed by taking a path on vertices $\{1, 2, \dots, k + 1\}$, adding s and t pendant vertices adjacent to vertex 1 and $k + 1$, respectively. The *path graph*, *complete graph* and the *star graph* on n

vertices are denoted by P_n , K_n and $K_{1,n-1}$, respectively.

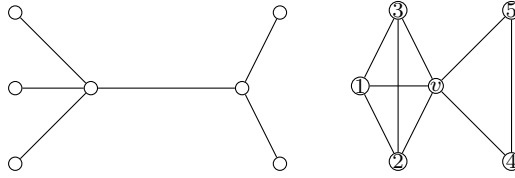


Figure 2: $T(1, 3, 2)$ and $L(T(1, 3, 2))$

Let A be a real symmetric matrix whose rows and columns are indexed by $X = \{1, 2, \dots, n\}$. Let $\Pi = \{X_1, X_2, \dots, X_m\}$ be a partition of X . Let A be partitioned according to Π as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix},$$

where A_{ij} denotes the submatrix of A formed by rows in X_i and the columns in X_j . If q_{ij} denotes the average row sum of A_{ij} , then the matrix $Q = [q_{ij}]$ is called the *quotient matrix* of A . If the row sum of each block A_{ij} is a constant, then the partition Π is called an *equitable partition*. $J_{m \times n}$ denotes the matrix of order $m \times n$ with all entries one.

Theorem 1 (Brouwer and Haemers, [3]). *Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition. Then the spectrum of A contains the spectrum of Q .*

Lemma 1. [17] *Let $B, C \in \mathbb{R}^{n \times n}$ and let D be any $n \times n$ matrix. Then*

$$\det \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = (-1)^n \det(B) \det(C).$$

Theorem 2. [4] *Let $C = \text{circ}(c_0, c_1, \dots, c_{n-1})$ be an $n \times n$ circulant matrix. Then the eigenvalues of C are $\lambda_j = f(\omega_j)$, where $f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}$ and $\omega_j = e^{\frac{2\pi j}{n}}$ are the n^{th} roots of unity for $j = 0, 1, 2, \dots, n-1$.*

Theorem 3. [8] *Let A be a square matrix of order n whose entries are nonnegative. Then A is irreducible if and only if $(I_n + A)^{n-1} > 0$.*

Theorem 4. [7] *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be an $n \times n$ Hermitian matrix, where A_{11} is nonsingular. Then

$$\text{In}(A) = \text{In}(A_{11}) + \text{In}(A/A_{11}),$$

where

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

is the Schur complement of A_{11} in A .

Lemma 2. For $a, b > 0$ and $l \geq 2$, the $2l \times 2l$ symmetric matrix partitioned as

$$A = \begin{pmatrix} a(J_l - I_l) & b(J_l - I_l) \\ b(J_l - I_l) & 0 \end{pmatrix},$$

where J_l and I_l denote the $l \times l$ matrix with all entries one and the identity matrix, respectively. Then the inertia of A is $(l, l, 0)$.

Proof. Let $A_1 = a(J_l - I_l)$ and $A_2 = b(J_l - I_l)$. Since A is symmetric and A_1 is nonsingular, Theorem 6 gives

$$\text{In}(A) = \text{In}(A_1) + \text{In}(-A_2A_1^{-1}A_2).$$

The inertia of A_1 is $(1, l-1, 0)$. Moreover, $-A_2A_1^{-1}A_2 = -\frac{b^2}{a}(J_l - I_l)$, whose inertia is $(l-1, 1, 0)$. $\square$

3. Line graph of trees

In this section, we determine the inertia of the eccentricity matrices of line graphs of trees. If T is a tree of diameter one, then $L(T) = K_1$ and the inertia is $(0, 0, 1)$. If T is a tree on n vertices of diameter two, then $L(T) = K_{n-1}$ and the inertia is $(1, n-2, 0)$. In the following theorems, we determine the inertia of $\mathcal{E}(L(T))$ for trees of odd and even diameter. As a result, we characterize all trees whose line graphs possess a nonsingular eccentricity matrix.

Theorem 5. Let T be a tree on n vertices with an odd diameter $d(\geq 3)$. Then the inertia of $\mathcal{E}(L(T))$ is

$$\begin{cases} (1, 2, n-4) & \text{if } d = 3, \\ (2, 2, n-5) & \text{otherwise.} \end{cases}$$

Proof. Case 1 : The trees of diameter three are of the form $T(1, s, t)$ and $L(T(1, s, t))$ is formed by the coalescence of K_{s+1} and K_{t+1} . Using the labeling shown in Figure 2, we obtain

$$\mathcal{E}(L(T)) = \begin{bmatrix} 0 & J_{1 \times s} & J_{1 \times t} \\ J_{s \times 1} & 0_{s \times s} & 2J_{s \times t} \\ J_{t \times 1} & 2J_{t \times s} & 0_{t \times t} \end{bmatrix}$$

The rank of $\mathcal{E}(L(T))$ is three. The quotient matrix of $\mathcal{E}(L(T))$ is $\begin{bmatrix} 0 & s & t \\ 1 & 0 & 2t \\ 1 & 2s & 0 \end{bmatrix}$. Its determinant is positive. Therefore, the inertia of $\mathcal{E}(L(T))$ is $(1, 2, n - 4)$.

