

# On the graphs attaining the minimum for the multiplicative variable Euler–Sombor index

Mehdi Eliasi

Department of Mathematics, Khansar Faculty, University of Isfahan, Isfahan, Iran  
[m.eliasi@khc.ui.ac.ir](mailto:m.eliasi@khc.ui.ac.ir)

*Received: 27 July 2026; Accepted: 30 September 2026*

*Published Online: 8 October 2026*

**Abstract:** We introduce the multiplicative variable Euler–Sombor index,

$$\Pi_{\text{ES}_\lambda}(G) = \prod_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + \lambda d_u d_v},$$

where  $\lambda$  is a real parameter. For  $0 \leq \lambda < 2$ , we characterize the graphs that attain the minimum of this quantity. A key intermediate result asserts that any connected graph attaining the minimum with fixed order and size must satisfy  $\Delta(G) - \delta(G) \leq 1$ . This structural fact yields a complete description of the graphs that attain the minimum in several families: paths among trees, cycles among unicyclic graphs, and, for bicyclic graphs, the book and single-bridge dumbbell graphs. In the general setting of  $k$ -cyclic graphs (with  $k \geq 3$  and  $n \geq 5(k-1)$ ), the graphs that attain the minimum are precisely the bidegreed graphs with degree set  $\{2, 3\}$ , and the value they attain is exactly

$$(18 + 9\lambda)^{\frac{3k-4}{2}} (13 + 6\lambda)(8 + 4\lambda)^{\frac{n-2k+1}{2}}.$$

Our framework provides the first extremal characterization for the multiplicative Euler–Sombor index (the case  $\lambda = 1$ ), together with a complete description for all  $0 \leq \lambda < 2$ . This simultaneously extends the known results for the classical multiplicative Sombor index ( $\lambda = 0$ ) and introduces a new parametric family that has not been previously investigated.

**Keywords:** Sombor index, Euler–Sombor index, multiplicative topological index, graphs attaining the minimum,  $k$ -cyclic graphs.

**AMS Subject Classification:** 05C09, 05C35, 92E10

## 1. Introduction

A graph invariant is a mapping from a class of graphs into the real numbers that is preserved under isomorphism [5]. In mathematical chemistry, invariants of this kind (usually called topological indices) are used as numerical descriptors of the structure of a molecular graph, in which vertices stand for atoms and edges for chemical bonds. This approach has been central to QSAR/QSPR methodology since the 1970s [21, 22], where the goal is to correlate physicochemical, biological, or pharmacological properties with the graph structure of a molecule.

Among the degree-based indices introduced in the past decade, the one proposed by Gutman [8] stands out for its geometric motivation: for a graph  $G$ , taking the degree pair  $(d(u), d(v))$  of an edge  $uv$  as a point in the plane and measuring its Euclidean distance from the origin yields

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}.$$

Several variants of  $SO(G)$  have since been proposed. Gutman, Furtula and Oz [10] replaced the point  $(d(u), d(v))$  by an ellipse, arriving at the elliptic Sombor index

$$ESO(G) = \sum_{uv \in E(G)} (d(u) + d(v)) \sqrt{d(u)^2 + d(v)^2}.$$

Gutman [9] pursued a related but different idea, based this time on the perimeter of an ellipse. Using Euler’s 1773 approximation formula for this perimeter [7], he arrived at the Euler–Sombor index

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}.$$

For a comprehensive survey on extremal results and bounds for elliptic Sombor and Euler–Sombor indices, see [2]. Its properties were studied further in [9, 20]; in the latter paper the index was shown to correlate (with  $R^2$  above 0.9) with the acentric factor and the entropy of octane isomers.

Shortly thereafter, a one-parameter extension was proposed [11] by replacing the coefficient of the cross term  $d(u)d(v)$  with a real parameter  $\lambda \in [-2, \infty)$ :

$$EU(\lambda, G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + \lambda d(u)d(v)}.$$

The restriction  $\lambda \geq -2$  ensures that the radicand is nonnegative for all degree pairs. This single parameter unifies several classical indices:  $\lambda = 0$  recovers  $SO(G)$ ;  $\lambda = 1$  yields the Euler–Sombor index  $EU(G)$ ;  $\lambda = 2$  gives  $\sum_{uv \in E(G)} (d(u) + d(v))$ , the

first Zagreb index [12]; and  $\lambda = -2$  produces  $\sum_{uv \in E(G)} |d(u) - d(v)|$ , the Albertson irregularity index [1]. The properties of this index were investigated in [6] over the interval  $\lambda \in [-2, 2]$ , yielding a complete characterization of the extremal graphs for the subintervals  $[0, 2)$  and  $[-2, 0)$  by different analytical methods. Gutman et al. [11] obtained bounds for several graph products; a geometric reading of the case  $\lambda \in [-2, 2]$ , via the law of cosines, was given later by Nyauli and Buragohain [19]. Alongside these additive constructions, multiplicative topological indices have developed into their own line of research. Kulli [15] defined the multiplicative Sombor index

$$\Pi_{SO}(G) = \prod_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}.$$

In many graph classes the structures attaining the minimum and maximum for  $\Pi_{SO}$  turn out to coincide with those for  $SO$  [4, 17]: the path  $P_n$  and the complete graph  $K_n$  are, respectively, the graphs attaining the minimum and the maximum among all graphs of given order, the star  $S_n$  attains the maximum among trees, and similar patterns persist for bipartite and unicyclic graphs. One place where this parallel breaks down, however, is the case of graphs with prescribed chromatic number: there the problem of identifying the graphs attaining the minimum has essentially only been settled for the additive index, and the multiplicative counterpart remains open.

