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Erratum

Corrections and further spectral results for the Graovac–Ghorbani matrix

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Abstract: The Graovac–Ghorbani index of a connected graph G is $\mathcal{A}(G) = \sum_{uv\in E(G)} \sqrt{\frac{n_u+n_v-2}{n_u\, n_v}}$, where n_u is the number of vertices closer to u than to v. Some statements in the paper of Alidehi-Ravandi and Ghorbani *(Commun. Comb. Optim., 2026)* contain several inaccuracies. We identify these inaccuracies, provide connected counterexamples, and give corrected statements. The corrections are the main part of the note. We also give an entrywise form of the bipartite bound $\rho(\widetilde{A}) \le \sqrt{n-2}$, which is already Theorem 9 of the original paper, a Randić-type bound for the index that follows from it, and a lower bound for the spectral radius of bipartite graphs, checked against the classical bounds on examples.

Keywords: Graovac–Ghorbani matrix, spectral radius, counterexample, strongly distance-balanced graph, bipartite graph, Randić index.

AMS Subject classification: 05C50, 05C12, 15A18

1. Introduction

Let $G = (V, E)$ be a finite, simple and connected graph with $n = |V|$ and $m = |E|$. For an edge $e = uv$ let $n_u = n_u(e \mid G) = \big|\{x \in V : d(x, u) < d(x, v)\}\big|$ and $n_v = n_v(e \mid G) = \big|\{x \in V : d(x, v) < d(x, u)\}\big|$. The Graovac–Ghorbani index [4] is

$$\mathcal{A}(G) = \sum_{e=uv\in E} \sqrt{\frac{n_u + n_v - 2}{n_u\, n_v}},$$

and the associated Graovac–Ghorbani matrix $\widetilde{A} = (\widetilde{A}_{uv})$ is defined by

$$\widetilde{A}_{uv} = \begin{cases} \sqrt{\dfrac{n_u + n_v - 2}{n_u\, n_v}}, & uv \in E, \\ 0, & \text{otherwise.} \end{cases}$$

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The eigenvalues of $\widetilde{A}$ are denoted by $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$, and θ_1 is the $\widetilde{A}$-spectral radius. The Laplacian version is $\widetilde{L} = \widetilde{D} - \widetilde{A}$, where $\widetilde{D}$ is the diagonal matrix of weighted degrees.

Since $\widetilde{A}$ is real, symmetric and nonnegative, its largest eigenvalue θ_1 equals its spectral radius $\rho(\widetilde{A})$. We write θ_1 in statements about the spectrum of the original paper and $\rho(\widetilde{A})$ in all new statements and wherever Perron–Frobenius theory is used; they are the same number.

Recently, Alidehi-Ravandi and Ghorbani [1] studied spectral properties of $\widetilde{A}$ and $\widetilde{L}$. In this note we show that several of their statements are incorrect. For each one we give a connected counterexample and, where appropriate, a corrected statement with proof. Section 4 contains some further results. Some of these follow directly from the definitions, some are standard, and one overlaps with a result of [1]; this is indicated at each place.

2. Preliminaries

We keep the notation of [1]. For an edge $e = uv$ we write

$$w_e = \sqrt{\frac{n_u + n_v - 2}{n_u\, n_v}},$$

so that $\mathcal{A}(G) = \sum_e w_e$ and $\widetilde{A}_{uv} = w_e$ for $uv \in E$. Recall that e is *perfectly distance balanced* (pdb) if $n_u = n_v = 1$; equivalently $w_e = 0$.

A graph G is *strongly distance-balanced* (SDB) if for every edge uv and every integer $k \geq 0$, the number of vertices at distance k from u and $k+1$ from v equals the number of vertices at distance $k+1$ from u and k from v. A graph G is *edge-transitive* if its automorphism group acts transitively on $E(G)$.

Lemma 1. *Let G be connected and let $e = uv \in E$. Then:*

(i) $w_e = 0$ if and only if $n_u = n_v = 1$;

(ii) if G is bipartite with parts X, Y, then for every edge uv we have $n_u + n_v = n$;

(iii) if G is regular of degree r, then $d_u + d_v = 2r$ for every edge.

Proof. (i) Obvious. (ii) In a bipartite graph every vertex is at different parity distance from u and v, so no ties occur; hence $n_u + n_v = n$. (iii) Clear. □

The reverse Szeged index is

$$\mathrm{RSz}(G) = 2 \sum_{e=uv \in E} \frac{1}{n_u\, n_v},$$

and the Randić index is

$$\mathrm{R}(G) = \sum_{e=uv\in E} \frac{1}{\sqrt{d_u d_v}}.$$

We shall also use that every connected SDB graph is regular: taking $k = 1$ in the definition gives $|N(u) \setminus N[v]| = |N(v) \setminus N[u]|$, hence $d_u = d_v$ for every edge uv. The notion of SDB graph is due to [6]. We also use the standard rank formula $\mathrm{rank}(R) = |V(H)| - c(H)$ for the (weighted) incidence matrix R of a graph H with positive edge weights; see, e.g., [2].

3. Corrections of published results

3.1. The rank assertion in the abstract

The abstract of [1] states that for every connected graph of order $n \geq 3$,

$$\mathrm{rank}(\widetilde{L}) = n - 1. \tag{3.1}$$

This statement, *as formulated in the abstract*, requires an additional assumption. Indeed, consider the complete graph $G = K_n$. For every edge $e = uv$ of K_n we have $n_u = n_v = 1$, and hence $w_e = 0$ by Lemma 1(i). Therefore,

$$\widetilde{A}(K_n) = 0, \qquad \widetilde{L}(K_n) = 0,$$

and consequently

$$\mathrm{rank}(\widetilde{L}(K_n)) = 0.$$

The complete graph K_n contradicts (3.1) only as it is written in the abstract. In the body of [1], Lemma 3 and Theorem 2 already assume $G \ncong K_n$, so they are not affected. Only the abstract lacks this hypothesis.

The rank can be stated for every connected graph in terms of the pdb edges. This is the standard incidence-matrix description of the rank of a weighted Laplacian [2], and we include it only to make the role of the pdb edges explicit.

Let F denote the set of pdb edges of G, and let $c(G)$ denote the number of connected components of the spanning subgraph $(V, E \setminus F)$.