Case 2 : As the diameter of T is odd, T has two adjacent vertices in the center. Observe that $L(T)$ has a unique central cut vertex, say v , and that $L(T) - v$ has exactly two components, G_1 and G_2 . Define

$$V_1 = \left\{ u' \in G_1 : d(u', v) = \frac{d-1}{2} \right\}, V_2 = \left\{ v' \in G_2 : d(u', v) = \frac{d-1}{2} \right\},$$

$$V_3 = \left\{ u' \in G_1 : 0 \leq d(u', v) < \frac{d-1}{2} \right\}, V_4 = \left\{ v' \in G_2 : 0 \leq d(v', v) < \frac{d-1}{2} \right\}$$

Then $V(L(T)) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup \{v\}$ and $\{V_1, V_2, V_3, V_4, v\}$ is a partition of $V(L(T))$. It follows that

$$\mathcal{E}(L(T)) = \begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & v \\ \hline V_1 & 0 & (d-1)J & 0 & P & \frac{d-1}{2} \\ V_2 & (d-1)J & 0 & Q & 0 & \frac{d-1}{2} \\ V_3 & 0 & Q^T & 0 & 0 & 0 \\ V_4 & P^T & 0 & 0 & 0 & 0 \\ v & \frac{d-1}{2} & \frac{d-1}{2} & 0 & 0 & 0 \end{array},$$

where J denotes the matrix with all entries one. One can see that all the rows of P and Q are identical. For $i = 1, 2$ the rows corresponding to every vertex in V_i are identical, so $\text{rank}(R[V_1]) = \text{rank}(R[V_2]) = 1$, where $R[V_i]$ denotes the rows corresponding to vertices in V_i . Thus $\text{rank}(R[V_3]) = \text{rank}(R[V_4]) = 1$. Pick vertices $v_3 \in V_3$ and $v_4 \in V_4$ such that $d(v_3, v) = d(v_4, v) = 1$, $\mathcal{E}(v_3) = \mathcal{E}(v_4) = \frac{d-1}{2} + 1$. Note that $R(v) = \frac{d-1}{d+1}(R(v_3) + R(v_4))$. This implies that the rank of $\mathcal{E}(L(T))$ is at most four. For $i = 1, 2, 3, 4$, choose $v_i \in V_i$. The principal submatrix Q of $\mathcal{E}(L(T))$ indexed by v_1, v_2, v_3, v_4 is as follows:

$$Q = \begin{bmatrix} 0 & d-1 & 0 & \frac{d-1}{2} + 1 \\ d-1 & 0 & \frac{d-1}{2} + 1 & 0 \\ 0 & \frac{d-1}{2} + 1 & 0 & 0 \\ \frac{d-1}{2} + 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\det(Q) \neq 0$, we have $\text{rank}(\mathcal{E}(L(T))) = 4$. By inspection, Q has two positive and two negative eigenvalues. By the interlacing theorem, $\mathcal{E}(L(T))$ has at least two positive and two negative eigenvalues. Consequently, the inertia of $\mathcal{E}(L(T))$ is $(2, 2, n - 5)$. $\square$

In the following result, we determine the inertia of the eccentricity matrix of the line graph of a tree with even diameter. To facilitate the proof, we also provide an illustrative example following the theorem.

Theorem 6. *Let T be a tree on n vertices with even diameter $d(\geq 4)$. Then the inertia of $\mathcal{E}(L(T))$ is $(l, l, n - 2l - 1)$, where $l = |D(T)|$.*

Proof. Since T has even diameter, it has a unique central vertex z . Let $t = \deg(z)$. It follows that $L(T)$ has t central vertices. Let $Q_z = \{v_1, v_2, \dots, v_t\}$ denote the set of central vertices of $L(T)$, and let $L(T)[Q_z]$ denote the subgraph induced by Q_z . Let $G_1, G_2, \dots, G_t$ be the components obtained from $L(T)$ by deleting all edges joining pairs of vertices in Q_z . The resulting graph has t components. For an edge $e \in E(T)$, define $\text{depth}(e)$ as the distance from z to the endpoint of e that is farther from z . Then the maximum branch depth of $L(T)$ from Q_z is equal to

$$\max_{e \in E(T)} d(v_i, e) = \frac{d}{2} - 1.$$

Suppose that exactly l of the vertices $v_1, v_2, \dots, v_t$ correspond to branches of maximum depth; after relabeling, assume that these are $v_1, v_2, \dots, v_l$. We now distinguish two cases.

