

Graphs with constant modular irregularity strength

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Abstract: Let $G = (V, E)$ be a simple and finite graph of order n . An edge k -labeling $\phi : E \rightarrow \{1, 2, \dots, k\}$ is called a modular irregular labeling of a graph G if the weight function, $\sigma : V(G) \rightarrow \mathbb{Z}_n$, defined by $\sigma(u) = wt_\phi(u) = \sum_{uv \in E(G)} \phi(uv) \pmod{n}$ is bijective, where $\mathbb{Z}_n$ denotes the group of integers modulo n . The value $\sigma(u)$ is referred to as the modular weight of the vertex u . The minimum integer k such that G admits a modular irregular labeling using labels from $\{1, 2, \dots, k\}$ is called the modular irregularity strength of G . In this paper, we determine the modular irregularity strength for complete tripartite graphs, cocktail party graphs, and crown graphs. Moreover, we study the behavior of this parameter under the addition of a regular factor, and we apply these results to decompositions of complete multipartite graphs.

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1. Introduction

Research on irregular labeling was initiated by Chartrand et al. [8] in 1988, motivated by the question: “Is there any irregular graph?” The existence of irregular graphs is guaranteed only in the context of multigraphs, where the number of multiple edges can be interpreted as the edge label. This gave rise to the concept of irregular labeling. An *irregular labeling* of a simple graph G is an edge labeling $\phi : E(G) \rightarrow \{1, 2, \dots, k\}$, where k is a positive integer, such that the weights of every pair of distinct vertices are different. The weight of a vertex $u \in V(G)$ is defined as $wt_\phi(u) = \sum_{uv \in E(G)} \phi(uv)$.

The notation $s(G)$ is used for the *irregularity strength* of G . It is the smallest positive integer k such that G admits an irregular labeling using labels from $\{1, 2, \dots, k\}$. The notation $s(G) = \infty$ is used if no such labeling of G exists. It is clear that $s(G) < \infty$ if and only if G has no isolated edges and contains at most one isolated vertex.

Several variations of irregular labeling have been investigated, including edge irregular labeling, vertex irregular total labeling, and edge irregular total labeling, see [1, 4]. One such variation, which is the focus of this paper, is modular irregular labeling. A *modular irregular labeling* of a graph G of order n is an edge k -labeling $\phi : E(G) \rightarrow \{1, 2, \dots, k\}$ such that the weight function $wt_\phi(u) : V(G) \rightarrow \mathbb{Z}_n$, defined by

$$wt_\phi(u) = \sum_{uv \in E(G)} \phi(uv),$$

is bijective, where $\mathbb{Z}_n$ denotes the group of integers modulo n . The function wt_ϕ is called the *modular weight* of the vertex u . The minimum value of k for which the graph G admits a modular irregular labeling using edge labels from 1 to k is called the *modular irregularity strength* of G , denoted $ms(G)$. If no such labeling of G exists, then $ms(G) = \infty$, see [6].

The modular irregularity strength has been determined for several classes of graphs. In [6], Bača, Muthugurupackiam, Kathiresan, and Ramya determined the modular irregularity strength for various classes of graphs, including paths, stars, triangular graphs, cycles, and gear graphs. Subsequently, in 2021, Bača et al. established the modular irregularity strength for fan graphs [5] and wheel graphs [3]. In the same year, Sugeng et al. [18] obtained the modular irregularity strength for double stars and friendship graphs. In 2022, Dewi [10] investigated the modular irregularity strength of the corona product of a cycle C_n and p isolates, denoted $C_n \odot \overline{K_p}$. More recently, in 2023, Sugeng et al. [19] determined the modular irregularity strength for certain flower graphs. The modular irregularity strength of some corona product of graphs was further explored by Barack et al. [7]. In 2024, Suparta et al. [20] established the exact values of the modular irregularity strength for several dense graphs, including complete graphs and complete bipartite graphs. Interestingly, these complete graphs and complete bipartite graphs, which are dense graphs, exhibit a constant modular irregularity strength. Motivated by these results, we aim to identify characteristics of families of graphs that possess constant modular irregularity strength.

2. Known results

Chartrand et al. in [8] obtained the lower bound of the irregularity strength in the following form

$$s(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i},$$

where n_i denotes the number of vertices of degree i and Δ is the maximum degree of the graph G . For any r -regular graph with $r \geq 2$ and order n this lower bound is reduced to

$$s(G) \geq \frac{n + r - 1}{r}. \quad (2.1)$$

Aigner and Triesch [2] and later Nierhoff [15] investigated the upper bound of the irregularity strength. For general graphs with no component of order at most 2 it was estimated that

$$s(G) \leq |V(G)| - 1.$$

This upper bound was improved by Cuckler and Lazebnik in [9]. The better upper bound was obtained by Przybyło in [16], where it was proved that $s(G) \leq 16\frac{n}{r} + 6$ for r -regular graphs with n vertices and $s(G) \leq 112\frac{n}{\delta} + 28$ in the general case for graphs with minimum degree $\delta \geq 1$. Further improvement in the estimation of the irregularity strength were obtained by Kalkowski et al. in [13], who made use of a deterministic algorithm to prove that $s(G) \leq 6 \lceil \frac{n}{\delta} \rceil$ for all graphs which do not contain isolated edges. Majerski and Przybyło [14] used the same approach and a more developed probabilistic method to prove that $s(G) \leq \frac{(4+o(1))n}{\delta} + 4$ for graphs of minimum degree $\delta \geq \sqrt{n \ln n}$.

It is evident that every modular irregular labeling of a graph is also an irregular labeling. Then

$$s(G) \leq ms(G) \quad (2.2)$$

i.e., the irregularity strength provides a lower bound for the modular irregularity strength.