The multiplicative Sombor index has been computed for specific nanostructures such as nanotubes [15]. More recently, its behaviour has been examined under a fixed chromatic number [14], in terms of sharp inequalities involving order, minimum degree and radius [3], and with respect to standard graph products [13]. Kulli [16] also introduced multiplicative analogues of the elliptic and modified elliptic Sombor indices, which suggests that the multiplicative framework can accommodate essentially any additive construction of this type.

What is missing from this picture is a parametric multiplicative index in the spirit of  $EU(\lambda, G)$ . To our knowledge, no multiplicative version of the variable Euler–Sombor index has been considered so far, and the results available for multiplicative indices are, almost without exception, proved index by index, with little in the way of a common method. We take this as our starting point and introduce the *multiplicative variable Euler–Sombor index*

$$\Pi_{ES_\lambda}(G) = \prod_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + \lambda d_u d_v}, \quad \lambda \in [-2, \infty).$$

The restriction  $\lambda \geq -2$  ensures that the radicand is nonnegative for all degree pairs, as in the additive case. Its kernel,

$$\varphi_\lambda(x, y) = \sqrt{x^2 + y^2 + \lambda xy},$$

contains a cross term  $\lambda xy$  that is absent from the classical multiplicative Sombor kernel, and it is precisely this term that makes the analysis of the graphs attaining the minimum considerably harder than in the case  $\lambda = 0$ .

The present paper is not simply a routine adaptation of the arguments used for  $\Pi_{SO}$ . Working directly with  $\varphi_\lambda$ , we obtain a sharp monotonicity statement together with a new difference inequality for  $\varphi_\lambda$ , strong enough to control edge-switching operations in situations where earlier, cruder estimates were not sufficient. Using these properties we are able to apply a general lemma of Liu et al. [18] and conclude that every graph attaining the minimum must be almost regular; however, checking that the hypotheses of that lemma actually hold for  $\varphi_\lambda$ , and carrying the resulting classification through to completion, requires arguments that are new to the multiplicative parametric setting. Rather than treating a single value of  $\lambda$ , we work over the whole range  $\lambda \in [0, 2)$  and track how the graphs attaining the minimum change with  $\lambda$ .

With this machinery we obtain a complete description of the graphs attaining the minimum of  $\Pi_{ES_\lambda}$  for  $\lambda \in [0, 2)$ . We show that every connected graph attaining the minimum of given order and size is almost regular, in the sense that  $\Delta(G) - \delta(G) \leq 1$ , and use this fact to determine the graphs attaining the minimum in several natural graph classes: the path  $P_n$  among trees, the cycle  $C_n$  among unicyclic graphs, and, among bicyclic graphs, the book graph together with the single-bridge dumbbell graph. For  $k$ -cyclic graphs with  $k \geq 3$  and  $n \geq 5(k - 1)$ , we show that the graphs attaining the minimum are exactly the bidegreed graphs with degree set  $\{2, 3\}$ , and we compute the corresponding minimum value in closed form:

$$\Pi_{ES_\lambda}(G) = (18 + 9\lambda)^{\frac{3k-4}{2}} (13 + 6\lambda)(8 + 4\lambda)^{\frac{n-2k+1}{2}}.$$

Beyond the classification itself, the analysis of  $\varphi_\lambda$  seems to be of some independent interest. The difference inequality we prove is sharp, with an explicit equality condition, and we expect the same estimate to be useful for other multiplicative indices built from similar two-variable kernels. Because our results are stated for a full range of  $\lambda$ , they also make it possible to compare the multiplicative and additive versions directly and to see how the interaction term  $\lambda d_u d_v$  shapes the structures that attain the minimum.

It should be stressed that, although the almost-regularity of graphs attaining the minimum is obtained via a general lemma taken from [18], everything else in the paper (the construction of the analytic framework around  $\varphi_\lambda$ , the verification that this lemma applies, the edge-switching arguments, and the full classification of graphs attaining the minimum) is developed here for the first time. In this sense our results extend the theory of multiplicative Sombor-type indices to a genuinely parametric setting.

The rest of the paper is organized as follows. Section 2 fixes notation and establishes the analytic properties of  $\varphi_\lambda$  that are used throughout. Section 3 contains the classification of graphs attaining the minimum among trees, unicyclic, bicyclic, and general  $k$ -cyclic graphs.

## 2. Preliminaries and notation

We begin by fixing the graph-theoretic terminology used throughout. All graphs under consideration are finite, undirected, and contain neither loops nor multiple edges. For a graph  $G$ , the degree of a vertex  $v \in V(G)$  is denoted by  $d_v$ ; the largest and smallest degrees occurring in  $G$  are denoted by  $\Delta(G)$  and  $\delta(G)$ , respectively. A graph is said to be *almost regular* whenever  $\Delta(G) - \delta(G) \leq 1$ .