Case 1 : Let $l \neq t$. We obtain the following partition of $V(L(T)) : V(L(T)) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_{2l+1}$, where

$$V_i = \left\{ u \in G_i \mid d(u, v_i) = \frac{d-2}{2} \right\}$$

$$V_{i+l} = \left\{ u \in G_i \mid 0 \leq d(u, v_i) < \frac{d-2}{2} \right\},$$

where $1 \leq i \leq l$ and V_{2l+1} contains the vertices of the components $G_{l+1}, G_{l+2}, \dots, G_t$. Therefore the eccentricity matrix of $L(T)$ can be partitioned as

	V_1	V_2	$\dots$	V_l	V_{l+1}	V_{l+2}	$\dots$	V_{2l}	V_{2l+1}
V_1	0	$(d-1)J$	$\dots$	$(d-1)J$	0	$P_{1,l+2}$	$\dots$	$P_{1,2l}$	$P_{1,2l+1}$
V_2	$(d-1)J$	0	$\dots$	$(d-1)J$	$P_{2,l+1}$	0	$\dots$	$P_{2,2l}$	$P_{2,2l+1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
V_l	$(d-1)J$	$(d-1)J$	$\dots$	0	$P_{l,l+1}$	$P_{l,l+2}$	$\dots$	0	$P_{l,2l+1}$
V_{l+1}	0	$P_{2,l+1}^T$	$\dots$	$P_{l,l+1}^T$	0	0	$\dots$	0	0
V_{l+2}	$P_{1,l+2}^T$	0	$\dots$	$P_{l,l+2}^T$	0	0	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
V_{2l}	$P_{1,2l}^T$	$P_{2,2l}^T$	$\dots$	0	0	0	$\dots$	0	0
V_{2l+1}	$P_{1,2l+1}^T$	$P_{2,2l+1}^T$	$\dots$	$P_{l,2l+1}^T$	0	0	$\dots$	0	0

where the rows of each matrix $P_{i,l+j}$ with $i \neq j$ and $(i, j) \in \{1, 2, \dots, l\} \times \{1, 2, \dots, l+1\}$ are proportional and nonzero. For $1 \leq i \leq l$, the rows corresponding to vertices in V_i are identical, thus $\text{rank}(R[V_i]) = 1$. Let $d = 2r$. For $1 \leq i \leq l$, choose $z_i \in V_{l+i}$

at distance $r - 2$ from v_i (equivalently, $e(z_i) = d - 2$). If $u \in V_{l+i}$ and $q = d(u, v_i)$, $0 \leq q \leq r - 2$, then one obtains

$$R(u) = \frac{q + r}{2r - 2} R(z_i).$$

Thus the rows belonging to V_{l+i} span a one-dimensional space. Similarly, let v belong to one of the components that does not attain maximum branch depth, and put $a = d(v, v_j)$ for the corresponding central vertex v_j . Then

$$R(v) = \frac{a + r}{(l - 1)(2r - 2)} \sum_{i=1}^l R(z_i).$$

Equivalently, in terms of d ,

$$R(v) = \frac{a + d/2}{(l - 1)(d - 2)} \sum_{i=1}^l R(z_i).$$

Therefore $\text{rank}(\mathcal{E}(L(T))) \leq 2l$. The principal submatrix indexed by suitable vertices from the V_i and V_{l+i} has the form:

$$Q = \begin{bmatrix} (d-1)(J-I) & (d-2)(J-I) \\ (d-2)(J-I) & 0 \end{bmatrix}.$$

By Lemma 2, its inertia is $(l, l, 0)$ and $\text{rank}(Q) = 2l$. Therefore $\text{rank}(\mathcal{E}(L(T))) = 2l$. By the interlacing theorem, $\mathcal{E}(L(T))$ has at least l positive and l negative eigenvalues. As a result, inertia of $\mathcal{E}(L(T))$ is $(l, l, n - 1 - 2l)$.

Case 2 : Let $l = t$. We obtain the following partition $V(L(T)) : V(L(T)) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_{2l}$, where V_i and V_{i+l} , $1 \leq i \leq l$, are defined as in Case 1. Therefore the eccentricity matrix of $L(T)$ can be partitioned as

	V_1	V_2	$\dots$	V_l	V_{l+1}	V_{l+2}	$\dots$	V_{2l}
V_1	0	$(d-1)J$	$\dots$	$(d-1)J$	0	$P_{1,l+2}$	$\dots$	$P_{1,2l}$
V_2	$(d-1)J$	0	$\dots$	$(d-1)J$	$P_{2,l+1}$	0	$\dots$	$P_{2,2l}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_l	$(d-1)J$	$(d-1)J$	$\dots$	0	$P_{l,l+1}$	$P_{l,l+2}$	$\dots$	0
V_{l+1}	0	$P_{2,l+1}^T$	$\dots$	$P_{l,l+1}^T$	0	0	$\dots$	0
V_{l+2}	$P_{1,l+2}^T$	0	$\dots$	$P_{l,l+2}^T$	0	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_{2l}	$P_{1,2l}^T$	$P_{2,2l}^T$	$\dots$	0	0	0	$\dots$	0

