

On detour edge cover pebbling number for some classes of graphs

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Abstract: Graph pebbling is a technique used to optimize the movement of resources through a network, focusing on efficient transit and allocation. Given a connected graph G , an edge pebbling move involves the removal of two pebbles from one edge and the addition of one pebble to an adjacent edge, while the other pebble will be discarded from the play. In this paper, we carry out the edge pebbling move in the concepts of detour pebbling and cover pebbling to arrive a new graph invariant called the detour edge cover pebbling number. We compute the edge detour distance between two edges with repeated vertices. An edge detour path between e_u and e_v is a sequence of edges $P_e^* = \{e_u, e_1, e_2, \dots, e_k, e_v\}$ such that $e_i \cap e_{i+1} \neq \emptyset$, no edge appears more than once in the sequence, and repeated vertices are allowed. Thus, the edge detour distance between e_u and e_v , denoted by $d^*(e_u, e_v) = |P_e^*| - 1$, where $|P_e^*|$ is the total number of edges in the edge detour path. The detour edge cover pebbling number, $f_{ec}^*(G)$, is the minimum number of pebbles required to cover all the edges of G with at least one pebble each on them after a sequence of pebbling moves by using edge detour paths. The strategy of detour edge cover pebbling number is applied to some standard graphs and staircase graphs $S_{1,n}$, $n \geq 4$. Also, we give an example that offers grounds for the results and some applications are being discussed.

Keywords: Edge pebbling move, edge detour path, edge detour distance, detour edge cover pebbling number, staircase graph.

AMS Subject classification: 05C12, 05C38, 05C57, 05C90

1. Introduction

Graph pebbling is a combinatorial game on graphs with practical applications in optimizing computer processes, such as reducing memory traffic, solving register allocation, and efficient data transmission. The pebbling move involves analyzing the cost

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of ‘pebble loss’ and has been extensively researched to establish computational lower bounds on graphs. Graph pebbling is one of the developing concepts in Graph theory and in which many of the fundamentals have been studied. For further reading on Graph theory concepts, readers can refer to [20]. The concept of pebbling in graphs was first mentioned by F.R.K. Chung [5] in 1989. Many of the authors did a detailed study on the fundamentals of graph pebbling, and a clear view of pebbling concept is explained by G. Hurlbert [9].

The pebbling number, $f(G)$, [5] is the minimum number of pebbles that are sufficient to reach any target vertex with one pebble irrespective of the initial configuration of the pebbles. The pebbling problem can be likened to transmitting data from a source node to a destination node.

There are many different types of pebbling concepts have been studied, including optimal pebbling, t-pebbling, cover pebbling and monophonic pebbling, please refer to [2, 8, 13, 16]. In 2020, P. Paul extended the concept of pebbling by finding the edge pebbling number for various kinds of graphs. Given a connected graph G , an edge pebbling move involves the removal of two pebbles from one edge and the addition of one pebble to an adjacent edge, while the other pebble will be discarded from the play. P. Paul has also introduced the covering cover edge pebbling number in graphs. To find more about these concepts, please refer to [15].

Topics relevant to distances in graphs are widely investigated in the literature, and a large number of its applications to real life problems are also found in the literature. For instance, Euclidean distance [6] is used in many of the Machine Learning Algorithms. Here, we use edge detour distance in graphs. Chartrand et al. [3] have investigated the fundamental concepts of detour distance in graphs. For any pair of vertices $x, y \in V(G)$, the detour distance $d^*(x, y)$ is the length of the longest x - y path between x and y in G . From [7], we compute the edge detour distance between two edges with repeated vertices. Let $G = (V, E)$ be a connected graph, and let $e_u, e_v \in E(G)$. An edge detour path between e_u and e_v is a sequence of edges $P_e^* = \{e_u, e_1, e_2, \dots, e_k, e_v\}$ such that $e_i \cap e_{i+1} \neq \emptyset$, no edge appears more than once in the sequence, and repeated vertices are allowed. Thus, the edge detour distance between e_u and e_v , $d^*(e_u, e_v) = |P_e^*| - 1$, where $|P_e^*|$ is the total number of edges in the edge detour path. Hurlbert et al. [10] have studied the secure vertex cover pebbling number in graphs. Lourdusamy et al. [14] have introduced the detour pebbling number using a detour path in any connected graph. The detour pebbling number, $f^*(G)$, is the minimum number such that these pebbles are placed on the vertices of G , it is possible to shift a pebble to a target vertex by a sequence of pebble moves regardless of the beginning configuration using a detour path. For the extension of detour pebbling concepts, please refer to [4, 11]. Vincylin et al. [18, 19] have introduced the detour edge pebbling number, $f_e^*(G)$, in graphs.