A connected graph on  $n$  vertices with exactly  $n - 1 + k$  edges is called *k-cyclic*; its cyclomatic number equals  $k$ . Trees ( $k = 0$ ), unicyclic graphs ( $k = 1$ ), and bicyclic graphs ( $k = 2$ ) are the three smallest cases.

For a fixed parameter  $\lambda \in [0, 2)$ , we define the *multiplicative variable Euler–Sombor index* of a graph  $G$  as

$$\Pi_{\text{ES}_\lambda}(G) := \prod_{uv \in E(G)} \varphi_\lambda(d_u, d_v),$$

where the function

$$\varphi_\lambda(x, y) := \sqrt{x^2 + y^2 + \lambda xy}, \quad x, y \geq 0,$$

will be referred to as the *kernel* of the index. This kernel contains a cross term  $\lambda xy$  that distinguishes it from the kernel of the classical multiplicative Sombor index. We also set

$$\psi_\lambda(x, y) := \log \varphi_\lambda(x, y).$$

Throughout the preliminary results, all logarithmic expressions involving the kernel are written in terms of  $\psi_\lambda$ .

We shall use  $P_n$  and  $C_n$  to denote, respectively, the path graph and the cycle graph on  $n$  vertices.

## 3. Minimal graphs for $\lambda \in [0, 2)$

In this section we study the kernel  $\psi_\lambda$  for  $\lambda \in [0, 2)$ . Our aim is to determine the structure of all  $k$ -cyclic graphs for which the minimum of the index  $\Pi_{\text{ES}_\lambda}$  is attained. We begin with the following lemma, which provides the elementary properties of the kernel needed for this purpose.

**Theorem 1.** *Let  $\lambda \in [0, 2)$ . Then the function  $\psi_\lambda$  is strictly increasing in each variable for  $x, y \geq 1$ .*

*Proof.* Since the map  $t \mapsto \ln(\sqrt{t}) = \frac{1}{2} \ln(t)$  is strictly increasing for  $t > 0$ , it suffices to prove that

$$f(x, y) = x^2 + y^2 + \lambda xy$$

is strictly increasing in each variable.

First, fix  $y$  and let  $x_1 > x_2 \geq 1$ . Then

$$\begin{aligned} f(x_1, y) - f(x_2, y) &= (x_1^2 + y^2 + \lambda x_1 y) - (x_2^2 + y^2 + \lambda x_2 y) \\ &= (x_1^2 - x_2^2) + \lambda y(x_1 - x_2) \\ &= (x_1 - x_2)(x_1 + x_2 + \lambda y). \end{aligned}$$

Because  $x_1 - x_2 > 0$  and  $x_1 + x_2 + \lambda y > 0$  (since  $x_1 + x_2 \geq 2$  and  $\lambda y \geq 0$ ), we obtain

$$f(x_1, y) - f(x_2, y) > 0.$$

Hence  $\psi_\lambda(x_1, y) > \psi_\lambda(x_2, y)$ . Therefore,  $\psi_\lambda$  is strictly increasing in  $x$ .

By symmetry of  $f$  in  $x$  and  $y$ , the same argument holds for the variable  $y$ . Indeed, fixing  $x$  and taking  $y_1 > y_2 \geq 1$  gives

$$f(x, y_1) - f(x, y_2) = (y_1 - y_2)(y_1 + y_2 + \lambda x) > 0,$$

so  $\psi_\lambda(x, y_1) > \psi_\lambda(x, y_2)$ . Thus,  $\psi_\lambda$  is strictly increasing in each variable.  $\square$

**Lemma 1.** *For every  $a > b \geq 0$  and  $\lambda \in [0, 2)$ , the function*

$$x \longmapsto \psi_\lambda(a, x) - \psi_\lambda(b, x)$$

*is strictly decreasing on  $(0, \infty)$ .*

*Proof.* Set

$$g(x) = \psi_\lambda(a, x) - \psi_\lambda(b, x) = \frac{1}{2} \ln \left( \frac{a^2 + x^2 + \lambda ax}{b^2 + x^2 + \lambda bx} \right).$$

Differentiating with respect to  $x$ , we get

$$g'(x) = \frac{1}{2} \left( \frac{2x + \lambda a}{a^2 + x^2 + \lambda ax} - \frac{2x + \lambda b}{b^2 + x^2 + \lambda bx} \right).$$

After putting over a common denominator, the numerator becomes

$$\begin{aligned} &(2x + \lambda a)(b^2 + x^2 + \lambda bx) - (2x + \lambda b)(a^2 + x^2 + \lambda ax) \\ &= -(a - b)(\lambda x^2 + 2(a + b)x + \lambda ab). \end{aligned}$$