where the rows of each matrix $P_{i,l+j}$ with $i \neq j$ and $(i, j) \in \{1, 2, \dots, l\} \times \{1, 2, \dots, l\}$ are proportional and nonzero. For $1 \leq i \leq l$, the rows corresponding to vertices in V_i are identical, whereas for V_{l+i} they are proportional (as in Case 1). Thus $\text{rank}(R[V_i]) = \text{rank}(R[V_{l+i}]) = 1$. Therefore $\text{rank}(\mathcal{E}(L(T))) \leq 2l$. The remainder of the proof is the same as in Case 1. $\square$

Example 1. 1. In Figure 3a, T_1 has a central vertex v of degree four. Therefore, $L(T_1)$ has four central vertices. Using the vertex labelling in Figure 3b, we have $Q_v = \{6, 7, 8, 9\}$. As in the proof of Theorem 6 we remove all the edges between vertices $\{6, 7, 8, 9\}$ and obtain components G_1, G_2, G_3, G_4 , where G_1, G_2, G_3 , and G_4 are the subgraphs induced by vertices $\{1, 2, 6\}$, $\{3, 7\}$, $\{4, 5, 8\}$ and $\{9\}$ respectively. Here $l = 3 \neq t = 4$. The partition used in the proof of Theorem 6 is

$$V_1 = \{1, 2\}, V_2 = \{3\}, V_3 = \{4, 5\}, V_4 = \{6\}, V_5 = \{7\}, V_6 = \{8\}, V_7 = \{9\}.$$

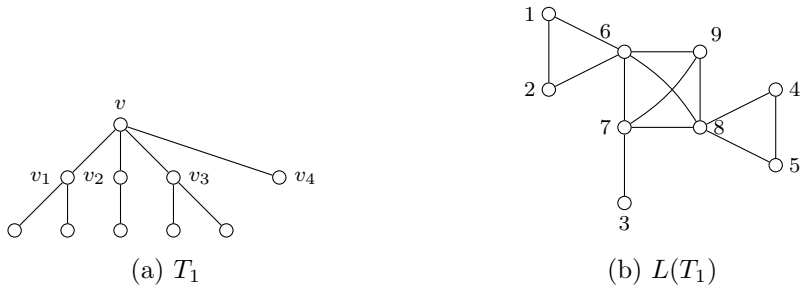


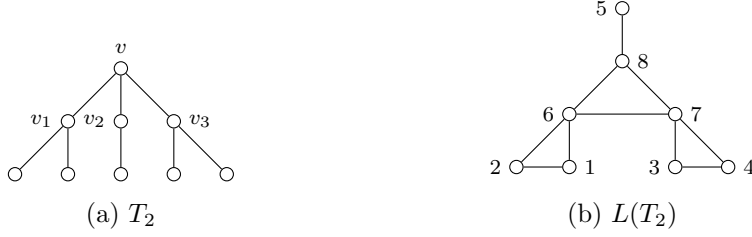
Figure 3: A tree T_1 and its corresponding $L(T_1)$

2. In Figure 4a, T_2 has a central vertex v of degree three. Therefore $L(T_2)$ has three vertices in the center. Using the vertex labelling in Figure 4b, we have $Q_v = \{6, 7, 8\}$. As in the proof of Theorem 6 we remove all the edges between vertices $\{6, 7, 8\}$ and obtain components G_1, G_2, G_3 , where G_1, G_2 , and G_3 are the subgraphs induced by vertices $\{1, 2, 6\}$, $\{3, 4, 7\}$, and $\{5, 8\}$, respectively. Here $l = t = 3$. The partition used in the proof of Theorem 6 is

$$V_1 = \{1, 2\}, V_2 = \{3, 4\}, V_3 = \{5\}, V_4 = \{6\}, V_5 = \{7\}, V_6 = \{8\}.$$

Corollary 1. *Let T be a tree on n vertices. Then $\mathcal{E}(L(T))$ is nonsingular if and only if T has one of the following forms:*

- $T = K_{1,m}$, with $m \geq 2$
- $T = P_4$
- T is obtained from $K_{1,m}$, with $m \geq 2$ by subdividing every edge exactly once.

Figure 4: A tree T_2 and its corresponding $L(T_2)$

Proof. We divide the proof into three cases:

Case 1 : If T is a tree of diameter two, then the inertia of $\mathcal{E}(L(T)) = (1, n - 2, 0)$.

Case 2 : Suppose that T has odd diameter. By Theorem 5, $\mathcal{E}(L(T))$ is nonsingular if and only if $T = P_4$.