To illustrate inequality (2.2), consider the complete bipartite graph $K_{1,4}$ with vertices v_1, v_2, v_3, v_4, w , where the edge labeling α is defined as $\alpha(v_i w) = i$, $i = 1, 2, 3, 4$. The corresponding vertex weights are $wt_\alpha(v_i) = i$, $i = 1, 2, 3, 4$, and $wt_\alpha(w) = 10 \equiv 0 \pmod{5}$. Hence, both the irregularity strength and the modular irregularity strength of $K_{1,4}$ are equal to 4, that is, $s(K_{1,4}) = ms(K_{1,4}) = 4$.

Now consider the graph G obtained from $K_{1,4}$ by adding the edge $v_3 v_4$. Define the edge labeling β by $\beta(v_1 w) = \beta(v_3 w) = 1$, $\beta(v_2 w) = \beta(v_4 w) = 2$, $\beta(v_3 v_4) = 2$. This

labeling yields vertex weights $wt_\beta(v_i) = i$, $i = 1, 2, 3, 4$, and $wt_\beta(w) = 6$. The edge labeling β is an irregular labeling, and therefore, $s(G) = 2$. However, β is not a modular irregular labeling. It is straightforward to verify that all modular irregular labels of G require at least label 3. Next, consider a different labeling γ of the same graph G , defined by $\gamma(v_1w) = 1$, $\gamma(v_2w) = \gamma(v_3w) = 2$, $\gamma(v_4w) = 3$, $\gamma(v_3v_4) = 2$. The induced vertex weights modulo 5 are: $wt_\gamma(v_1) = 1$, $wt_\gamma(v_2) = 2$, $wt_\gamma(v_3) = 4$, $wt_\gamma(v_4) \equiv 0$, $wt_\gamma(w) \equiv 3$. Thus, γ is a modular irregular labeling, and we obtain $ms(G) = 3 > s(G)$. This example demonstrates that the converse of inequality (2.2) is not valid in general. Nevertheless, a condition under which equality holds is provided in the following statement.

Theorem 1. [6] *Let G be a graph with no component of order at most 2 and let $s(G) = k$. If there exists an irregular labeling of G with edge values at most k , where weights of vertices constitute a set of consecutive integers, then $s(G) = ms(G) = k$.*

Bača, Muthugurupackiam, Kathiresan and Ramya [6] proved the following theorem, which gives a sufficient condition for the modular irregularity strength of a graph to be infinite.

Theorem 2. [6] *If G is a graph of order n , $n \equiv 2 \pmod{4}$, then G has no modular irregular k -labeling, i.e., $ms(G) = \infty$.*

Suparta et al. [20] determined the precise values of the modular irregularity strength of the complete graph K_n , $n \geq 3$, and the complete bipartite graph $K_{n,n}$, $n \geq 2$, as follows:

Theorem 3. [20] *Let K_n be a complete graph on n vertices, $n \geq 3$. Then*

$$ms(K_n) = \begin{cases} 3, & \text{if } n \not\equiv 2 \pmod{4}, \\ \infty, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Theorem 4. [20] *Let $K_{n,n}$ be a complete bipartite graph on $2n$ vertices with $n \geq 2$. Then*

$$ms(K_{n,n}) = \begin{cases} 3, & \text{if } n \text{ is even,} \\ \infty, & \text{if } n \text{ is odd.} \end{cases}$$

The following two theorems, proven in [20], provide the exact values of the modular irregularity strength of complete bipartite graphs $K_{n,n+1}$ and $K_{n,n+2}$.

Theorem 5. [20] *Let $K_{n,n+1}$ be a complete bipartite graph on $2n+1$ vertices with $n \geq 1$. Then*

$$ms(K_{n,n+1}) = \begin{cases} 2, & \text{if } n = 1, \\ 3, & \text{if } n \geq 2. \end{cases}$$

Theorem 6. [20] Let $K_{n,n+2}$ be a complete bipartite graph on $2n+2$ vertices with $n \geq 1$. Then

$$\text{ms}(K_{n,n+2}) = \begin{cases} 4, & \text{if } n = 1, \\ 3, & \text{if } n \geq 3 \text{ is odd,} \\ \infty, & \text{if } n \text{ is even.} \end{cases}$$

A modular irregular labeling with $k = n$ is equivalent to a *modular edge-graceful labeling*, see for instance [12]. When such a modular graceful labeling exists in a graph G , we clearly have $\text{ms}(G) < \infty$. Hence, as a corollary of their results, we have the following:

Corollary 1. Let G be a connected graph of order $n \geq 3$. Then $\text{ms}(G) \leq n$ if and only if $n \not\equiv 2 \pmod{4}$.

Given a graph G , we denote its *density* by $D(G) = \frac{2|E|}{|V|(|V|-1)}$. G is said to be *dense* if $\frac{1}{2} < D(G) \leq 1$, and it is *sparse* otherwise, see [20]. Based on the previous results, we propose the following conjecture.

Conjecture 7. Let G be a dense regular graph G of order n , $n \not\equiv 2 \pmod{4}$. Then $\text{ms}(G) = 3$.

In subsequent sections of the paper, we present results that support the validity of the proposed conjecture.

3. Tripartite graphs

For the regular graph G , we have $\text{ms}(G) \geq 3$ as stated in the following lemma.

Lemma 1. If G is an r -regular graph with $r \geq 1$, then $\text{ms}(G) \geq 3$.