Since  $a > b \geq 0$  and  $x > 0$ , we have  $a - b > 0$  and

$$\lambda x^2 + 2(a + b)x + \lambda ab > 0.$$

Hence the numerator is strictly negative. The denominator

$$2(a^2 + x^2 + \lambda ax)(b^2 + x^2 + \lambda bx)$$

is positive for  $x > 0$  because both quadratic factors are positive. Therefore

$$g'(x) < 0 \quad \forall x > 0.$$

Thus  $g$  is strictly decreasing on  $(0, \infty)$ . □

**Lemma 2.** *Let  $A = 2 + \lambda$  with  $\lambda \in [0, 2)$ , and define for integers  $n \geq 2$*

$$A_n := \frac{n}{2} \ln \left( \frac{An^2}{An^2 - An + 1} \right), \quad B_n := \frac{n}{2} \ln \left( \frac{An^2 + An + 1}{An^2} \right).$$

*Then  $A_n > B_{n-1}$ .*

*Proof.* Set  $D_n := An^2 - An + 1$ . The inequality  $A_n > B_{n-1}$  is equivalent to

$$(An^2)^n (A(n-1)^2)^{n-1} > D_n^{2n-1}.$$

Taking the  $(2n-1)$ -th root, this becomes

$$G(A) := (An^2)^{\frac{n}{2n-1}} (A(n-1)^2)^{\frac{n-1}{2n-1}} > D_n.$$

We prove that  $G(A) > D_n$  for every  $A \geq 2$ . Consider

$$P(A) := \ln G(A) - \ln D_n = \frac{n}{2n-1} \ln(An^2) + \frac{n-1}{2n-1} \ln(A(n-1)^2) - \ln D_n.$$

A direct differentiation yields

$$\begin{aligned} P'(A) &= \frac{n}{2n-1} \frac{1}{A} + \frac{n-1}{2n-1} \frac{1}{A} - \frac{n^2-n}{D_n} \\ &= \frac{1}{A} - \frac{n(n-1)}{D_n} = \frac{D_n - An(n-1)}{AD_n} = \frac{1}{AD_n} > 0. \end{aligned}$$

Hence  $P$  is strictly increasing on  $[2, \infty)$ . Thus  $P(A) > P(2)$  for all  $A > 2$ . It remains to verify  $P(2) > 0$ .

At  $A = 2$ , we have  $D_n = 2n^2 - 2n + 1 = n^2 + (n-1)^2$ , and

$$G(2) = 2(n^2)^{\frac{n}{2n-1}} ((n-1)^2)^{\frac{n-1}{2n-1}}.$$

We need to show  $G(2) > n^2 + (n-1)^2$ . Divide both sides by  $(n-1)^2$  and put

$$x := \frac{n}{n-1} > 1, \quad d := \frac{1}{2n-1} > 0.$$

Then  $x^{2n/(2n-1)} = x^{1+d}$ , and the required inequality becomes

$$2x^{1+d} > x^2 + 1.$$

Let  $t := x - 1 = 1/(n-1) > 0$ . By Bernoulli's inequality,

$$(1+t)^{1+d} > 1 + (1+d)t.$$

Thus

$$2x^{1+d} > 2 + 2(1+d)t.$$

It is enough to show

$$2 + 2(1+d)t \geq 1 + 2t + t^2,$$

which is equivalent to

$$1 + 2dt - t^2 > 0.$$

Since  $d = 1/(2n-1)$  and  $t = 1/(n-1)$ , we have

$$1 + 2dt - t^2 = 1 + \frac{2}{(2n-1)(n-1)} - \frac{1}{(n-1)^2}.$$

For  $n = 2$ , this equals  $2/3 > 0$ . For  $n \geq 3$ , we have  $1 - \frac{1}{(n-1)^2} > 0$  and the middle term is positive; hence the expression is strictly positive. Therefore  $2x^{1+d} > x^2 + 1$ , so  $G(2) > D_n$ . Thus  $P(2) > 0$ , and consequently  $P(A) > 0$  for all  $A \geq 2$ . This proves  $G(A) > D_n$ , which is exactly  $A_n > B_{n-1}$ .  $\square$

**Lemma 3.** *The sequence  $(B_n)_{n \geq 1}$  is strictly increasing.*

*Proof.* Consider the real function

$$F(x) := \frac{x}{2} \ln \left( 1 + \frac{1}{x} + \frac{1}{Ax^2} \right), \quad x > 0.$$

Then  $B_n = F(n)$ . Put  $u = 1/x$  and define

$$H(u) := \ln \left( 1 + u + \frac{u^2}{A} \right) - \frac{u + \frac{2u^2}{A}}{1 + u + \frac{u^2}{A}}, \quad u > 0.$$

A direct calculation gives

$$H'(u) = \frac{u \left(1 - \frac{2}{A} + \frac{2u}{A} + \frac{2u^2}{A^2}\right)}{\left(1 + u + \frac{u^2}{A}\right)^2} > 0,$$

because for  $A \geq 2$  and  $u > 0$ : - If  $A = 2$ , then the bracket becomes  $u + \frac{u^2}{2} > 0$ . - If  $A > 2$ , then  $1 - \frac{2}{A} > 0$  and all other terms are non-negative, so the bracket is strictly positive. Thus  $H'(u) > 0$  for all  $u > 0$ . Since  $\lim_{u \rightarrow 0^+} H(u) = 0$ , we have  $H(u) > 0$  for  $u > 0$ . Therefore

$$F'(x) = \frac{1}{2}H(1/x) > 0,$$

so  $F$  is strictly increasing. Hence  $B_n = F(n)$  is strictly increasing.  $\square$