Proposition 1. *Let G be a connected graph of order n. Then*

$$\mathrm{rank}(\widetilde{L}) = n - c(G), \qquad \mathrm{nullity}(\widetilde{L}) = c(G).$$

In particular, $\mathrm{rank}(\widetilde{L}) = n - 1$ *if and only if* $(V, E \setminus F)$ *is connected.*

Proof. Choose an arbitrary orientation of G and let R be the associated incidence matrix, so that for each edge $e = uv$ oriented $u \to v$ the corresponding column of R has $\sqrt{w_e}$ at position u, $-\sqrt{w_e}$ at position v, and 0 elsewhere. By the definition of $\widetilde{L}$, $\widetilde{L} = RR^T$. Columns of R corresponding to pdb edges are zero, so $\operatorname{rank}(R) = \operatorname{rank}(R')$, where R' is obtained by deleting these zero columns, i.e., R' is the weighted incidence matrix of $(V, E\setminus F)$ on which all weights w_e are positive. By the standard rank formula for a weighted incidence matrix, $\operatorname{rank}(R') = n - c(G)$. Hence $\operatorname{rank}(\widetilde{L}) = n - c(G)$, and the rank–nullity theorem gives $\operatorname{nullity}(\widetilde{L}) = c(G)$. □

Remark 1. For $G = K_n$ every edge is pdb, so $c(G) = n$ and $\operatorname{rank}(\widetilde{L}) = 0$, as computed above. That $(V, E \setminus F)$ is connected whenever $G \ncong K_n$ is the content of Lemma 3 of [1], which we do not examine here.

3.2. Theorem 5 and Theorem 6(ii): SDB graphs

Theorem 5 of [1] claims that if G is strongly distance-balanced (SDB), then

$$\theta = \frac{r\sqrt{2d-2}}{d} \tag{3.2}$$

is an eigenvalue of $\widetilde{A}$, where G is r-regular and "the contribution of each edge is d". Theorem 6(ii) further claims that if G is SDB, then

$$\theta_i = \frac{\sqrt{2d-2}}{d}\,\lambda_i \tag{3.3}$$

for every adjacency eigenvalue λ_i.

Both statements are not correct. The error is the implicit assumption that every edge of an SDB graph has the same value of n_u (and hence the same weight w_e). SDB forces G to be regular (Section 2), but it does not force n_u to be constant across edges.

An SDB graph is distance-balanced, so on each edge e we have $n_u(e) = n_v(e) =: d_e$ and $w_e = \sqrt{2d_e - 2}/d_e$, but d_e may change from one edge to another. Both (3.2) and (3.3) assume that $d_e = d$ for all edges, that is, that the weights are constant and $\widetilde{A} = cA$. This assumption is what fails; it is not just a matter of a wrong constant in the formula.

Example 1 (Counterexample to (3.2) and (3.3)). Let $G = C_4[K_2]$ be the graph obtained from the 4-cycle by replacing each vertex by a pair of adjacent vertices and joining every vertex of one pair to every vertex of the adjacent pairs. Then:

- G has $n = 8$ vertices and is 5-regular.

- G is SDB. Its edges split into two orbits with the following parameters:

edge type	(n_u, n_v)
inside a fiber	$(1, 1)$
between consecutive fibers	$(3, 3)$

  A direct check of the distance layers for both orbits verifies the SDB condition.

- There are 4 inside-pair edges, each of weight $w_e = 0$, and 16 cross-pair edges, each of weight

$$w_e = \sqrt{\frac{3+3-2}{3\cdot 3}} = \frac{2}{3}.$$

The all-ones vector gives $\widetilde{A}\mathbf{j} = \frac{8}{3}\mathbf{j}$, so $\theta_1 = \frac{8}{3}$. In fact the full spectrum of $\widetilde{A}$ is

$$\mathrm{Spec}(\widetilde{A}) = \Big\{-\frac{8}{3},\, 0^{(6)},\, \frac{8}{3}\Big\}.$$

Since G is 5-regular ($r = 5$) and the two edge orbits have $d = 1$ and $d = 3$ respectively, formula (3.2) can only be tested with these values:

$$d = 1: \quad \frac{5\sqrt{0}}{1} = 0, \qquad\qquad d = 3: \quad \frac{5\sqrt{4}}{3} = \frac{10}{3} \approx 3.333,$$

and neither value equals $\theta_1 = \frac{8}{3} \approx 2.667$. So (3.2) is wrong. Theorem 5(i) of [1] asserts only that the number in (3.2) is an eigenvalue, not that it is the largest one, so a comparison with θ_1 alone is not sufficient and we compare with the whole spectrum. For $d = 1$ the predicted value 0 does occur in $\mathrm{Spec}(\widetilde{A})$, but with multiplicity 6, contrary to Theorem 5(ii). For $d = 3$ the predicted value $10/3$ does not occur in the spectrum displayed above. Consequently (3.3) is also wrong, because $\widetilde{A} \neq cA$ for any constant c: the scalarization criterion of Proposition 3 below fails for G (the edge weights take the two distinct values 0 and $2/3$).
The failure of (3.3) can also be seen from the spectra, since $\widetilde{A} \neq cA$ alone does not exclude proportional spectra. Here

$$\mathrm{Spec}(A(C_4[K_2])) = \{5,\, 1^{(2)},\, (-1)^{(4)},\, -3\},$$

whereas $\widetilde{A}$ is nonzero and has nullity 6. A scalar multiple cA with $c \neq 0$ has no eigenvalue 0, and $c = 0$ would give $\widetilde{A} = 0$. Hence no scalar multiple of the adjacency spectrum equals the spectrum of $\widetilde{A}$, which gives the required spectral contradiction.