In this paper, by considering the concepts of vertex cover, detour pebbling and edge pebbling move, we define the detour edge cover pebbling number and compute the detour edge cover pebbling number for the path graph, wheel graph, n -star graph, cycle graph and staircase graphs.

The detour edge cover pebbling number, $f_{ec}^*(G)$, is the minimum number of pebbles

required to cover all the edges of G with at least one pebble each on them after a sequence of pebbling moves by using edge detour paths.

We give an example in the next section that paves the way for finding out the detour edge cover pebbling number for any graph. In section 3, we find the detour edge cover pebbling number for some standard graphs. In section 4, we discuss the staircase graph and find the detour edge cover pebbling for staircase graphs. In section 5, some authentic applications are discussed. In the last section, pre-existing limitations are addressed, and the detour edge cover pebbling concept is compared to some existing theories using a comparative interpretation table.

2. Example

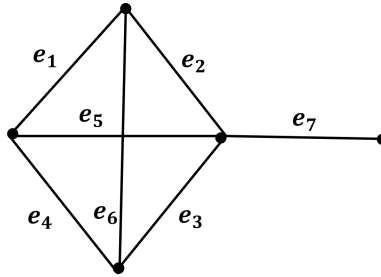


Figure 1. A connected graph G with 7 edges.

From Figure 1, let $\dot{s}_e = e_1$, where $\dot{s}_e$ is the source edge. The possible edge detour paths from e_1 to e_2 are $P_{e,1}^* : e_1, e_6, e_4, e_5, e_2$ and $P_{e,2}^* : e_1, e_5, e_3, e_6, e_2$. Thus, the edge detour distance from e_1 to e_2 is 4. Therefore, to cover e_2 , we require 16 pebbles. The edge detour distance from e_1 to e_3 , e_4 , e_5 , or e_6 is 4. Therefore, to cover e_3 , e_4 , e_5 , and e_6 using the edge detour paths $P_{e,1}^* : e_1, e_6, e_4, e_5, e_3$, $P_{e,2}^* : e_1, e_6, e_3, e_5, e_4$, $P_{e,3}^* : e_1, e_2, e_3, e_4, e_5$, and $P_{e,4}^* : e_1, e_2, e_5, e_4, e_6$, we require $4 \cdot 2^4$ pebbles. The edge detour distance from e_1 to e_7 is 5. Therefore, to cover e_7 using the edge detour path $P_{e,1}^* : e_1, e_2, e_5, e_4, e_3, e_7$, we require 32 pebbles. Moreover, to cover e_1 , we require 1 pebble. Therefore, altogether 113 pebbles are required to cover $E(G)$ using the edge detour paths. Hence $\dot{s}_e = \sum_{e_k \in E(G)} 2^{d^*(e_1, e_k)}$. Similarly, we can consider any edge to be the source edge and determine the number of pebbles needed to cover $E(G)$ using the edge detour paths. Therefore, if $\dot{s}_e = e_2, e_3, e_4, e_5$, or e_6 , we require 113 pebbles. If $\dot{s}_e = e_7$, we require 193 pebbles. Table 1 shows the edge detour distance between the edges of G , and Table 2 shows the possible edge detour paths between the edges of G and the number of pebbles needed to cover each edge from the source edges. Therefore, from Tables 1 and 2, we can conclude that the detour edge cover pebbling number for the graph G is $f_{ec}^*(G) = 193$.