Case 3 : Let T be a tree of even diameter $d = 2r \geq 4$ and suppose that $\mathcal{E}(L(T))$ is nonsingular. By Theorem 6 and the nonsingularity of $\mathcal{E}(L(T))$, we have $n - 1 = 2l$. Hence $n = 2l + 1$. Each of the l diametrically distinguished branches contains at least r vertices different from the center. Therefore

$$n \geq 1 + lr.$$

Combining this with $n = 2l + 1$ yields

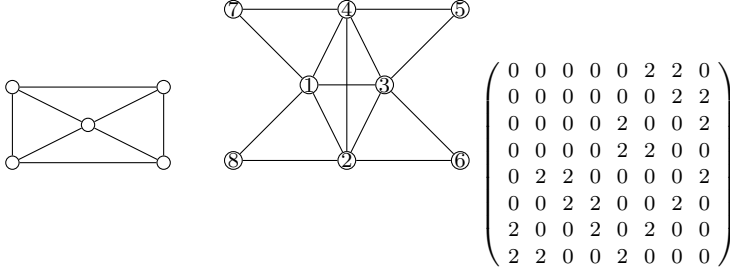
$$2l + 1 \geq 1 + lr,$$

and hence $r \leq 2$. Since $r \geq 2$, it follows that $r = 2$. Moreover, equality must hold throughout. Consequently, there can be no additional short branches or additional vertices in the distinguished branches. Each distinguished branch consists of exactly two edges from the center. Therefore T is precisely a star in which every edge has been subdivided once.

Conversely, let T be a tree obtained from $K_{1,m}$, with $m \geq 2$ by subdividing every edge exactly once. Then

$$\mathcal{E}(L(T)) = \begin{bmatrix} 0_k & 2S \\ 2S & 3S \end{bmatrix},$$

where $S = J_k - I_k$. By Lemma 1, $\det(\mathcal{E}(L(T))) = (-1)^k \det(4S^2) = (-4)^k (k - 1)^2$. This shows $\mathcal{E}(L(T))$ is nonsingular. $\square$

Figure 5: W_5 , $L(W_5)$ and $\mathcal{E}(L(W_5))$

4. Eccentricity matrices of line graphs of wheel graphs

Let C_n be a cycle graph with n vertices. The wheel graph W_{n+1} is the join of a single vertex and the cycle C_n . The eccentricity matrix of the line graph of a wheel graph for $n = 3, 4, 5$ and $n \geq 6$ is equal to $\begin{pmatrix} 0_3 & B \\ B^T & 0_3 \end{pmatrix}$, $\begin{pmatrix} 0_4 & B \\ B^T & \mathcal{E}(C_4) \end{pmatrix}$, $\begin{pmatrix} 0_5 & B \\ B^T & 2A(C_5) \end{pmatrix}$ and $\begin{pmatrix} 0_n & B \\ B^T & C \end{pmatrix}$, where $\mathcal{E}(C_4) = \begin{pmatrix} 0 & 2I_2 \\ 2I_2 & 0 \end{pmatrix}$, B and C are circulant matrices with first row $(0, \underbrace{2, 2, \dots, 2}_{n-2}, 0)$ and $(0, 0, 0, \underbrace{3, 3, \dots, 3}_{n-5}, 0, 0)$, respectively.

4.1. Irreducibility of $\mathcal{E}(L(W_{n+1}))$

The matrix $\mathcal{E}(L(W_4))$ is reducible because its associated graph is disconnected. For $n \geq 4$, the matrix $\mathcal{E}(L(W_{n+1}))$ is irreducible, as shown in the following proposition.

Proposition 1. *The eccentricity matrix of $L(W_{n+1})$ for $n \geq 4$ is irreducible.*

Proof. The graphs associated with $\mathcal{E}(L(W_5))$ and $\mathcal{E}(L(W_6))$ are connected. Therefore, both matrices are irreducible. For $n \geq 6$,

$$(I + \mathcal{E}(L(W_{n+1})))^2 = \begin{pmatrix} I_n + B \cdot B^T & 2B + BC \\ 2B^T + CB^T & B^T B + (C + I_n)^2 \end{pmatrix}.$$

As B is circulant, BB^T is circulant with first row

$$(4(n-2), 4(n-3), \underbrace{4(n-4), \dots, 4(n-4)}_{n-3}, 4(n-3)),$$

Therefore each entry of $I_n + B \cdot B^T$ and $B^T B + (C + I_n)^2$ is positive. It is easy to check that each entry of $2B + BC$ and $2B^T + B^T C$ is positive. The result follows from Theorem 3. $\square$

4.2. Eigenvalues of $\mathcal{E}(L(W_{n+1}))$

In this subsection, we explicitly determine the spectrum of $\mathcal{E}(L(W_{n+1}))$. The following table displays the eccentricity spectrum for $n = 3, 4, 5$. The next theorem gives the spectrum for $n \geq 6$. We relate its spectrum to the Dirichlet kernel $D_N(x)$ for $N = 2$, where

$$D_N(x) = 1 + 2 \sum_{k=1}^N \cos(kx) = \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})} \text{ for } x \notin 2\pi\mathbb{Z}.$$

The Dirichlet kernels $D_N, N \geq 0$, form a family of 2π -periodic trigonometric polynomials used to represent the partial sums of a Fourier series via convolution.