Remark 2 (Correct version of Theorem 5). A sufficient condition for the scalarization $\widetilde{A} = cA$ is edge-transitivity. Strong distance-balance alone is not sufficient, as Example 1 shows. More precisely, if G is edge-transitive, then there exist positive integers a, b such that

$$\{n_u(e \mid G), n_v(e \mid G)\} = \{a, b\}$$

for every edge $e \in E(G)$. Consequently all edge weights are equal to

$$c = \sqrt{\frac{a+b-2}{ab}},$$

and hence $\widetilde{A}(G) = cA(G)$, so

$$\theta_i = c\lambda_i \qquad (1 \le i \le n).$$

If G is connected and $c > 0$, then $\theta_1 = c\,\lambda_1$ is the spectral radius of $\widetilde{A}$ and has multiplicity one. For an r-regular edge-transitive graph this gives

$$\theta_1 = r\sqrt{\frac{a+b-2}{ab}}.$$

The star $K_{1,n-1}$ shows why one cannot replace $\{a, b\}$ by $\{d, d\}$ in general: for the star, $\{a, b\} = \{1, n-1\}$ on every edge. The complete graph K_n shows that the hypothesis $c > 0$ is necessary; there $a = b = 1$ and $c = 0$, so $\widetilde{A}(K_n) = 0$ and no positive eigenvalue exists.

Remark 3. Edge-transitivity is a natural *sufficient* condition for $\widetilde{A} = cA$, whereas Proposition 3 below gives the exact necessary and sufficient condition.

3.3. The spectrum of K_4/e

After Theorem 6, the authors of [1] consider the graph $\mathcal{K} = K_4/e$ (the diamond graph: K_4 with one edge deleted) and claim that its $\widetilde{A}$-spectrum is

$$\{[-2]^1,\ [0]^2,\ [2]^1\}. \tag{3.4}$$

The notation K_4/e usually means contraction of e, but here, as in [1], it denotes the diamond graph, that is, K_4 with the edge e deleted. We write $K_4 - e$ for it below and also treat the contraction.

Example 2 (Counterexample to (3.4)). Label the vertices of the diamond $\mathcal{K} = K_4 - e$ by $1, 2, 3, 4$ with edges $12, 23, 34, 41, 24$, so that $1, 3$ have degree 2 and $2, 4$ have degree 3. On each rim edge (say 12) we have $n_u = 1$, $n_v = 2$ (vertex 1 is closer only to itself, vertex 2 is closer to itself and to 3), so

$$w_{12} = w_{23} = w_{34} = w_{41} = \sqrt{\frac{1+2-2}{1\cdot 2}} = \frac{1}{\sqrt{2}}.$$

On the diagonal edge 24, however, vertices 1 and 3 are equidistant from 2 and from 4, so neither is counted for either endpoint; hence $n_u = n_v = 1$ and 24 is a pdb edge with $w_{24} = 0$. Consequently

$$\widetilde{A}(\mathcal{K}) = \frac{1}{\sqrt{2}}\,A(C_4),$$

where C_4 is the rim 1-2-3-4-1 (the zero weight on 24 contributes nothing to $\widetilde{A}$), and since $\mathrm{Spec}(A(C_4)) = \{2, 0, 0, -2\}$ we get

$$\mathrm{Spec}(\widetilde{A}) = \Big\{\sqrt{2},\ 0,\ 0,\ -\sqrt{2}\Big\},$$

not (3.4). If instead K_4/e is read as K_4 with one edge *contracted*, the resulting graph is K_3, whose $\widetilde{A}$-spectrum is $\{0, 0, 0\}$, again not (3.4).

3.4. Theorem 7(ii)

Theorem 7(ii) of [1] claims that if G is bipartite, then

$$\sum_{i=1}^{n} \theta_i^2 = 2(n-2)\mathrm{RSz}(G). \tag{3.5}$$

Remark 4. This is not correct. We have

$$\mathrm{tr}(\widetilde{A}^2) = \sum_i \theta_i^2 = 2 \sum_{uv \in E} \frac{n_u + n_v - 2}{n_u\, n_v}.$$

For bipartite G, Lemma 1(ii) gives $n_u + n_v = n$, so

$$\sum_i \theta_i^2 = 2(n-2) \sum_{uv \in E} \frac{1}{n_u\, n_v} = (n-2)\mathrm{RSz}(G),$$

because $\mathrm{RSz}(G) = 2\sum_{uv \in E} 1/(n_u\, n_v)$. This only uses $\mathrm{tr}(\widetilde{A}^2)$ and $n_u + n_v = n$.

3.5. Theorem 7(iii) and Theorem 7(iv)

Theorem 7(iii) of [1] claims that if G has no pdb edge, then

$$\mathcal{A}(G) \le \sum_{e \in E} \sqrt{\frac{n_u + n_v - 2}{n_u + n_v}}. \tag{3.6}$$

Theorem 7(iv) claims that if G is bipartite without pdb edge, then

$$\mathcal{A}(G) \le \frac{\sqrt{n-2}}{n}\, m. \tag{3.7}$$

Example 3 (Counterexample to (3.6) and (3.7)). Take $G = S_4 = K_{1,3}$. Every edge has $(n_u, n_v) = (1, 3)$, so S_4 has no pdb edge. Then

$$\mathcal{A}(S_4) = 3\sqrt{\frac{2}{3}} = \sqrt{6} \approx 2.449.$$

For (3.6) the right-hand side is

$$3\sqrt{\frac{1+3-2}{1+3}} = 3\sqrt{\frac{1}{2}} = \frac{3}{\sqrt{2}} \approx 2.121 < \sqrt{6}.$$

For (3.7) the right-hand side is

$$\frac{\sqrt{2}}{4} \cdot 3 = \frac{3\sqrt{2}}{4} \approx 1.061 < \sqrt{6}.$$

So both inequalities fail.

The argument in [1] uses the inequality $n_u + n_v \le n_u n_v$. For positive integers a, b, $a + b \le ab$ is equivalent to $(a-1)(b-1) \ge 1$; it holds if $a, b \ge 2$, but fails when one of them equals 1. For $(n_u, n_v) = (1, 3)$ the inequality is reversed since $4 > 3$. The hypotheses of the published statements (no pdb edge, bipartite) do not give $a, b \ge 2$, which the proof needs, and the star shows that the conclusion itself is false. A valid (if much weaker) replacement is the bound $\mathcal{A}(G) < m$ of Proposition 4 below.