Proof. Suppose that G is a r -regular graph with n vertices that admit a modular irregular labeling $\phi : E(G) \rightarrow \{1, 2\}$. Under this labeling, the vertex weight for every vertex v is given by: $wt_\phi(v) = r + n_v$ where n_v is the number of edges incident with v having label 2. Since ϕ is a modular irregular labeling, then $\{wt_\phi(v) : v \in V(G)\} = \{0, 1, \dots, n-1\} \pmod{n}$. Thus, $\{r + n_v : v \in V(G)\}$ is equivalent to $\{0, 1, \dots, n-1\} \pmod{n}$. This implies that there exists a vertex w that must have $n-1$ incident edges with label 2, while another vertex u having no incident edges with label 2, which is a contradiction. So, there exists no regular graph G with $\text{ms}(G) = 2$. $\square$

To simplify the notation, we denote the m -partite graph $K_{n, n, \underbrace{n, \dots, n}_{m \text{ times}}}$ as $K_{m \times n}$. In

this notation, the complete bipartite graph $K_{n,n}$ is denoted by $K_{2 \times n}$. In this section, we determine the modular irregularity strength of complete tripartite graphs $K_{3 \times n}$.

In Figure 1, we can see the modular irregularity labeling for $K_{3 \times 4}$, where the color black, blue, and red represent the edge label 1, 2 and 3, respectively. This labeling is found from the pattern of the adjacency-label matrix in Table 1. Adjacency-label means that the element of the matrix is the label of the representing edge.

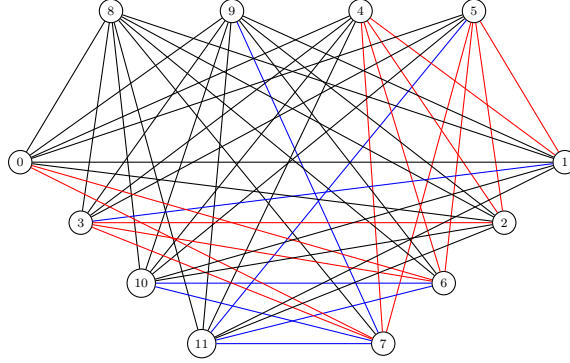


Figure 1. A modular irregular labeling and corresponding vertex weights of $K_{3 \times 4}$. Edges with label 1, 2, and 3 are depicted in black, blue, and red, respectively.

In Table 1, we can see the idea on how to find the pattern on the edge label. We start by filling all the elements from row 1, that is, represent the edge color from vertex x_1 , with label 1. To make the vertex weights distinct, we cannot use all 1 labels for another row. Thus, in the second row, which represents the vertex x_2 , we add element 2. Since the size of the matrix is 12×12 , and to make the vertex weight are all distinct, the labels 1 and 2 are not enough, so we need the label 3. The next process is to try to find the pattern where to put the labels 1, 2, and 3, so that it can also be used for the general graph $K_{3 \times n}$.

Table 1. A modular irregular labeling and corresponding vertex weights of $K_{3 \times 4}$.

	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	z_1	z_2	z_3	z_4	$wt_\phi(u_i)$
x_1	0	0	0	0	1	1	1	1	1	1	1	1	8
x_2	0	0	0	0	1	1	1	1	1	1	1	2	9
x_3	0	0	0	0	1	1	1	1	3	3	3	3	4
x_4	0	0	0	0	1	1	1	2	3	3	3	3	5
y_1	1	1	1	1	0	0	0	0	1	1	3	3	0
y_2	1	1	1	1	0	0	0	0	2	3	3	3	3
y_3	1	1	1	1	0	0	0	0	1	1	2	2	10
y_4	1	1	1	2	0	0	0	0	1	1	2	2	11
z_1	1	1	3	3	1	2	1	1	0	0	0	0	1
z_2	1	1	3	3	1	3	1	1	0	0	0	0	2
z_3	1	1	3	3	3	3	2	3	0	0	0	0	6
z_4	1	2	3	3	3	3	2	2	0	0	0	0	7

Theorem 8. *Let $K_{3 \times n}$ be a complete tripartite graph with $n \geq 1$. Then*

$$\text{ms}(K_{3 \times n}) = \begin{cases} 3, & \text{if } n \not\equiv 2 \pmod{4}, \\ \infty, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. If $n \equiv 2 \pmod{4}$ then $|V(K_{3 \times n})| = 3n \equiv 2 \pmod{4}$ and from Theorem 2, it follows $\text{ms}(K_{3 \times n}) = \infty$.

The complete tripartite graph $K_{3 \times n}$ is a $2n$ -regular graph with three partition sets of vertices, let us say $V_1 = \{x_1, x_2, \dots, x_n\}$, $V_2 = \{y_1, y_2, \dots, y_n\}$, and $V_3 = \{z_1, z_2, \dots, z_n\}$. According to Lemma 1,

$$\text{ms}(K_{3 \times n}) \geq 3.$$

To show that the value 3 is also the upper bound for the irregularity strength of $K_{3 \times n}$, $n \not\equiv 2 \pmod{4}$, we describe an appropriate 3-labeling in three cases, according to the order of the complete tripartite graph.

Case I. For $n \equiv 0 \pmod{4}$, define the edge labeling ϕ as follows.

$$\begin{aligned} \phi(x_i y_j) &= \begin{cases} 2, & \text{if } i + j \geq \frac{3}{2}n + 2, \\ 1, & \text{otherwise,} \end{cases} \\ \phi(x_i z_k) &= \begin{cases} 2, & \text{if } i + k \geq n + 2 \text{ and } i \leq \frac{n}{2}, \\ 3, & \text{if } \frac{n}{2} + 1 \leq i \leq n, \\ 1, & \text{otherwise,} \end{cases} \\ \phi(y_j z_k) &= \begin{cases} 2, & \text{if } j + k = \frac{n}{2} + 1, k \leq \frac{n}{4}, \\ 2, & \text{if } j \geq \frac{n}{2} + 1, k \geq \frac{n}{2} + 1, \\ 3, & \text{if } j + k \geq \frac{n}{2} + 2, j \leq \frac{n}{2}, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

The maximum label of the edge labeling ϕ is equal to three. Then, the edge labeling ϕ is an edge 3-labeling of the complete tripartite graph $K_{3 \times n}$ with $n \equiv 0 \pmod{4}$.