W_{n+1}	Spectrum of $\mathcal{E}(L(W_{n+1}))$
W_4	$\begin{Bmatrix} -2 & 2 \\ 3 & 3 \end{Bmatrix}$
W_5	$\begin{Bmatrix} 0 & -4 & 2 & 1 \pm \sqrt{17} \\ 1 & 2 & 3 & 1 \end{Bmatrix}$
W_6	$\begin{Bmatrix} 2 & 2 \pm \sqrt{40} & -3 \pm \sqrt{5} \\ 4 & 1 & 2 \end{Bmatrix}$

Table 1

Theorem 7. *Let $n \geq 6$. Then the eigenvalues of $\mathcal{E}(L(W_{n+1}))$ are as follows:*

$$\left\{ \begin{array}{l} \begin{Bmatrix} \frac{3(n-5) \pm \sqrt{9(n-5)^2 + 16(n-2)^2}}{2} & \frac{-3D_2(\frac{2\pi j}{n}) \pm \sqrt{9[D_2(\frac{2\pi j}{n})]^2 + 64 \cos^2(\frac{\pi j}{n})}}{2} \\ 1 & 2 \end{Bmatrix} & \text{if } n \text{ is odd and } j = 1, 2, \dots, \frac{n-1}{2} \\ \begin{Bmatrix} \frac{3(n-5) \pm \sqrt{9(n-5)^2 + 16(n-2)^2}}{2} & 0 & -3 & \frac{-3D_2(\frac{2\pi j}{n}) \pm \sqrt{9[D_2(\frac{2\pi j}{n})]^2 + 64 \cos^2(\frac{\pi j}{n})}}{2} \\ 1 & 1 & 1 & 2 \end{Bmatrix} & \text{if } n \text{ is even and } j = 1, 2, \dots, \frac{n-2}{2}. \end{array} \right.$$

Proof. The Characteristic polynomial of $\mathcal{E}(L(W_{n+1}))$ is $\phi(\lambda) = \det(\lambda I - \mathcal{E}(L(W_{n+1})))$. Hence the eigenvalues are the solutions of $\phi(\lambda) = 0$. For $\lambda \neq 0$, the Schur complement gives

$$\phi(\lambda) = \det \begin{pmatrix} \lambda I_n & -B \\ -B^T & \lambda I_n - C \end{pmatrix} = \det(\lambda^2 I_n - \lambda C - B^T B).$$

Since B is circulant, $B^T B = B B^T$. Thus equivalently,

$$\phi(\lambda) = (-1)^n \det(\lambda(C - \lambda I_n) + B B^T).$$

Clearly $\lambda(C - \lambda I_n) + B B^T$ is circulant with the following first row

$$(-\lambda^2 + 4(n-2), 4(n-3), 4(n-4), \underbrace{4(n-4) + 3\lambda, 4(n-4) + 3\lambda, \dots, 4(n-4) + 3\lambda}_{n-5}, 4(n-4), 4(n-3)).$$

By Theorem 2, $\phi(\lambda) = \prod_{j=0}^{n-1} f(\omega_j) = 0$, where $\omega_j = e^{\frac{2\pi i j}{n}}$ are the n^{th} roots of unity for $j = 0, 1, 2, \dots, n-1$. A direct computation shows that

$$f(\omega_0) = c_0 + c_1 + \dots + c_{n-1} = -\lambda^2 + 3(n-5)\lambda + 4(n-2)^2$$

$$\begin{aligned} f(\omega_j) &= -\lambda^2 + 4(n-2) + 4(n-3)\omega_j + 4(n-4)\omega_j^2 + (4(n-4) + 3\lambda) \sum_{k=3}^{n-3} \omega_j^k + 4(n-4)\omega_j^{n-2} + 4(n-3)\omega_j^{n-1} \\ &= -\lambda^2 + 4(n-2) + 4(n-3)(\omega_j + \omega_j^{n-1}) + 4(n-4)(\omega_j^2 + \omega_j^{n-2}) - (4(n-4) + 3\lambda) \left(\sum_{k=0}^2 \omega_j^k + \sum_{k=n-2}^{n-1} \omega_j^{n-2} \right) \\ &= -\lambda^2 - 3D_2 \left(\frac{2\pi j}{n} \right) \lambda + 16\cos^2 \left(\frac{\pi j}{n} \right), \end{aligned}$$

where $D_2 \left(\frac{2\pi j}{n} \right) = 1 + 2 \left(\cos \frac{2\pi j}{n} + \cos \frac{4\pi j}{n} \right)$. As $f(\omega_j) = f(\overline{\omega_j})$, the corresponding eigenvalues have multiplicity two. Solving $f(w_j) = 0$ yields the stated eigenvalues. $\square$