3.6. Theorem 7(v) and Corollary 1

Theorem 7(v) of [1] claims that for bipartite G with n odd,

$$\sqrt{\frac{n_u + n_v - 2}{n_u\, n_v}} \ge \frac{2\sqrt{n-2}}{n-1}. \tag{3.8}$$

Example 4 (Counterexample to (3.8)). Take $G = K_{2,3}$ with $n = 5$. For every edge, $n_u = 3$ and $n_v = 2$ (or vice versa), so

$$\sqrt{\frac{3+2-2}{3\cdot 2}} = \sqrt{\frac{1}{2}} \approx 0.707.$$

The right-hand side of (3.8) is

$$\frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \approx 0.866 > 0.707.$$

So (3.8) is false.

The correct bound for odd n uses $\max ab = (n^2-1)/4$, not $\lfloor n/2 \rfloor^2$:

Proposition 2. *Let G be a bipartite graph of odd order n with $n \ge 3$. Then, for every edge $uv \in E(G)$,*

$$\sqrt{\frac{n_u + n_v - 2}{n_u\, n_v}} \ge \frac{2\sqrt{n-2}}{\sqrt{n^2-1}}.$$

Consequently,

$$\mathcal{A}(G) \ge \frac{2m\sqrt{n-2}}{\sqrt{n^2-1}}.$$

Both bounds are sharp: equality holds in the edgewise bound when $\{n_u, n_v\} = \{(n-1)/2, (n+1)/2\}$, and in the index bound if and only if this holds on every edge of G. The graph $K_{(n-1)/2,(n+1)/2}$ is one such graph, but not the only one: for $n = 7$ there are two further extremal graphs. The first is obtained from C_6 by adding a false twin of one vertex, that is, a new vertex adjacent to the two cycle neighbours of that vertex; it has 8 edges. The second is the cube Q_3 with one vertex deleted; it has 9 edges. Both graphs are connected and bipartite and have $\{n_u, n_v\} = \{3, 4\}$ on every edge, so they attain equality in both bounds.

Proof. Since G is bipartite, Lemma 1(ii) gives $n_u + n_v = n$ for every edge. Since n is odd and n_u, n_v are positive integers with $n_u + n_v = n$, we have

$$n_u\, n_v \le \frac{n^2-1}{4},$$

with equality iff $\{n_u, n_v\} = \{(n-1)/2, (n+1)/2\}$. Therefore

$$\sqrt{\frac{n_u + n_v - 2}{n_u\, n_v}} = \sqrt{\frac{n-2}{n_u\, n_v}} \ge \frac{2\sqrt{n-2}}{\sqrt{n^2-1}}.$$

Summing over all m edges proves the second inequality. For $G = K_{(n-1)/2,(n+1)/2}$ the parameters on every edge are $n_u = (n+1)/2$, $n_v = (n-1)/2$ (up to relabeling), and

$$m = \frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4},$$

so equality holds throughout. For the index bound, equality holds exactly when it holds on every edge. □

Corollary 1 of [1], which is derived from Theorem 7(v), inherits the error.

3.7. Theorem 8(a)

Theorem 8(a) of [1] claims that for every tree T on $n \ge 7$ vertices,

$$\theta_1 \ge \frac{2\pi\sqrt{n-2}}{n}. \tag{3.9}$$

Example 5 (Counterexample to (3.9)). Take $T = P_7$. For the path P_7 the weights of the six edges are

$$\sqrt{\tfrac{5}{6}},\ \sqrt{\tfrac{1}{2}},\ \sqrt{\tfrac{5}{12}},\ \sqrt{\tfrac{5}{12}},\ \sqrt{\tfrac{1}{2}},\ \sqrt{\tfrac{5}{6}}.$$

The largest row sum of $\widetilde{A}$ is attained at vertex 2 (or 6) and equals

$$\sqrt{\tfrac{5}{6}} + \sqrt{\tfrac{1}{2}} \approx 1.620.$$

Since $\widetilde{A}$ is a nonnegative matrix,

$$\rho(\widetilde{A}) \le \max_i \sum_j \widetilde{A}_{ij},$$

so $\theta_1 \le 1.620$. On the other hand, the right-hand side of (3.9) is

$$\frac{2\pi\sqrt{5}}{7} \approx 2.007 > 1.620.$$

So (3.9) is false.

The claimed bound appears to result from treating the asymptotic relation $\mathcal{A}(P_n) \sim \pi\sqrt{n-2}$ (see [3]) as an exact identity. An asymptotic relation cannot give an inequality at the fixed value $n = 7$.

3.8. Theorem 8(b)

Theorem 8(b) claims that for a unicycle bipartite graph G with $n \geq 8$,

$$\theta_1 \geq \frac{4\sqrt{n-3}}{n-1}. \tag{3.10}$$

Example 6 (Counterexample to (3.10)). Take $G = C_8$. Every edge has weight $\sqrt{6}/4$, so the row sum at every vertex is $\sqrt{6}/2 \approx 1.225$, hence

$$\theta_1 = \frac{\sqrt{6}}{2} \approx 1.225.$$

The right-hand side of (3.10) is

$$\frac{4\sqrt{5}}{7} \approx 1.278 > 1.225.$$

So (3.10) is false.

3.9. A remark on Theorem 4 of the original paper

Theorem 4 of [1] states that no connected graph has exactly two distinct $\widetilde{A}$-eigenvalues. Its proof uses that the rows of $\widetilde{A}$ at the two endpoints of a pdb edge are equal. This does not follow directly from the definition, so we prove it. We have not checked the rest of the proof.

Lemma 2. *Let uv be a pdb edge of a connected graph G. Then $N[u] = N[v]$ (so u, v are adjacent twins), the transposition $(u\,v)$ is an automorphism of G, and the rows of $\widetilde{A}$ indexed by u and v are equal.*

Proof. Since $n_u = n_v = 1$, the only vertex closer to u than to v is u itself, and the only vertex closer to v is v; hence $d(x,u) = d(x,v)$ for every $x \notin \{u,v\}$. If $x \neq v$ is adjacent to u, then $d(x,v) = d(x,u) = 1$, so x is adjacent to v, and conversely. Thus $N[u] = N[v]$, and swapping u and v preserves adjacency, i.e. it is an automorphism σ of G. Automorphisms preserve all numbers $n_u(e \mid G)$, so $w_{\sigma(e)} = w_e$; in particular $w_{ux} = w_{vx}$ for every $x \in N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Finally $\widetilde{A}_{uv} = w_{uv} = 0$ and $\widetilde{A}_{uu} = \widetilde{A}_{vv} = 0$, so rows u and v agree in all columns. □

4. Further results

The counterexamples of the previous section show that the bipartite condition alone does not force the edge weights of the Graovac–Ghorbani matrix to be constant. For example, the path P_4 already has two different edge weights. Nevertheless, several exact structural and spectral results remain valid and are developed in this section.