**Theorem 2.** For every  $\lambda \in [0, 2)$  and all integers  $a > b + 1 \geq 2$ ,

$$a[\psi_\lambda(a, a) - \psi_\lambda(a - 1, a)] > b[\psi_\lambda(b + 1, b) - \psi_\lambda(b, b)].$$

*Proof.* Since

$$\psi_\lambda(x, y) = \frac{1}{2} \ln(x^2 + y^2 + \lambda xy),$$

let  $A = 2 + \lambda$ . For  $n \geq 2$ ,

$$n[\psi_\lambda(n, n) - \psi_\lambda(n - 1, n)] = \frac{n}{2} \ln\left(\frac{An^2}{An^2 - An + 1}\right) = A_n,$$

and for  $m \geq 1$ ,

$$m[\psi_\lambda(m + 1, m) - \psi_\lambda(m, m)] = \frac{m}{2} \ln\left(\frac{Am^2 + Am + 1}{Am^2}\right) = B_m.$$

Now assume  $a > b + 1 \geq 2$ . Then  $a - 1 > b$ . By Lemma 2 with  $n = a$ ,

$$A_a > B_{a-1}.$$

Since  $B_n$  is strictly increasing (Lemma 3) and  $a - 1 > b$ , we have

$$B_{a-1} > B_b.$$

Combining the two inequalities yields  $A_a > B_b$ , which is precisely

$$a[\psi_\lambda(a, a) - \psi_\lambda(a - 1, a)] > b[\psi_\lambda(b + 1, b) - \psi_\lambda(b, b)].$$

This completes the proof.  $\square$

**Theorem 3.** *Let  $\lambda \in [0, 2)$ , and let  $G$  be a connected graph with  $n$  vertices and  $m$  edges. If  $G$  attains the minimum value of  $\Pi_{\text{ES}_\lambda}$  among all connected graphs with the same numbers of vertices and edges, then  $G$  is almost regular.*

*Proof.* Based on the results established above, the function  $\psi_\lambda$  satisfies the three required conditions of [18, Lemma 2.2]: condition (i) follows from the strict monotonicity proven in Theorem 1, condition (ii) follows from the decreasing difference property proven in Lemma 1, and condition (iii) follows from the switching inequality proven in Theorem 2.

Since the logarithm is strictly increasing on  $(0, \infty)$ , the problem of attaining the minimum of  $\Pi_{\text{ES}_\lambda}(G)$  is equivalent to attaining the minimum of its logarithm:

$$\ln \Pi_{\text{ES}_\lambda}(G) = \sum_{uv \in E(G)} \psi_\lambda(d_u, d_v).$$

By [18, Lemma 2.2], if a symmetric function  $I: \mathbb{Z}_{\geq 0}^2 \rightarrow \mathbb{R}$  satisfies conditions (i)–(iii), then any connected graph attaining the minimum of

$$TI(G) = \sum_{uv \in E(G)} I(d_u, d_v)$$

must satisfy  $\Delta(G) - \delta(G) \leq 1$  (that is, it must be almost regular).

Applying this lemma to the function  $I = \psi_\lambda$ , every connected graph attaining the minimum of  $\sum \psi_\lambda(d_u, d_v)$  is almost regular. Because of the logarithmic equivalence, the same property holds for the graphs attaining the minimum of  $\Pi_{\text{ES}_\lambda}$ . Hence  $G$  is almost regular.  $\square$

**Corollary 1.** *Let  $\lambda \in [0, 2)$  and  $n \geq 3$ . Among all trees of order  $n$ , the path  $P_n$  uniquely attains the minimum value of  $\Pi_{\text{ES}_\lambda}$ .*

*Proof.* Let  $T$  be a tree of order  $n$  that attains the minimum of  $\Pi_{\text{ES}_\lambda}$  over all trees on  $n$  vertices. Since every tree is connected and has exactly  $n - 1$  edges, Theorem 3 applies and yields that  $T$  is almost regular, i.e.,

$$\Delta(T) - \delta(T) \leq 1.$$

Every tree has at least two leaves, so  $\delta(T) = 1$ . The almost regularity condition then implies  $\Delta(T) \leq 2$ . Since  $T$  is connected and  $n \geq 3$ , we must have  $\Delta(T) = 2$ .

A connected graph with maximum degree at most 2 is either a path or a cycle. Because  $T$  is a tree, it cannot contain a cycle; hence  $T$  must be the path  $P_n$ . Therefore, the unique graph attaining the minimum is  $P_n$ .  $\square$

**Corollary 2.** *Let  $\lambda \in [0, 2)$  and  $n \geq 3$ . Among all unicyclic graphs of order  $n$ , the cycle  $C_n$  is the unique graph attaining the minimum value of  $\Pi_{\text{ES}_\lambda}$ .*

*Proof.* Let  $G$  be a unicyclic graph of order  $n$  that attains the minimum of  $\Pi_{\text{ES}_\lambda}$  over all unicyclic graphs on  $n$  vertices. Every unicyclic graph is connected and has exactly  $n$  edges, so Theorem 3 applies and yields

$$\Delta(G) - \delta(G) \leq 1.$$

We first show that  $\delta(G) \geq 2$ . Suppose, to the contrary, that  $\delta(G) = 1$ . Then almost regularity forces  $\Delta(G) \leq 2$ , so every vertex has degree 1 or 2. Let  $n_1$  be the number of vertices of degree 1. By the Handshaking Lemma,

$$\sum_{v \in V(G)} d_v = n_1 \cdot 1 + (n - n_1) \cdot 2 = 2n - n_1.$$

But for a unicyclic graph, the sum of degrees is  $2n$  (since it has  $n$  edges). Hence  $n_1 = 0$ , contradicting  $\delta(G) = 1$ . Therefore  $\delta(G) \geq 2$ .