Corollary 2. *Let $n \geq 6$. Then the inertia of $\mathcal{E}(L(W_{n+1}))$ is given by*

$$\begin{cases} (n, n, 0) & \text{if } n \text{ is odd,} \\ (n-1, n, 1) & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let n be odd. Clearly $\frac{3(n-5) + \sqrt{9(n-5)^2 + 16(n-2)^2}}{2}$ is positive, and $\frac{3(n-5) - \sqrt{9(n-5)^2 + 16(n-2)^2}}{2}$ is negative, as $9(n-5)^2 < 9(n-5)^2 + 16(n-2)^2$. For convenience, set $s = D_2 \left(\frac{2\pi j}{n} \right)$. Suppose $\lambda_j = \frac{-3s + \sqrt{9s^2 + 64\cos^2 \left(\frac{\pi j}{n} \right)}}{2} \leq 0$, which implies $64\cos^2 \left(\frac{\pi j}{n} \right) \leq 0$, a contradiction. Similarly, one obtains $\frac{-3s - \sqrt{9s^2 + 64\cos^2 \left(\frac{\pi j}{n} \right)}}{2} \leq 0$. Each index $j \in \{1, 2, \dots, \frac{n-1}{2}\}$ contributes one positive and one negative eigenvalue, each with multiplicity two. Since there are $\frac{n-1}{2}$ such indices, we obtain $n-1$ positive and $n-1$ negative eigenvalues. Therefore, the inertia of the matrix is $(n, n, 0)$. The argument for even n is analogous. For $j = \frac{n}{2}$, the corresponding eigenvalues are 0 and -3 . Hence the inertia is $(n-1, n, 1)$. $\square$

4.3. Determinant of $\mathcal{E}(L(W_{n+1}))$

Theorem 8. *Let $n \geq 3$. Then*

$$\det(\mathcal{E}(L(W_{n+1}))) = \begin{cases} -2^{2n}(n-2)^2, & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is even, the determinant is zero by Corollary 2. If n is odd, then $\mathcal{E}(L(W_{n+1})) = \begin{bmatrix} 0 & B \\ B^T & D \end{bmatrix}$ is a $2n \times 2n$ matrix. By Lemma 1, $\det(\mathcal{E}(L(W_{n+1}))) = (-1)^n \det(B.B^T)$. As shown above, $B.B^T$ is a circulant matrix with first row

$$(4(n-2), 4(n-3), \underbrace{4(n-4), \dots, 4(n-4)}_{n-3}, 4(n-3)).$$

By Theorem 2,

$$\begin{aligned} \det(B.B^T) &= \prod_{j=0}^{n-1} 4(n-2) + 4(n-3)(\omega_j + \omega_j^{n-1}) + 4(n-4)(\omega_j^2 + \omega_j^3 + \dots + \omega_j^{n-2}) \\ &= 4(n^2 - 4n + 4) \prod_{j=1}^{n-1} (4(n-2) + 4(n-3)(\omega_j + \omega_j^{n-1}) + 4(n-4)(-1 - \omega_j - \omega_j^{n-1})) \\ &= 4(n^2 - 4n + 4) \prod_{j=1}^{n-1} (8 + 4(\omega_j + \omega_j^{n-1})) \\ &= 4(n^2 - 4n + 4) \prod_{j=1}^{n-1} \left(8 + 8\cos\frac{2\pi j}{n} \right) \\ &= 4^{\frac{n+1}{2}} . 8^{n-1} (n^2 - 4n + 4) \prod_{j=1}^{\frac{n-1}{2}} \left(\cos\frac{\pi j}{n} \right)^4 \end{aligned}$$

Since $\prod_{j=1}^{\frac{n-1}{2}} \left(\cos\frac{\pi j}{n} \right)^4 = \frac{1}{2^{2(n-1)}}$, it follows that $\det(B.B^T) = 2^{2n}(n-2)^2$. $\square$

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References

- [1] S. Balamoorthy, S.V. Bharanedhar, Y. Shang, and R. Gurusamy, *On the eccentric spectra of the line graph of starlike trees*, Discrete Dynamics in Nature and Society **2025** (2025), no. 1, 9972973.
<https://doi.org/10.1155/ddns/9972973>.
- [2] L.W. Beineke and J.S. Bagga, *Line Graphs and Line Digraphs*, Springer, 2021.