One can see that the vertex weights are as follows.

$$\begin{aligned} wt(x_i) &= \begin{cases} 2n + i - 1, & \text{if } 1 \leq i \leq \frac{n}{2}, \\ \frac{7n}{2} + i - 1, & \text{if } \frac{n}{2} + 1 \leq i \leq n, \end{cases} \\ wt(y_j) &= \begin{cases} 3n + 2j - 2, & \text{if } 1 \leq j \leq \frac{n}{4}, \\ 3n + 2j - 1, & \text{if } \frac{n}{4} + 1 \leq j \leq \frac{n}{2}, \\ 2n + j - 1, & \text{if } \frac{n}{2} + 1 \leq j \leq n, \end{cases} \\ wt(z_k) &= \begin{cases} 3n + 2k - 1, & \text{if } 1 \leq k \leq \frac{n}{4}, \\ 3n + 2k - 2, & \text{if } \frac{n}{4} + 1 \leq k \leq \frac{n}{2}, \\ 4n + k - 1, & \text{if } \frac{n}{2} + 1 \leq k \leq n. \end{cases} \end{aligned}$$

We can see that in this case the weights of vertices of the complete tripartite graph $K_{3 \times n}$ attain the consecutive integers from $2n$ to $5n - 1$.

Case II. For $n \equiv 1 \pmod{4}$, $n \geq 5$, define the edge labeling ϕ in the following way.

$$\begin{aligned}\phi(x_i y_j) &= \begin{cases} 2, & \text{if } i + j \geq \frac{3n+3}{2}, \\ 1, & \text{otherwise.} \end{cases} \\ \phi(x_i z_k) &= \begin{cases} 2, & \text{if } i + k \geq n + 1 \text{ and } i \leq \frac{n+1}{2}, \\ 3, & \text{if } i \geq \frac{n+3}{2}, \\ 1, & \text{otherwise,} \end{cases} \\ \phi(y_j z_k) &= \begin{cases} 2, & \text{if } j + k = \frac{n+1}{2}, k \leq \frac{n-1}{4}, \\ 2, & \text{if } j \geq \frac{n+3}{2}, k \geq \frac{n+1}{2}, \\ 3, & \text{if } j + k \geq \frac{n+3}{2}, j \leq \frac{n+1}{2}, \\ 1, & \text{otherwise.} \end{cases}\end{aligned}$$

The maximal edge labels are 3 and the vertex weights are the following

$$\begin{aligned}wt(x_i) &= \begin{cases} 2n + i, & \text{if } 1 \leq i \leq \frac{n+1}{2}, \\ \frac{7n+1}{2} + i - 1, & \text{if } \frac{n+3}{2} \leq i \leq n, \end{cases} \\ wt(y_j) &= \begin{cases} 3n + 2j - 1, & \text{if } 1 \leq j \leq \frac{n-1}{4}, \\ 3n + 2j, & \text{if } \frac{n+3}{4} \leq j \leq \frac{n-1}{2}, \\ 4n, & \text{if } j = \frac{n+1}{2}, \\ 2n + j, & \text{if } \frac{n+3}{2} \leq j \leq n, \end{cases} \\ wt(z_k) &= \begin{cases} 3n + 2k, & \text{if } 1 \leq k \leq \frac{n-1}{4}, \\ 3n + 2k - 1, & \text{if } \frac{n+3}{4} \leq k \leq \frac{n-1}{2}, \\ 4n + k, & \text{if } \frac{n+1}{2} \leq k \leq n. \end{cases}\end{aligned}$$

Thus, in this case the weights of vertices of the complete tripartite graph $K_{3 \times n}$ successively admit consecutive integers from $2n + 1$ to $5n$.

For $n = 1$ we have the complete tripartite graph $K_{3 \times 1}$ as the cycle on three vertices x_1, y_1, z_1 . Consider an edge labeling ϕ with the edge values $\phi(x_1 y_1) = 1$, $\phi(x_1 z_1) = 2$ and $\phi(y_1 z_1) = 3$. The corresponding weights of vertices are $wt(x_1) = 3$, $wt(y_1) = 4$ and $wt(z_1) = 5$. Hence, the edge labeling ϕ is appropriate labeling which proves that $s(K_{3 \times 1}) = 3$.

Case III. For $n \equiv 3 \pmod{4}$, $n \geq 3$, define the edge labeling ϕ as follows.

$$\phi(x_i y_j) = \begin{cases} 2, & \text{if } i + j \geq \frac{3n+3}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

$$\phi(x_i z_k) = \begin{cases} 2, & \text{if } i + k \geq n + 2 \text{ and } i \leq \frac{n+1}{2}, \\ 3, & \text{if } i \geq \frac{n+3}{2}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\phi(y_j z_k) = \begin{cases} 2, & \text{if } j + k = \frac{n+3}{2}, \frac{n+5}{4} \leq k \leq \frac{n+1}{2}, \\ 2, & \text{if } j \geq \frac{n+3}{2}, k \geq \frac{n+3}{2}, \\ 3, & \text{if } j + k = \frac{n+3}{2}, 1 \leq k \leq \frac{n+1}{4}, \\ 3, & \text{if } j + k \geq \frac{n+5}{2}, j \leq \frac{n+1}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

The maximum label of the edge labeling ϕ is equal to three. Then, the edge labeling ϕ is an edge 3-labeling of the complete tripartite graph on $K_{3 \times n}$ with $n \equiv 3 \pmod{4}$. We obtain that the vertex weights are as follows.