Now, since  $\delta(G) \geq 2$  and  $\Delta(G) - \delta(G) \leq 1$ , all degrees must be equal to 2. Thus  $G$  is a 2-regular connected graph, which is precisely a cycle. Hence  $G \cong C_n$ .

Consequently,  $C_n$  is the unique graph attaining the minimum of  $\Pi_{\text{ES}_\lambda}$  among all unicyclic graphs of order  $n$ .  $\square$

**Theorem 4.** *Let  $\lambda \in [0, 2)$  and  $n \geq 4$ . Among all bicyclic graphs of order  $n$ , a graph  $G$  attains the minimum of  $\Pi_{\text{ES}_\lambda}$  if and only if  $G$  has exactly two vertices of degree 3, they are adjacent, and all remaining vertices have degree 2. The graphs with this property are precisely:*

- (i) *two cycles sharing a common edge (the “book” graph), existing for all  $n \geq 4$ ;*
- (ii) *two cycles joined by a single edge (a bridge of length 1) (the “dumbbell” graph), existing for  $n \geq 6$ .*

*In particular, for  $n \in \{4, 5\}$  the minimum is uniquely attained by type (i), while for  $n \geq 6$  both types attain the minimum simultaneously.*

*Proof.* By Theorem 3, any graph  $G$  attaining the minimum of  $\Pi_{\text{ES}_\lambda}$  among all connected graphs with the same number of vertices and edges is almost regular. Hence

$$\Delta(G) - \delta(G) \leq 1.$$

Every bicyclic graph on  $n$  vertices is connected and has exactly  $n + 1$  edges. Since all graphs in this class share the same number of edges, the above bound applies to every bicyclic graph  $G$  attaining the minimum.

Let  $G$  be a bicyclic graph attaining the minimum. The Handshaking Lemma gives. The Handshaking Lemma gives

$$\sum_{v \in V(G)} d_v = 2(n + 1) = 2n + 2.$$

We first determine the minimum degree. If  $\delta(G) \geq 3$ , then  $\sum d_v \geq 3n > 2n + 2$  for  $n \geq 4$ , impossible. If  $\delta(G) = 1$ , then almost regularity gives  $\Delta(G) \leq 2$ . Since  $G$  is connected and has at least one vertex of degree 1, we have

$$\sum_{v \in V(G)} d_v \leq 1 + 2(n-1) = 2n - 1 < 2n + 2,$$

a contradiction. Hence  $\delta(G) = 2$ , and therefore  $\Delta(G) \leq 3$ .

Let  $n_3$  be the number of vertices of degree 3. Since all degrees are 2 or 3, the degree sum gives

$$2(n - n_3) + 3n_3 = 2n + 2 \implies n_3 = 2.$$

Thus  $G$  has exactly two degree-3 vertices, say  $u$  and  $v$ , and all other vertices have degree 2.

We now classify the possible structures based on the adjacency of  $u$  and  $v$ .

Case 1:  $u$  and  $v$  are non-adjacent. In this case, there are two possible configurations. Either  $u$  and  $v$  are connected by three internally vertex-disjoint paths (a theta graph  $\Theta(a, b, c)$ ), or they lie on two cycles joined by a path of length at least 2 (a dumbbell graph with a bridge of length  $\geq 2$ ). In both configurations, the multiset of edge degree-pairs is the same:

$$(3, 3) : 0, \quad (3, 2) : 6, \quad (2, 2) : n - 5.$$

Indeed, since  $u$  and  $v$  are not adjacent, there is no  $(3, 3)$ -edge. Each of the two degree-3 vertices is incident to exactly three edges whose other endpoints have degree 2, giving  $3 + 3 = 6$  edges of type  $(3, 2)$ . The remaining  $n + 1 - 6 = n - 5$  edges are of type  $(2, 2)$ . Thus, for the purpose of comparing the index, these two configurations can be treated together as the non-adjacent case.