Subsections 4.1–4.2 (constant weights, an elementary domination by A, complete bipartite graphs) follow from the definition of $\widetilde{A}$ and are used as tools for Section 3; in particular Proposition 4 supplies a valid replacement for the false bounds of Theorem 7(iii)–(iv). Subsection 4.3 gives a lower bound that follows from the corrected identity of Theorem 7(ii) (Section 3.4). In Subsection 4.4, the bound $\rho(\widetilde{A}) \le \sqrt{n-2}$ and its equality case are Theorem 9 of [1]; what we add is the entrywise inequality $\widetilde{A} \le \sqrt{n-2}\, D^{-1/2} A D^{-1/2}$, which implies it, and a proof of the equality case. Subsection 4.5 gives a corollary for the index, and Subsection 4.6 a lower bound that we compare with known ones on examples.
For an edge $e = uv$ of a connected graph G put

$$w_e = \sqrt{\frac{n_u(e \mid G) + n_v(e \mid G) - 2}{n_u(e \mid G)\, n_v(e \mid G)}},$$

so that $\widetilde{A}_{uv} = w_{uv}$ for $uv \in E(G)$ and $\widetilde{A}_{uv} = 0$ otherwise.

4.1. The constant-weight principle and the scalarization criterion

Lemma 3 (Constant-weight principle). *Let G be a connected graph. If there exists a constant $c \ge 0$ such that $w_e = c$ for every $e \in E(G)$, then*

$$\widetilde{A}(G) = cA(G), \qquad \widetilde{L}(G) = cL(G).$$

Consequently,

$$\operatorname{Spec}(\widetilde{A}) = c \operatorname{Spec}(A), \qquad \operatorname{Spec}(\widetilde{L}) = c \operatorname{Spec}(L).$$

Proof. By assumption, every nonzero entry of $\widetilde{A}$ equals c, hence $\widetilde{A} = cA$. The weighted degree of u is $\widetilde{d}(u) = \sum_{v \sim u} \widetilde{A}_{uv} = c\, d(u)$, so $\widetilde{D} = cD$ and therefore $\widetilde{L} = \widetilde{D} - \widetilde{A} = c(D - A) = cL$. Multiplying a matrix by the nonnegative scalar c multiplies every eigenvalue by c. □

Proposition 3 (Scalarization criterion). *Let G be a connected graph. Then $\widetilde{A}(G) = cA(G)$ for some constant $c \ge 0$ if and only if the quantity*

$$\frac{n_u(e \mid G) + n_v(e \mid G) - 2}{n_u(e \mid G)\, n_v(e \mid G)}$$

is independent of the edge $e \in E(G)$. In this case $\rho(\widetilde{A}) = c\, \rho(A)$.

Proof. If $\widetilde{A} = cA$, then every edge uv satisfies $\widetilde{A}_{uv} = c$, so the displayed ratio equals c^2 on every edge. Conversely, if this ratio equals γ on every edge, then $w_e = \sqrt{\gamma}$ for every edge, and Lemma 3 gives $\widetilde{A} = c\, A$ with $c = \sqrt{\gamma}$, whence $\rho(\widetilde{A}) = c\rho(A)$. □

Proposition 3 restates the definition of $\widetilde{A}$. It shows that SDB is not sufficient for $\widetilde{A} = cA$ (Example 1), while edge-transitivity is (Remark 2). Complete bipartite graphs satisfy the criterion (Proposition 5), and many regular graphs fail it (Example 1).

Proposition 4. *Let G be a connected graph with at least one edge. Then $0 \le w_e < 1$ for every edge e, with $w_e = 0$ if and only if e is pdb. Consequently $\widetilde{A} \le A$ entrywise, with strict inequality on every edge (so $\widetilde{A} \ne A$), and*

$$\rho(\widetilde{A}) < \rho(A), \qquad \mathcal{A}(G) < m.$$

Proof. For an edge $e = uv$ put $a = n_u$, $b = n_v$. Then $ab - (a + b - 2) = (a - 1)(b - 1) + 1 > 0$, so $w_e^2 = \frac{a+b-2}{ab} < 1$, and $w_e = 0$ iff $a = b = 1$, i.e. iff e is pdb. Thus $0 \le w_e < 1 = A_{uv}$ on every edge, and both matrices vanish off the edge set. Since G is connected, A is irreducible, so the strict Perron–Frobenius comparison theorem [5] gives $\rho(\widetilde{A}) < \rho(A)$. Summing $w_e < 1$ over the m edges gives $\mathcal{A}(G) < m$. □

Remark 5. This is elementary. Its use here is that $\mathcal{A}(G) < m$ is a valid replacement for the false bounds (3.6) and (3.7) (for S_4: $\sqrt{6} < 3$). The constant 1 cannot be improved: for the star, $\mathcal{A}(K_{1,n-1})/m = \sqrt{(n-2)/(n-1)} \to 1$.