$$wt(x_i) = \begin{cases} 2n + i - 1, & \text{if } 1 \leq i \leq \frac{n+1}{2}, \\ \frac{7n+1}{2} + i - 1, & \text{if } \frac{n+3}{2} \leq i \leq n. \end{cases}$$

$$wt(y_j) = \begin{cases} 3n + 2j - 2, & \text{if } 1 \leq j \leq \frac{n+1}{4}, \\ 3n + 2j - 1, & \text{if } \frac{n+5}{4} \leq j \leq \frac{n+1}{2}, \\ 2n + j - 1, & \text{if } \frac{n+3}{2} \leq j \leq n. \end{cases}$$

$$wt(z_k) = \begin{cases} 3n + 2k - 1, & \text{if } 1 \leq k \leq \frac{n+1}{4}, \\ 3n + 2k - 2, & \text{if } \frac{n+5}{4} \leq k \leq \frac{n+1}{2}, \\ 4n + k - 1, & \text{if } \frac{n+3}{2} \leq k \leq n. \end{cases}$$

In any of the previous three cases we have described the appropriate irregular labeling 3 that shows that the value 3 is also the upper bound for the irregularity strength of $K_{3 \times n}$, $n \not\equiv 2 \pmod{4}$, and then $s(K_{3 \times n}) = 3$. Since in each case the weights of vertices constitute a set of consecutive integers, then according to Theorem 1 we have $s(K_{3 \times n}) = ms(K_{3 \times n}) = 3$ for $n \not\equiv 2 \pmod{4}$. This completes the proof. $\square$

4. Removing perfect matching of regular graph

By removing a perfect matching from a complete graph of order $2m$, we obtain the complete m -partite graph $K_{m \times 2}$. This graph is known as the cocktail party graph of order $2m$, see [21] and is also referred to as the Turán graph $T(2m, m)$, see [17]. In the following theorem, we characterize the (modular) irregularity strength of the cocktail party graph.

Theorem 9. *Let $K_{m \times 2}$ be a cocktail party graph, $m \geq 2$. Then*

$$ms(K_{m \times 2}) = \begin{cases} 3, & \text{if } m \text{ is even,} \\ \infty, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let the vertex set $V(K_{2m}) = \{x_1, x_2, \dots, x_{2m}\}$ and the edge set $E(K_{2m}) = \bigcup_{i=1}^{2m-1} \{x_i x_{i+j} : 1 \leq j \leq 2m-i\}$ except for the perfect matching edges $x_i x_{i+1}$, $i = 1, 3, \dots, 2m-1$.

If m is odd, $m \geq 3$, then $|V(K_{m \times 2})| = 2m \equiv 2 \pmod{4}$ and it follows from Theorem 2 that $\text{ms}(K_{m \times 2}) = \infty$.

Since $K_{m \times 2}$ is a $(2m-2)$ -regular graph then from (2.1) it follows that

$$s(K_{m \times 2}) \geq \frac{4m-3}{2m-2}.$$

The irregularity strength of $K_{m \times 2}$ must be larger or equal than the smallest integer greater or equal to $\frac{4m-3}{2m-2}$. Hence $s(K_{m \times 2}) \geq 3$. To show that the value 3 is also an upper bound for the irregularity strength of the cocktail party graph, we describe an appropriate edge labeling.

For $m = 2$ the cocktail party graph $K_{2 \times 2}$ is the cycle $C_4 = (x_1, x_2, x_3, x_4)$. Evidently, the edge labeling λ with values $\lambda(x_1 x_2) = \lambda(x_1 x_4) = 1$, $\lambda(x_2 x_3) = 2$ and $\lambda(x_3 x_4) = 3$ is appropriate 3-labeling with vertex weights $wt(x_1) = 2$, $wt(x_2) = 3$, $wt(x_3) = 5$ and $wt(x_4) = 4$. Consequently $s(K_{2 \times 2}) = \text{ms}(K_{2 \times 2}) = 3$.

For $m \geq 4$ we define an edge labeling $\phi : E(K_{m \times 2}) \rightarrow \{1, 2, 3\}$ in the following way.

$$\begin{aligned} \phi(x_i x_j) &= \phi(x_j x_i) = 1, & \text{if } i+j \leq 2m+1, i \neq j \text{ and } i+1 \neq j \text{ for } i \text{ odd,} \\ \phi(x_i x_j) &= \phi(x_j x_i) = 3, & \text{if } i+j = 3m \text{ and } m+1 \leq i \leq \frac{3m-2}{2}, \\ \phi(x_i x_{2m}) &= \phi(x_{2m} x_i) = 3, & \text{if } i = m+1, \\ \phi(x_i x_{i+1}) &= \phi(x_{i+1} x_i) = 3, & \text{if } m+2 \leq i \leq 2m-2 \text{ and } i \text{ is even,} \\ \phi(x_i x_{2m}) &= \phi(x_{2m} x_i) = 3, & \text{if } i = \frac{3m}{2}, \\ \phi(x_i x_j) &= \phi(x_j x_i) = 2, & \text{otherwise.} \end{aligned}$$

Now, consider a $2m \times 2m$ matrix $A = [a_{i,j}]$, $i, j = 1, 2, \dots, 2m$, with entry $a_{i,j} = \phi(x_i x_j)$. Thus, the symmetric matrix A describes the edge labeling ϕ of the cocktail party graph $K_{m \times 2}$ if it also contains zero elements $a_{i,i} = 0$, $i = 1, 2, \dots, 2m$, respectively $a_{i,i+1} = a_{i+1,i} = 0$, $i = 1, 3, 5, \dots, 2m-1$, which correspond to non-existent edges $x_i x_i$, $i = 1, 2, \dots, 2m$, respectively, the perfect matching edges $x_i x_{i+1}$ and $x_{i+1} x_i$, $i = 1, 3, 5, \dots, 2m-1$.

Clearly, the weight of vertex x_i , $i = 1, 2, \dots, 2m$, under the edge labeling ϕ is the sum of entries in the i th row of the matrix A . Consequently $wt(x_i) = \sum_{j=1}^{2m} a_{i,j}$, $i = 1, 2, \dots, 2m$.