Case 2:  $u$  and  $v$  are adjacent. If the edge  $uv$  lies on two distinct cycles, we obtain the *book graph*. If  $uv$  is a bridge connecting the two cycles, we obtain the *dumbbell graph* (with a bridge of length exactly 1). In both subcases, the multiset of edge degree-pairs is identical:

$$(3, 3) : 1, \quad (3, 2) : 4, \quad (2, 2) : n - 4.$$

Let  $G_{\text{bd}}$  be a graph of book or dumbbell type, and let  $G_{\text{non-adj}}$  be a graph in the non-adjacent case above. Using the edge-pair counts,

$$\ln \Pi_{\text{ES}_\lambda}(G_{\text{bd}}) - \ln \Pi_{\text{ES}_\lambda}(G_{\text{non-adj}}) = \psi_\lambda(3, 3) + \psi_\lambda(2, 2) - 2\psi_\lambda(3, 2).$$

Since  $\psi_\lambda(x, y) = \frac{1}{2} \ln(x^2 + y^2 + \lambda xy)$ , we obtain

$$\ln \Pi_{\text{ES}_\lambda}(G_{\text{bd}}) - \ln \Pi_{\text{ES}_\lambda}(G_{\text{non-adj}}) = \frac{1}{2} \ln \left( \frac{(18 + 9\lambda)(8 + 4\lambda)}{(13 + 6\lambda)^2} \right).$$

Now

$$(18 + 9\lambda)(8 + 4\lambda) = 144 + 144\lambda + 36\lambda^2 = (12 + 6\lambda)^2,$$

so

$$\frac{(18 + 9\lambda)(8 + 4\lambda)}{(13 + 6\lambda)^2} = \frac{(12 + 6\lambda)^2}{(13 + 6\lambda)^2}.$$

For  $\lambda \in [0, 2)$ , we have  $12 + 6\lambda < 13 + 6\lambda$ , hence the fraction is strictly less than 1. Therefore

$$\ln \Pi_{\text{ES}_\lambda}(G_{\text{bd}}) - \ln \Pi_{\text{ES}_\lambda}(G_{\text{non-adj}}) < 0,$$

which means  $\Pi_{\text{ES}_\lambda}(G_{\text{bd}}) < \Pi_{\text{ES}_\lambda}(G_{\text{non-adj}})$ . Hence the non-adjacent case cannot be a graph attaining the minimum.

Consequently, the two degree-3 vertices must be adjacent, and the graphs attaining the minimum are exactly the book graph and the dumbbell graph with a single-edge bridge. The book graph exists for every  $n \geq 4$  (two cycles sharing an edge, with total order  $n$ ), while the single-edge dumbbell graph requires two cycles, each of length at least 3, plus the bridge edge, so it exists only for  $n \geq 6$ . For  $n = 4, 5$ , only the book graph is possible, giving a unique graph attaining the minimum; for  $n \geq 6$ , both types occur and attain the same minimum value.  $\square$

**Theorem 5.** *Let  $\lambda \in [0, 2)$ ,  $k \geq 3$ , and  $n \geq 5(k - 1)$ . Among all  $k$ -cyclic graphs of order  $n$ , a graph  $G$  attains the minimum value of  $\Pi_{\text{ES}_\lambda}(G)$  if and only if  $G$  is bidegreed with degrees in  $\{2, 3\}$  and the multiset of its edge degree-pairs consists of:*

- (i) two edges of type  $(2, 3)$ ;
- (ii)  $n - 2k + 1$  edges of type  $(2, 2)$ ;
- (iii)  $3k - 4$  edges of type  $(3, 3)$ .

The minimum value is

$$\Pi_{\text{ES}_\lambda}(G) = (\varphi_\lambda(3, 3))^{3k-4} (\varphi_\lambda(2, 3))^2 (\varphi_\lambda(2, 2))^{n-2k+1},$$

which, upon substituting  $\varphi_\lambda(x, y) = \sqrt{x^2 + y^2 + \lambda xy}$ , becomes

$$\Pi_{\text{ES}_\lambda}(G) = (18 + 9\lambda)^{\frac{3k-4}{2}} (13 + 6\lambda)(8 + 4\lambda)^{\frac{n-2k+1}{2}}.$$

*Proof.* As verified in the proof of Theorem 3, the function  $\psi_\lambda$  satisfies conditions (i)–(iii) of [18, Lemma 2.2] for every  $\lambda \in [0, 2)$ .

Since the natural logarithm is strictly increasing on  $(0, \infty)$ , attaining the minimum of the product  $\Pi_{\text{ES}_\lambda}(G)$  is equivalent to attaining the minimum of its logarithm:

$$\ln \Pi_{\text{ES}_\lambda}(G) = \sum_{uv \in E(G)} \ln(\varphi_\lambda(d_u, d_v)) = \sum_{uv \in E(G)} \psi_\lambda(d_u, d_v).$$

Thus, the graph structure (the degree sequence and the edge-pair multiplicities) that attains the minimum of the product is exactly the same as the one that attains the minimum of  $\sum \psi_\lambda$ .

Applying [18, Theorem 2.1] (which establishes the structure for the sum of  $\psi_\lambda$ ), we obtain that any graph  $G$  attaining the minimum must be bidegreed with degrees  $\{2, 3\}$ , containing exactly two  $(2, 3)$  edges,  $n - 2k + 1$  edges of type  $(2, 2)$ , and  $3k - 4$  edges of type  $(3, 3)$ .