4.2. Exact reduction for complete bipartite graphs

Proposition 5. *Let $G = K_{a,b}$ with $a, b \ge 1$ and $n = a + b$. Then every edge uv with u in the part of size a and v in the part of size b satisfies $n_u = b$ and $n_v = a$, so*

$$\widetilde{A}(K_{a,b}) = \sqrt{\frac{n-2}{ab}}\, A(K_{a,b}), \qquad \widetilde{L}(K_{a,b}) = \sqrt{\frac{n-2}{ab}}\, L(K_{a,b}).$$

In particular,

$$\operatorname{Spec}(\widetilde{A}(K_{a,b})) = \sqrt{\frac{n-2}{ab}}\, \operatorname{Spec}(A(K_{a,b})).$$

Proof. Let X, Y be the parts of $K_{a,b}$ with $|X| = a$ and $|Y| = b$, and let $u \in X$, $v \in Y$. Every $x \in Y \setminus \{v\}$ satisfies $d(x, u) = 1 < 2 = d(x, v)$, and every $x \in X \setminus \{u\}$ satisfies $d(x, v) = 1 < 2 = d(x, u)$; moreover u is closer to u and v is closer to v. Thus the vertices closer to u are exactly those of $\{u\} \cup (Y \setminus \{v\})$, and the vertices closer to v are exactly those of $\{v\} \cup (X \setminus \{u\})$. Hence $n_u = b$, $n_v = a$, so $n_u n_v = ab$ and $n_u + n_v = n$, and

$$w_{uv} = \sqrt{\frac{a+b-2}{ab}} = \sqrt{\frac{n-2}{ab}}$$

for every edge. Apply Lemma 3. □

Corollary 1. *For $G = K_{a,b}$,*

$$\mathcal{A}(K_{a,b}) = ab\sqrt{\frac{n-2}{ab}} = \sqrt{(n-2)\,ab}.$$

In particular, for fixed n this value is maximized when the two parts are as balanced as possible.

Proof. $K_{a,b}$ has $m = ab$ edges, each of weight $\sqrt{(n-2)/(ab)}$. Summing gives the formula. □

4.3. A lower bound from the bipartite trace identity

Corollary 2. *If G is a connected bipartite graph of order n, then*

$$\rho(\widetilde{A}) \;\geq\; \sqrt{\frac{(n-2)\,\mathrm{RSz}(G)}{n}}.$$

Proof. Since $\widetilde{A}$ is symmetric, $\sum_i \theta_i^2 = \mathrm{tr}(\widetilde{A}^2) = 2\sum_{uv\in E} w_{uv}^2$, and for bipartite G this equals $(n-2)\mathrm{RSz}(G)$ by Section 3.4. Since $\rho(\widetilde{A})^2 \geq \frac{1}{n}\sum_i \theta_i^2$ (as $\rho(\widetilde{A}) = \theta_1 = \|\widetilde{A}\|_2$ and $\|\widetilde{A}\|_F^2 \leq n\|\widetilde{A}\|_2^2$), the claim follows. □

4.4. A normalized-adjacency domination

The inequality $\rho(\widetilde{A}) \leq \sqrt{n-2}$ for connected bipartite graphs, with equality only for complete bipartite graphs, is Theorem 9 of [1]. Its proof there uses the test vector $(\sqrt{d_1},\ldots,\sqrt{d_n})^T$ and the same estimate $n_u \geq d_u$, $n_v \geq d_v$ that we use. Here the estimate is written as an entrywise inequality between matrices. This is stronger than the spectral bound and also gives the bound for the index in Corollary 3.

Theorem 1. *Let G be a connected bipartite graph of order $n \geq 3$. Then, entrywise,*

$$\widetilde{A}(G) \;\leq\; \sqrt{n-2}\, D^{-1/2} A(G) D^{-1/2},$$

where D is the (ordinary) degree matrix of G. Consequently,

$$\rho(\widetilde{A}(G)) \;\leq\; \sqrt{n-2}.$$

Equality holds if and only if G is complete bipartite. (The spectral statement is Theorem 9 of [1].)

Proof. Step 1 (edgewise estimate). Let X, Y be the two parts of G, and let $uv \in E(G)$ with $u \in X$ and $v \in Y$. By Lemma 1(ii), $n_u + n_v = n$, and hence $w_{uv}^2 = \frac{n-2}{n_u\, n_v}$.

Every neighbor $w \in Y$ of u with $w \neq v$ satisfies $d(w,u) = 1$. Since $w, v \in Y$, the distance $d(w,v)$ is even; as $w \neq v$, we have $d(w,v) \geq 2$. Thus $d(w,u) < d(w,v)$. Together with u itself, this shows that

$$n_u \geq d_u.$$

Similarly,

$$n_v \geq d_v.$$

Consequently,

$$w_{uv}^2 = \frac{n-2}{n_u\, n_v} \leq \frac{n-2}{d_u d_v} = (n-2)\big(D^{-1/2}AD^{-1/2}\big)_{uv}^2.$$

Since both sides are nonnegative,

$$\widetilde{A}_{uv} = w_{uv} \leq \sqrt{n-2}\,\big(D^{-1/2}AD^{-1/2}\big)_{uv}.$$

Thus, $\widetilde{A}(G) \leq \sqrt{n-2}\, D^{-1/2}A(G)D^{-1/2}$ entrywise.
Step 2 (spectral radius). Now let $N := D^{-1/2}AD^{-1/2}$. Since G is connected, N is irreducible. Moreover,

$$N\sqrt{d} = \sqrt{d}, \qquad \sqrt{d} = (\sqrt{d_1}, \ldots, \sqrt{d_n})^T > 0.$$

Hence, by the Perron–Frobenius theorem, $\rho(N) = 1$. The entrywise inequality above therefore gives $\rho(\widetilde{A}) \leq \sqrt{n-2}\,\rho(N) = \sqrt{n-2}$.
Step 3 (equality forces $n_u = d_u$, $n_v = d_v$ on every edge). Suppose that $\rho(\widetilde{A}) = \sqrt{n-2}$. Since N is irreducible and $0 \leq \widetilde{A} \leq \sqrt{n-2}\, N$ entrywise, the strict Perron–Frobenius comparison theorem (if $0 \leq B \leq C$ entrywise, C irreducible and $B \neq C$, then $\rho(B) < \rho(C)$; see [5]) implies $\widetilde{A} = \sqrt{n-2}\, N$. Therefore, for every edge uv,

$$\frac{n-2}{n_u\, n_v} = \frac{n-2}{d_u d_v}, \qquad \text{hence} \qquad n_u\, n_v = d_u d_v.$$