It is a matter of routine checking to see that the first row of the matrix A contains two zeros, $2m-2$ integers 1 and no integers 2 and 3. Then $wt(x_1) = 2m-2$.

For every i , $2 \leq i \leq m$, the i th row of the matrix A contains two zeros, $2m-i-1$ integers 1, $i-1$ integers 2 and no integer 3. Then $wt(x_i) = 2m-i-1+2(i-1) = 2m+i-3$.

For every i , $m + 1 \leq i \leq 2m$, the i th row of the matrix A contains two zeros, two integers 3, $2m - i + 1$ integers 1 and $i - 5$ integers 2. Then $wt(x_i) = 6 + 2m - i + 1 + 2(i - 5) = 2m + i - 3$.

The determined vertex weights constitute a sequence of consecutive integers from $2m - 2$ to $4m - 3$, i.e., the edge labeling ϕ is the required irregular 3-labeling with $s(K_{m \times 2}) = 3$. Immediately from Theorem 1 it follows that $s(K_{m \times 2}) = ms(K_{m \times 2}) = 3$ and we are done. $\square$

In this part, we consider the class of graphs obtained by removing a perfect matching from a complete bipartite graph of order $2n$. Let $K_{n,n}$ denote the complete bipartite graph with vertex set $V(K_{n,n}) = \{u_i, v_j : 1 \leq i, j \leq n\}$ and edge set $E(K_{n,n}) = \{u_i v_j : 1 \leq i, j \leq n\}$. In [20], Suparta, Candisa, Prasancika, and Bača provided the following edge labeling φ of $K_{n,n}$ for even $n \geq 4$ to prove that $ms(K_{n,n}) = 3$.

$$\begin{aligned} \varphi(u_2 v_n) &= 2, \\ \varphi(u_{n-1} v_1) &= 2, \\ \varphi(u_i v_j) &= 2, & \text{if } i + j = n + 1, \frac{n}{2} + 1 \leq i \leq n, \\ \varphi(u_i v_j) &= 3, & \text{if } i + j \geq n + 2, 3 \leq i \leq n, \\ \varphi(u_i v_j) &= 1, & \text{otherwise.} \end{aligned} \quad (4.1)$$

A crown graph $H_{n,n}$ is obtained from $K_{n,n}$ by removing a perfect matching, see [11]. More formally, the graph $H_{n,n}$ is an $(n - 1)$ -regular graph with vertex set $V(H_{n,n}) = V(K_{n,n})$ and edge set $E(H_{n,n}) = \{u_i v_j : 1 \leq i, j \leq n \text{ and } i \neq j\}$. The following theorem establishes the exact value of the modular irregularity strength for $H_{n,n}$.

Theorem 10. *Let $H_{n,n}$ be a crown graph on $2n$ vertices with $n \geq 3$. Then*

$$ms(H_{n,n}) = \begin{cases} 4, & \text{if } n \text{ is even,} \\ \infty, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is odd, $n \geq 3$, then $|V(H_{n,n})| = 2n \equiv 2 \pmod{4}$ and by Theorem 2 we have $ms(H_{n,n}) = \infty$.

The fact that the value 4 is a lower bound for the irregularity strength of the $(n - 1)$ -regular crown graph $H_{n,n}$ follows from (2.1). In order to show that value 4 is an upper bound for the irregularity strength of $H_{n,n}$, it only remains to describe a suitable edge 4-labeling for every even $n \geq 4$.

First, for $H_{4,4}$ with vertices u_i, v_j , $1 \leq i, j \leq 4$, we define a special edge labeling ϕ such that

$$\begin{aligned} \phi(u_1 v_j) &= 1, & \text{if } 2 \leq j \leq 4, \\ \phi(u_2 v_1) &= 4, \\ \phi(u_i v_1) &= 3, & \text{if } 3 \leq i \leq 4, \end{aligned}$$

$$\begin{aligned}
\phi(u_3v_2) &= 1, \\
\phi(u_4v_2) &= 2, \\
\phi(u_iv_3) &= 3, \text{ if } i = 2 \text{ or } i = 4, \\
\phi(u_iv_4) &= 2, \text{ if } 2 \leq i \leq 3.
\end{aligned}$$

To facilitate a clearer visualization, in Table 2 the edge labels are presented in the form of a matrix, where the entry in the row u_i and column v_j corresponds to the edge label connecting the vertices u_i and v_j , if such an edge exists. An entry of 0 indicates the absence of an edge between the respective vertices. The weights of the vertices u_i , for $i = 1, 2, 3, 4$, are listed in the final column of Table 2, while the weights of the vertices v_j , for $j = 1, 2, 3, 4$, are provided in the final row. Since the vertex weights form a sequence of consecutive integers, it follows directly from Theorem 1 that $s(H_{4,4}) = \text{ms}(H_{4,4}) = 4$.

Table 2. A modular irregular labeling and corresponding vertex weights of $H_{4,4}$.

	v_1	v_2	v_3	v_4	$wt_\phi(u_i)$
u_1	0	1	1	1	3
u_2	4	0	3	2	9
u_3	3	1	0	2	6
u_4	3	2	3	0	8
$wt_\phi(v_i)$	10	4	7	5	

Now, for $H_{6,6}$ with vertices u_i, v_j , $1 \leq i, j \leq 6$, we construct a special edge labeling ϕ as follows.

$$\begin{aligned}
\phi(u_iv_j) &= 1, & \text{if } i + j \leq 7, 1 \leq i \leq 3 \text{ and } i \neq j, \\
\phi(u_4v_j) &= 1, & \text{if } j = 1, 2, \\
\phi(u_2v_6) &= \phi(u_5v_1) = 2, \\
\phi(u_iv_j) &= 2, & \text{if } i + j = 7, 4 \leq i \leq 6, \\
\phi(u_3v_j) &= 3, & \text{if } 5 \leq j \leq 6, \\
\phi(u_6v_j) &= 3, & \text{if } 2 \leq j \leq 3, \\
\phi(u_5v_3) &= 3, \\
\phi(u_iv_j) &= 4, & \text{if } 4 \leq i, j \leq 6 \text{ and } i \neq j.
\end{aligned}$$

A labeling of $H_{6,6}$ is presented analogously in Table 3. Once again, the vertex weights are different and range from 5 to 16, which implies $s(H_{6,6}) = \text{ms}(H_{6,6}) = 4$.