Substituting these multiplicities into the product definition yields the stated minimum value:

$$\begin{aligned}\Pi_{\text{ES}_\lambda}(G) &= (\varphi_\lambda(3, 3))^{3k-4} (\varphi_\lambda(2, 3))^2 (\varphi_\lambda(2, 2))^{n-2k+1} \\ &= (\sqrt{18+9\lambda})^{3k-4} (\sqrt{13+6\lambda})^2 (\sqrt{8+4\lambda})^{n-2k+1} \\ &= (18+9\lambda)^{\frac{3k-4}{2}} (13+6\lambda)(8+4\lambda)^{\frac{n-2k+1}{2}}.\end{aligned}$$

This completes the proof. □

**Acknowledgements:** The author thanks the anonymous referees for their constructive comments.

**Conflict of Interest:** The author declares that he has no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## References

- [1] M.O. Albertson, *The irregularity of a graph*, Ars Combin. **46** (1997), 219–226.
- [2] A. Ali, I. Gutman, T. Réti, A.M. Albalahi, and A.E. Hamza, *Survey on extremal results and bounds for elliptic Sombor and Euler–Sombor indices*, Discrete Math. Lett. **17** (2026), 101–112.  
<https://doi.org/10.47443/dml.2026.056>.
- [3] M. Azari and F. Falahati-Nezhad, *Inequalities involving multiplicative Sombor index*, J. Math. Inequal. **20** (2026), 425–448.  
<https://doi.org/10.7153/jmi-2026-20-26>.
- [4] K.C. Das, *Extremal graphs on the multiplicative Sombor index*, Discrete Math. **349** (2026), no. 10, 115192.  
<https://doi.org/10.1016/j.disc.2026.115192>.
- [5] R. Diestel, *Graph Theory*, Grad. Texts in Math., vol. 173, Springer, Heidelberg, 2005.
- [6] M. Eliasi, *On the minimum variable Euler–Sombor index*, MATCH Commun. Math. Comput. Chem. **97** (2027), no. 2, 785–815.  
<https://doi.org/10.46793/match.97-2.23926>.
- [7] L. Euler, *Nova series infinita maxime convergens perimetrum ellipsis exprimens*, Novi Comm. Acad. Sci. Petrop. **18** (1773), 71–84.

- [8] I. Gutman, *Geometric approach to degree-based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem. **86** (2021), 11–16.
- [9] ———, *Relating Sombor and Euler indices*, Milit. Techn. Courier **72** (2024), no. 1, 1–12.  
<https://doi.org/10.5937/vojtehg72-48818>.
- [10] I. Gutman, B. Furtula, and M.S. Oz, *Geometric approach to vertex-degree-based topological indices – elliptic Sombor index, theory and application*, Int. J. Quantum Chem. **124** (2024), e27346.  
<https://doi.org/10.1002/qua.27346>.
- [11] I. Gutman, I. Redžepović, G.Ö. Kizilirmak, and V. R. Kulli, *Euler–Sombor index and its congeners*, Open J. Math. Sci. **9** (2025), 141–148.  
<https://doi.org/10.30538/oms2025.0249>.
- [12] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total  $\pi$ -electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972), no. 4, 535–538.  
[https://doi.org/10.1016/0009-2614\(72\)85099-1](https://doi.org/10.1016/0009-2614(72)85099-1).
- [13] D. Haji Mohammed, *The Sombor index and multiplicative Sombor index of some products of graphs*, Commun. Comb. Optim. **11** (2026), no. 3, 787–826.  
<https://doi.org/10.22049/cco.2024.30123.2321>.
- [14] D.-Y. Huh, *Extremal multiplicative Sombor index of graphs with a given chromatic number*, Math. Rep. **27(77)** (2025), no. 1-2, 103–120.  
<https://doi.org/10.59277/mrar.2025.27.77.1.2.103>.
- [15] V.R. Kulli, *Multiplicative Sombor indices of certain nanotubes*, Int. J. Math. Arch. **12** (2021), no. 3, 1–5.
- [16] ———, *Multiplicative elliptic Sombor and multiplicative modified elliptic Sombor indices*, Ann. Pure Appl. Math. **29** (2024), 19–23.  
<https://doi.org/10.22457/apam.v29n1a03927>.
- [17] H. Liu, *Multiplicative Sombor index of graphs*, Discrete Math. Lett. **9** (2022), 80–85.
- [18] H. Liu, Z. Du, Y. Huang, H. Chen, and S. Elumalai, *Note on the minimum bond incident degree indices of  $k$ -cyclic graphs*, MATCH Commun. Math. Comput. Chem. **91** (2024), 255–266.  
<https://doi.org/10.46793/match.91-1.255L>.
- [19] G.G. Nyauli and J. Buragohain, *Geometric approach to degree-based topological indices: Cosine-rule generalized Sombor indices*, MATCH Commun. Math. Comput. Chem. **96** (2026), no. 1, 295–324.  
<https://doi.org/10.46793/match.96-1.24725>.
- [20] Z. Tang, Y. Li, and H. Deng, *The Euler Sombor index of a graph*, Int. J. Quantum Chem. **124** (2024), no. 9, e27387.  
<https://doi.org/10.1002/qua.27387>.
- [21] R. Todeschini and V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley-VCH, Weinheim, 2009.
- [22] K. Varmuza, M. Dehmer, and D. Bonchev, *Statistical Modelling of Molecular Descriptors in QSAR/QSPR*, Wiley, 2012.