Since $n_u \geq d_u$ and $n_v \geq d_v$, it follows that $n_u = d_u$ and $n_v = d_v$ for every edge uv.
Step 4 (equality forces diameter at most 2). Suppose, to the contrary, that $\operatorname{diam}(G) \geq 3$. Choose a shortest path $u - z - w - y$ with $d(u,y) = 3$, and consider the edge uz. By Step 3 applied to this edge, $n_z(uz) = d_z$. On the other hand, the proof of Step 1 (with the roles of the endpoints exchanged) shows that the d_z vertices consisting of z itself and the neighbors of z other than u are all counted by $n_z(uz)$; since $n_z(uz) = d_z$, the set of vertices counted by $n_z(uz)$ is *exactly* this set. But

$$d(y,z) = 2 < 3 = d(y,u),$$

so y is also counted by $n_z(uz)$, while $y \neq z$ and y is not adjacent to z (because $d(y,z) = 2$). Thus y is a vertex counted by $n_z(uz)$ that does not belong to this set, a contradiction.
Thus, $\operatorname{diam}(G) \leq 2$. A connected bipartite graph of diameter at most 2 is complete bipartite. Hence $G = K_{a,b}$.
Step 5 (converse). Conversely, if $G = K_{a,b}$, then $n_u = b$ and $n_v = a$ for every edge uv with u in the part of size a and v in the part of size b (Proposition 5); in particular $n_u n_v = ab$. Therefore,

$$\widetilde{A} = \sqrt{\frac{n-2}{ab}}\, A$$

by Proposition 5, and consequently

$$\rho(\widetilde{A}) = \sqrt{\frac{n-2}{ab}}\, \rho(A) = \sqrt{\frac{n-2}{ab}}\sqrt{ab} = \sqrt{n-2}.$$

Hence equality holds if and only if G is complete bipartite. □

4.5. A Randić-type bound for the Graovac–Ghorbani index

The next inequality is obtained by summing the edgewise estimate of Step 1 of Theorem 1. We did not find it in the literature, though our search was not exhaustive. It is known that among bipartite graphs of order n the index is largest for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ [3], which agrees with Corollary 1. The estimate $n_u \geq d_u$, $n_v \geq d_v$ for bipartite graphs is also used in the proof of Theorem 3.1 there.

Corollary 3. *Let G be a connected bipartite graph of order $n \geq 3$. Then*

$$\mathcal{A}(G) \;\leq\; \sqrt{n-2}\,\mathrm{R}(G),$$

where $\mathrm{R}(G) = \sum_{uv \in E(G)} (d_u d_v)^{-1/2}$ is the Randić index of G. Equality holds if and only if G is complete bipartite.

Proof. By Step 1 of the proof of Theorem 1, for every edge $uv \in E(G)$,

$$w_{uv} \leq \frac{\sqrt{n-2}}{\sqrt{d_u d_v}}.$$

Summing over all edges gives

$$\mathcal{A}(G) = \sum_{uv \in E} w_{uv} \leq \sqrt{n-2} \sum_{uv \in E} \frac{1}{\sqrt{d_u d_v}} = \sqrt{n-2}\,\mathrm{R}(G).$$

Equality holds if and only if $n_u = d_u$ and $n_v = d_v$ for every edge. By Steps 3–4 of Theorem 1, this is equivalent to G being complete bipartite. □

Corollary 4. *For $G = K_{a,b}$,*

$$\mathcal{A}(K_{a,b}) = \sqrt{n-2}\,\mathrm{R}(K_{a,b}) = \sqrt{(n-2)ab}.$$

In particular, the same graphs that attain the bipartite spectral-radius bound $\rho(\widetilde{A}) \le \sqrt{n-2}$ also attain the Randić-type index bound $\mathcal{A} \le \sqrt{n-2}\,\mathrm{R}$.

Proof. For $K_{a,b}$ with parts of sizes a, b, each edge has $d_u d_v = ab$, so $\mathrm{R}(K_{a,b}) = ab/\sqrt{ab} = \sqrt{ab}$, and $\sqrt{n-2}\,\sqrt{ab} = \sqrt{(n-2)ab} = \mathcal{A}(K_{a,b})$ (Corollary 1). □

4.6. A two-sided spectral bound

The upper bound below is Theorem 1; the lower bounds combine the second-moment estimate, the constant-vector Rayleigh quotient (cf. [1, Lemma 5]) and a degree-weighted Rayleigh quotient.

Theorem 2. *Let G be a connected bipartite graph of order $n \ge 3$. Then*

$$\max\left\{\sqrt{\frac{(n-2)\mathrm{RSz}(G)}{n}},\ \frac{2\mathcal{A}(G)}{n},\ \frac{\sqrt{n-2}}{m}\sum_{uv\in E(G)}\sqrt{\frac{d_u d_v}{n_u\, n_v}}\right\} \le \rho(\widetilde{A}) \le \sqrt{n-2}.$$

The upper equality holds if and only if G is complete bipartite.

Proof. The upper bound $\rho(\widetilde{A}) \le \sqrt{n-2}$ is Theorem 1. The first lower bound is Corollary 2. The second lower bound is the constant-vector bound: for the all-ones vector $\mathbf{j}$ the Rayleigh quotient gives $\rho(\widetilde{A}) \ge \mathbf{j}^T\widetilde{A}\mathbf{j}/\mathbf{j}^T\mathbf{j} = 2\sum_{uv\in E} w_{uv}/n = 2\mathcal{A}(G)/n$ (this is [1, Lemma 5]). It remains to prove the third lower bound.
For each edge $uv \in E(G)$ define

$$r_{uv} := \sqrt{\frac{d_u d_v}{n_u\, n_v}}.$$

Since $n_u \ge d_u$ and $n_v \ge d_v$ (Step 1 of the proof of Theorem 1), we have $0 < r_{uv} \le 1$. Moreover, since $n_u + n_v = n$ on every edge of a bipartite graph,

$$\widetilde{A}_{uv} = \sqrt{\frac{n-2}{n_u\, n_v}} = \sqrt{n-2}\,\frac{r_{uv}}{\sqrt{d_u d_v}}.$$

Let $x = (\sqrt{d_1}, \dots, \sqrt{d_n})^T$. Then

$$x^T x = \sum_{v\in V} d_v = 2m,$$

and

$$x^T \widetilde{A} x = 2 \sum_{uv \in E} \widetilde{A}_{uv} \sqrt{d_u d_v} = 2\sqrt{n-2} \sum_{uv \in E} r_{uv}.$$

By the Rayleigh quotient theorem,

$$\rho(\widetilde{A}) \;\geq\; \frac{x^T \widetilde{A} x}{x^T x} = \frac{2\sqrt{n-2} \sum_{uv \in E} r_{uv}}{2m} = \frac{\sqrt{n-2}}{m} \sum_{uv \in E} r_{uv}.$$

Taking the maximum of the three lower bounds gives the claim.