- [3] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012.
- [4] P.J. Davis, *Circulant Matrices*, American Mathematical Soc., 2013.
- [5] T. Divyadevi and I. Jeyaraman, *On the eccentricity matrices of Certain bi-block graphs*, Bull. Malays. Math. Sci. Soc. **47** (2024), no. 3, 92.
<https://doi.org/10.1007/s40840-024-01687-3>.
- [6] F. Harary, *Graph Theory*, Addison-Wesley, 1969.
- [7] E.V. Haynsworth, *Determination of the inertia of a partitioned Hermitian matrix*, Linear Algebra Appl. **1** (1968), no. 1, 73–81.
[https://doi.org/10.1016/0024-3795\(68\)90050-5](https://doi.org/10.1016/0024-3795(68)90050-5).
- [8] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 2012.
- [9] I. Jeyaraman and T. Divyadevi, *On eccentricity matrices of wheel graphs*, Indian J. Pure Appl. Math. **56** (2025), no. 4, 1507–1522.
<https://doi.org/10.1007/s13226-024-00717-x>.
- [10] X. Lei and J. Wang, *Spectral determination of graphs with one positive anti-adjacency eigenvalue*, Appl. Math. Comput. **422** (2022), 126995.
<https://doi.org/10.1016/j.amc.2022.126995>.
- [11] X.H. Li, J.F. Wang, and M. Brunetti, *Inertia and spectral symmetry for the eccentricity matrices of clique trees*, Acta Math. Appl. Sin. Engl. Ser. **42** (2026), no. 1, 23–38.
<https://doi.org/10.1007/s10255-024-1140-5>.
- [12] I. Mahato and M.R. Kannan, *On the eccentricity matrices of trees: Inertia and spectral symmetry*, Discrete Math. **345** (2022), no. 11, 113067.
<https://doi.org/10.1016/j.disc.2022.113067>.
- [13] ———, *Extremal problems for the eccentricity matrices of complements of trees*, Electron. J. Linear Algebra **39** (2023), 339–354.
<https://doi.org/10.13001/ela.2023.7781>.
- [14] S. Pandey, L. Selvaganesh, and J. Pervin, *Eccentricity matrix of corona of two graphs*, Discrete Appl. Math. **359** (2024), 354–363.
<https://doi.org/10.1016/j.dam.2024.08.017>.
- [15] A.K. Patel, L. Selvaganesh, and S.K. Pandey, *Energy and inertia of the eccentricity matrix of coalescence of graphs*, Discrete Math. **344** (2021), no. 12, 112591.
<https://doi.org/10.1016/j.disc.2021.112591>.
- [16] M. Randić, *DMAX-matrix of dominant distances in a graph*, MATCH Commun. Math. Comput. Chem. **70** (2013), no. 1, 221–238.
- [17] J.R. Sylvester, *Determinants of block matrices*, Math. Gaz. **84** (2000), no. 501, 460–467.
- [18] Y. Song, M. Brunetti, and J. Wang, *On graphs with third largest eccentricity eigenvalue in the interval $[-2, -1]$* , Discrete Appl. Math. **378** (2026), 671–681.
<https://doi.org/10.1016/j.dam.2025.09.027>.
- [19] Y. Song, Y. Li, M. Brunetti, and J. Wang, *On the third largest eigenvalue of eccentricity matrices of graphs*, Discrete Appl. Math. **372** (2025), 237–259.
<https://doi.org/10.1016/j.dam.2025.04.039>.
- [20] J. Wang, X. Lei, M. Lu, S. Sorgun, and H. Küçük, *On graphs with exactly one*

- anti-adjacency eigenvalue and beyond*, Discrete Math. **346** (2023), no. 6, 113373.
<https://doi.org/10.1016/j.disc.2023.113373>.
- [21] J. Wang, X. Lei, W. Wei, X. Luo, and S. Li, *On the eccentricity matrix of graphs and its applications to the boiling point of hydrocarbons*, Chemometrics and Intelligent Laboratory Systems **207** (2020), 104173.
<https://doi.org/10.1016/j.chemolab.2020.104173>.
- [22] J. Wang, M. Lu, F. Belardo, and M. Randić, *The anti-adjacency matrix of a graph: Eccentricity matrix*, Discrete Appl. Math. **251** (2018), 299–309.
<https://doi.org/10.1016/j.dam.2018.05.062>.
- [23] J. Wang, M. Lu, M. Brunetti, L. Lu, and X. Huang, *Spectral determinations and eccentricity matrix of graphs*, Adv. Appl. Math. **139** (2022), 102358.
<https://doi.org/10.1016/j.aam.2022.102358>.
- [24] J. Wang, M. Lu, L. Lu, and F. Belardo, *Spectral properties of the eccentricity matrix of graphs*, Discrete Appl. Math. **279** (2020), 168–177.
<https://doi.org/10.1016/j.dam.2019.10.015>.
- [25] J.F. Wang, X.Y. Lei, S.C. Li, and Z. Stanić, *The complete classification of graphs whose second largest eigenvalue of the eccentricity matrix is less than 1*, Acta Math. Sin. (Engl. Ser.) **40** (2024), no. 7, 1741–1766.
<https://doi.org/10.1007/s10114-024-2413-x>.
- [26] H. Whitney, *Congruent graphs and the connectivity of graphs*, Hassler Whitney Collected Papers, Springer, 1992, pp. 61–79.