For n even, $n \geq 8$, we use the edge label φ defined in (4.1) and construct the label

Table 3. A modular irregular labeling and corresponding vertex weights of $H_{6,6}$.

	v_1	v_2	v_3	v_4	v_5	v_6	$wt_\phi(u_i)$
u_1	0	1	1	1	1	1	5
u_2	1	0	1	1	1	2	6
u_3	1	1	0	1	3	3	9
u_4	1	1	2	0	4	4	12
u_5	2	2	3	4	0	4	15
u_6	2	3	3	4	4	0	16
$wt_\phi(v_i)$	7	8	10	11	13	14	

$\varphi' : E(H_{n,n}) \rightarrow \{1, 2, 3, 4\}$ as follows.

$$\begin{aligned}
\varphi'(u_{\frac{n}{2}}v_1) &= 2, \\
\varphi'(u_i v_{i+1}) &= 2, & \text{if } 1 \leq i \leq \frac{n}{2} - 1, \\
\varphi'(u_i v_{i+1}) &= 4, & \text{if } \frac{n}{2} + 1 \leq i \leq n - 1, \\
\varphi'(u_i v_{i+2}) &= 4, & \text{if } \frac{n}{2} + 1 \leq i \leq n - 2, \\
\varphi'(u_i v_{i+3}) &= 4, & \text{if } \frac{n}{2} + 1 \leq i \leq n - 3, \\
\varphi'(u_i v_{\frac{n}{2}+1}) &= 4, & \text{if } i = n - 2, n - 1, n, \\
\varphi'(u_i v_{\frac{n}{2}+2}) &= 4, & \text{if } i = n - 1, n, \\
\varphi'(u_n v_{\frac{n}{2}+3}) &= 4, \\
\varphi'(u_i v_j) &= \varphi(u_i v_j), & \text{otherwise.}
\end{aligned}$$

It is clear that the labeling φ' is a 4-labeling. Now, we split the vertex set of $H_{n,n}$ into four mutually disjoint subsets A, B, C, D , where

$$\begin{aligned}
A &= \{u_1, u_2, v_1, v_2\}, \\
B &= \{u_i : i = 3, 4, \dots, \frac{n}{2}\} \cup \{v_j : j = 3, 4, \dots, \frac{n}{2}\}, \\
C &= \{u_i : i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n - 3\} \cup \{v_j : j = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n - 1\}, \\
D &= \{u_{n-2}, u_{n-1}, u_n, v_n\}.
\end{aligned}$$

We proceed by systematically verifying the vertex weights in each subset under the labeling φ' .

1. For the weights of the vertices from the set A we get

$$\begin{aligned}
wt_{\varphi'}(u_1) &= n, \\
wt_{\varphi'}(u_2) &= n + 1, \\
wt_{\varphi'}(v_1) &= n + 2, \\
wt_{\varphi'}(v_2) &= n + 3.
\end{aligned}$$

The weights obtained from the set A create the set $W_A = \{n, n+1, n+2, n+3\}$.

2. In the set B we first obtain weights for the vertices $u_i, i = 3, 4, \dots, \frac{n}{2} - 1$, then for the vertex $u_{\frac{n}{2}}$, and finally for the vertices $v_j, j = 3, 4, \dots, \frac{n}{2}$.

$$\begin{aligned} wt_{\varphi'}(u_i) &= n - 2 + 2i, \\ wt_{\varphi'}(u_{\frac{n}{2}}) &= 2n - 2, \\ wt_{\varphi'}(v_j) &= n - 1 + 2j. \end{aligned}$$

One can see that the vertices of the set B admit the weights of the set $W_B = \{n+4, n+5, n+6, \dots, 2n-2, 2n-1\}$.

3. In the set C we first obtain weights for the vertices $u_i, i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n-3$, then step by step for the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}$ and finally for the vertices $v_i, i = \frac{n}{2} + 4, \frac{n}{2} + 5, \dots, n-1$.

$$\begin{aligned} wt_{\varphi'}(u_i) &= n + 2i - 1, \\ wt_{\varphi'}(v_{\frac{n}{2}+1}) &= 2n, \\ wt_{\varphi'}(v_{\frac{n}{2}+2}) &= 2n + 2, \\ wt_{\varphi'}(v_{\frac{n}{2}+3}) &= 2n + 4, \\ wt_{\varphi'}(v_j) &= n + 2j - 2. \end{aligned}$$

Observe that the weights of the vertices of the set C form the set $W_C = \{2n, 2n+1, \dots, 3n-7, 3n-6, 3n-4\}$.

4. The vertices of the set D receive the following weights.

$$\begin{aligned} wt_{\varphi'}(u_{n-2}) &= 3n - 5, \\ wt_{\varphi'}(u_{n-1}) &= 3n - 2, \\ wt_{\varphi'}(u_n) &= 3n - 1, \\ wt_{\varphi'}(v_n) &= 3n - 3. \end{aligned}$$

The vertices of the set D admit the weights $W_D = \{3n-5, 3n-3, 3n-2, 3n-1\}$.