Equality case. If $G = K_{a,b}$, then $n_u = d_u$ and $n_v = d_v$ on every edge, so $r_{uv} = 1$ on every edge. Hence the third lower bound equals

$$\frac{\sqrt{n-2}}{m} \cdot m = \sqrt{n-2},$$

which coincides with the upper bound $\sqrt{n-2}$. Conversely, if $\rho(\widetilde{A}) = \sqrt{n-2}$, then by Theorem 1 (Steps 3–4 of its proof) $G \cong K_{a,b}$. □

Example 7 (Comparison of the lower bounds). Let $L_{\mathrm{mom}} = \sqrt{(n-2)\mathrm{RSz}(G)/n}$ be the second-moment bound, $L_{\mathrm{con}} = 2\mathcal{A}(G)/n$ the bound of [1, Lemma 5], and $L_{\mathrm{deg}} = \frac{\sqrt{n-2}}{m} \sum_{uv} \sqrt{d_u d_v/(n_u n_v)}$ the bound of Theorem 2. We compute them for the paths P_4 and P_5; $\rho(\widetilde{A})$ comes from the characteristic polynomial of the weighted path.

P_4: $(n_u, n_v) = (1,3), (2,2), (3,1)$, so the weights are $\sqrt{2/3}$, $1/\sqrt{2}$, $\sqrt{2/3}$ and $\rho(\widetilde{A}) = \sqrt{(11+\sqrt{57})/12} \approx 1.2433$.

P_5: $(n_u, n_v) = (1,4), (2,3), (3,2), (4,1)$, so the weights are $\sqrt{3}/2$, $1/\sqrt{2}$, $1/\sqrt{2}$, $\sqrt{3}/2$ and $\rho(\widetilde{A}) = \sqrt{7}/2 \approx 1.3229$.

G	L_{mom}	L_{con}	L_{deg}	$\rho(\widetilde{A})$	$\sqrt{n-2}$
P_4	0.9574	1.1700	1.2412	1.2433	1.4142
P_5	1.0000	1.2585	1.3195	1.3229	1.7321

In both cases L_{deg} is larger than L_{mom} and L_{con} and within 0.3% of $\rho(\widetilde{A})$. For P_6 the same holds ($L_{\mathrm{deg}} \approx 1.3383$, $\rho \approx 1.3398$).

The ordering of the lower bounds is not universal. For the path P_n put

$$w_i = \sqrt{\frac{n-2}{i(n-i)}} \qquad (1 \leq i \leq n-1).$$

Then

$$L_{\mathrm{con}} = \frac{2}{n} \sum_{i=1}^{n-1} w_i, \qquad L_{\mathrm{deg}} = \frac{2\sum_{i=1}^{n-1} w_i - 2(2-\sqrt{2})\, w_1}{n-1}.$$

For P_{20} one obtains $L_{\mathrm{mom}} \approx 0.799120$ and

$$L_{\mathrm{deg}} \approx 1.050290 < L_{\mathrm{con}} \approx 1.054792.$$

Thus L_{deg} can improve on the constant-vector bound L_{con}, but it can also be weaker; this is why L_{con} is included in the maximum in Theorem 2.

Remark 6. L_{deg} is not always strictly the best of the three. If G is regular, then $(\sqrt{d_i})$ is proportional to the all-ones vector, so $L_{\mathrm{deg}} = L_{\mathrm{con}}$, regardless of whether the Graovac–Ghorbani edge weights are constant. If in addition all edge weights are equal, this common value equals $\rho(\widetilde{A})$ (for example $G = C_{2k}$), while L_{mom} is smaller. Consequently an improvement of L_{deg} over L_{con} can only occur for irregular graphs, as in Example 7; for irregular graphs L_{deg} can also be weaker than L_{con} (the path P_{20}). The bound L_{deg} comes from a standard Rayleigh-quotient argument.

4.7. A conceptual distinction

The two central phenomena established above should not be conflated.

1. The identity $\widetilde{A} = cA$ is an *edge-weight uniformity phenomenon*: by Proposition 3 it holds if and only if $(n_u + n_v - 2)/(n_u n_v)$ is constant on $E(G)$.
2. The inequality $\rho(\widetilde{A}) \le \sqrt{n-2}$ for connected bipartite graphs is a *global spectral inequality*: by Theorem 1 it holds without any uniformity assumption on the edge weights (this inequality is Theorem 9 of [1]).

This distinction explains why a scalar reduction of $\widetilde{A}$ cannot be used as a proof of the general bipartite upper bound, and vice versa. It also explains, in hindsight, why Theorem 5 of [1] fails for merely SDB graphs (Example 1): SDB guarantees regularity but not the edge-weight uniformity that item 1 requires, whereas item 2 needs no such uniformity at all.

5. Conclusion

We have examined the main spectral statements of [1] and identified several errors, for each of which we provided a connected counterexample.
The corrections concern the SDB claims (Theorems 5 and 6(ii)), the spectrum of the diamond graph, Theorem 7(ii)–(v) with Corollary 1, and Theorem 8(a),(b). For the rank, the abstract of [1] omits the hypothesis $G \ncong K_n$ that Lemma 3 and Theorem 2 contain, so K_n does not affect the body of that paper. We also proved the equality of rows at the endpoints of a pdb edge used in Theorem 4 (Lemma 2).
In Section 4, the bound $\rho(\widetilde{A}) \le \sqrt{n-2}$ and its equality case are Theorem 9 of [1]; we have written it as the entrywise inequality $\widetilde{A} \le \sqrt{n-2}\, D^{-1/2} A D^{-1/2}$. The bound $\mathcal{A}(G) \le \sqrt{n-2}\,\mathrm{R}(G)$ follows from the same estimate. The elementary inequality $\mathcal{A}(G) < m$ (Proposition 4) replaces the false bounds of Theorem 7(iii)–(iv). The lower bound L_{deg} improves on L_{mom} and L_{con} for the paths P_4 and P_5.

Conflict of Interest: The authors declare that they have no conflict of interest.

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