It is a matter of performing a routine check to see that the determined weights of the vertices of $H_{n,n}$, $n \geq 8$ even, constitute a sequence of consecutive integers from n to $3n-1$. This means that the edge labeling φ' is an irregular 4-labeling and with respect to Theorem 1 we get $s(H_{n,n}) = ms(H_{n,n}) = 4$. $\square$

5. Factor addition and multipartite decomposition

In the search for graphs with constant modular irregularity strength, we investigated the effect of adding factors to some classes of dense graphs. We then applied these results to particular classes of multipartite graphs, obtaining for several families of pairs m, n the value of $ms(K_{m \times n})$. The following result, inspired by Theorem 10 of [12], shows how the modular irregularity strength behaves under the addition of a factor.

Theorem 11. *Let G' be a graph of order n , and let $G' = G \cup F$, where G is a spanning subgraph of G' and F is a k -factor of G' , $E(G) \cap E(F) = \emptyset$. Then $ms(G') \leq ms(G)$.*

Proof. The statement clearly holds if $ms(G) = \infty$, and hence we may assume that $ms(G)$ is finite. Let then $\phi : E(G) \rightarrow \{1, \dots, ms(G)\}$ be a modular irregular labeling that realizes the modular irregularity strength of G . Hence $\{wt_\phi(v) : v \in V(G)\} \equiv \{0, 1, \dots, n-1\} \pmod{n}$. Now, consider the following labeling $\Phi_c : E(G') \rightarrow \{1, \dots, ms(G)\}$, where c can be any fixed element of $\{1, \dots, ms(G)\}$.

$$\Phi_c(e) = \begin{cases} \phi(e), & e \in E(G), \\ c, & e \in E(F). \end{cases}$$

Since F is a k -factor of G' and G is spanning (hence G and G' have the same order n and the weight calculation has the same modulo), it follows that: $wt_{\Phi_c}(v) = wt_\phi(v) + c \cdot k \pmod{n}$. Thus, $\{wt_{\Phi_c}(v) : v \in V(G')\} \equiv \{wt_\phi(v) + c \cdot k \pmod{n} : v \in V(G)\} \equiv \{0, 1, \dots, n-1\} \pmod{n}$. Hence, Φ_c is a modular irregular labeling of G' , and the statement follows. $\square$

Corollary 2. *Let $G' = G \cup F$ be a regular graph of order n , with G is a spanning graph of G' and F be a k -factor of G' . Then $ms(G) = 3$ gives $ms(G') = 3$.*

For even m , the cocktail party graph $K_{m \times 2}$ contains a spanning copy of $K_{m,m}$, while the remaining edges form an $(m-2)$ -factor. Hence, Theorem 11, together with Theorem 4 and Lemma 1, gives an alternative proof of Theorem 9.

In the remainder of this section, we compute the value of $ms(K_{m \times n})$ for some pairs m, n using Theorem 11.

Theorem 12. *Let n, m, d be positive integers. If $d|n$, then*

$$ms(K_{dm \times \frac{n}{d}}) \leq ms(K_{m \times n}).$$

Moreover, if $m = 2$ and n are even or $m = 3$ and $n \not\equiv 2 \pmod{4}$ then $ms(K_{dm \times \frac{n}{d}}) = 3$.

Proof. It can be easily seen that $K_{dm \times \frac{n}{d}} = K_{m \times n} \cup F$, where F is a $(n - \frac{n}{d})$ -factor. So, by Theorem 11 we have

$$\text{ms}(K_{dm \times \frac{n}{d}}) \leq \text{ms}(K_{m \times n})$$

Now, if $m = 2$ and n are even or $m = 3$ and $n \not\equiv 2 \pmod{4}$, by Theorems 4 and 8 it respectively follows $\text{ms}(K_{m \times n}) = 3$; since $K_{dm \times \frac{n}{d}}$ is regular, by Lemma 1 we have:

$$3 \leq \text{ms}(K_{dm \times \frac{n}{d}}) \leq \text{ms}(K_{m \times n}) = 3,$$

hence the statement. $\square$

Corollary 3. *Let m, n be positive integers such that $mn \not\equiv 2 \pmod{4}$, and $2 \mid m$ or $3 \mid m$. Then $\text{ms}(K_{m \times n}) = 3$.*

Proof. Let m, n be as in the statement. It is clear from Lemma 1 that $\text{ms}(K_{m \times n}) \geq 3$. Assume first that m is even, hence $m = 2d$ for some integer d . Since $mn \not\equiv 2 \pmod{4}$ we deduce that nd is even. By applying Theorems 4 and 12, we have $\text{ms}(K_{2d \times n}) \leq \text{ms}(K_{2 \times nd}) = 3$. A similar argument can be done for $m = 3d$, where now the conclusion follows from Theorem 8. $\square$

6. Conclusion

We conjecture that every dense and regular graph of order n , $n \not\equiv 2 \pmod{4}$, has a constant modular irregularity strength. In the paper, we determined the exact values of the modular irregularity strength for complete tripartite graphs, providing evidence to support the proposed conjecture. In addition, we establish the precise values of the modular irregularity strength for the cocktail party graph $K_{m \times 2}$, obtained by removing a perfect matching from the complete graph K_{2m} , and the crown graph $H_{n,n}$ obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$. Furthermore, we determined the exact modular irregularity strength for a broad family of graphs obtained through the decomposition of complete multipartite graphs.

The known results, together with those presented in this paper, suggest that regular multipartite graphs $K_{m \times n}$ satisfying the necessary condition $nm \not\equiv 2 \pmod{4}$ are strong candidates for having a constant modular irregularity strength. It would be of particular interest to investigate this conjecture further for the remaining cases $m \equiv 1, 5 \pmod{6}$. Moreover, our findings concerning graphs derived from dense graphs by the removal of a perfect matching open an additional promising direction for future research. Specifically, one may ask: What is the effect of removing a regular factor (such as a collection of cycles) from a dense graph? Will the resulting graph still have constant modular irregularity strength? Based on the observed patterns, we believe that this is indeed the case.

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