Table 1. Edge detour distance between the edges of G

-	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	4	4	4	4	4	5
e_2	4	0	4	4	4	4	5
e_3	4	4	0	4	4	4	5
e_4	4	4	4	0	4	4	5
e_5	4	4	4	4	0	4	5
e_6	4	4	4	4	4	0	5
e_7	5	5	5	5	5	5	0

Table 2. Edge detour paths between the edges of G and pebbles distributed on $E(G)$

-	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$s_e = e_1$	1 pebble	16 pebbles $P_1: e_1, e_6,$ e_4, e_5, e_2 & $P_2: e_1, e_5,$ e_3, e_6, e_2	16 pebbles $P_1: e_1, e_6,$ $e_4, e_5, e_3,$ $P_2: e_1, e_2,$ e_5, e_4, e_3 & $P_3: e_1, e_4,$ e_6, e_2, e_3	16 pebbles $P_1: e_1, e_6,$ e_3, e_5, e_4 & $P_2: e_1, e_5,$ e_2, e_6, e_4	16 pebbles $P_1: e_1, e_2,$ e_3, e_4, e_5 & $P_2: e_1, e_4,$ e_6, e_2, e_5	16 pebbles $P_1: e_1, e_2,$ e_5, e_4, e_6 & $P_2: e_1, e_4,$ e_3, e_2, e_6	32 pebbles $P_1: e_1, e_2,$ $e_5, e_4,$ e_3, e_7 & $P_2: e_1, e_5,$ $e_3, e_6,$ e_2, e_7
$s_e = e_2$	16 pebbles $P_1: e_2, e_5,$ e_4, e_6, e_1 & $P_2: e_2, e_6,$ e_3, e_5, e_1	1 pebble	16 pebbles $P_1: e_2, e_5,$ e_1, e_6, e_3 & $P_2: e_2, e_6,$ e_4, e_5, e_3	16 pebbles $P_1: e_2, e_6,$ $e_3, e_5, e_4,$ $P_2: e_2, e_5,$ e_1, e_6, e_4 & $P_3: e_2, e_1,$ e_5, e_3, e_4	16 pebbles $P_1: e_2, e_1,$ e_4, e_3, e_5 & $P_2: e_2, e_3,$ e_6, e_1, e_5	16 pebbles $P_1: e_2, e_1,$ e_5, e_3, e_6 & $P_2: e_2, e_3,$ e_4, e_1, e_6	32 pebbles $P_1: e_2, e_5,$ $e_1, e_6,$ e_3, e_7 & $P_2: e_2, e_3,$ $e_6, e_1,$ e_5, e_7
$s_e = e_3$	16 pebbles $P_1: e_3, e_5,$ $e_4, e_6, e_1,$ $P_2: e_3, e_4,$ e_5, e_2, e_1 & $P_3: e_3, e_2,$ e_6, e_4, e_1	16 pebbles $P_1: e_3, e_6,$ e_1, e_5, e_2 & $P_2: e_3, e_4,$ e_5, e_6, e_2	1 pebble	16 pebbles $P_1: e_3, e_5,$ e_1, e_6, e_4 & $P_2: e_3, e_6,$ e_2, e_5, e_4	16 pebbles $P_1: e_3, e_4,$ e_1, e_2, e_5 & $P_2: e_3, e_2,$ e_6, e_4, e_5	16 pebbles $P_1: e_3, e_2,$ e_1, e_4, e_6 & $P_2: e_3, e_4,$ e_5, e_2, e_6	32 pebbles $P_1: e_3, e_5,$ $e_4, e_6,$ e_2, e_7 & $P_2: e_3, e_2,$ $e_6, e_4,$ e_5, e_7
$s_e = e_4$	16 pebbles $P_1: e_4, e_5,$ e_3, e_6, e_1 & $P_2: e_4, e_6,$ e_2, e_5, e_1	16 pebbles $P_1: e_4, e_5,$ $e_3, e_6, e_2,$ $P_2: e_4, e_6,$ e_1, e_5, e_2 & $P_3: e_4, e_3,$ e_5, e_1, e_2	16 pebbles $P_1: e_4, e_6,$ e_1, e_5, e_3 & $P_2: e_4, e_5,$ e_2, e_6, e_3	1 pebble	16 pebbles $P_1: e_4, e_3,$ e_2, e_1, e_5 & $P_2: e_4, e_1,$ e_6, e_3, e_5	16 pebbles $P_1: e_4, e_1,$ e_2, e_3, e_6 & $P_2: e_4, e_3,$ e_5, e_1, e_6	32 pebbles $P_1: e_4, e_3,$ $e_5, e_1,$ e_2, e_7 & $P_2: e_4, e_5,$ $e_2, e_6,$ e_3, e_7
$s_e = e_5$	16 pebbles $P_1: e_5, e_4,$ e_3, e_2, e_1 & $P_2: e_5, e_2,$ e_6, e_4, e_1	16 pebbles $P_1: e_5, e_3,$ e_4, e_1, e_2 & $P_2: e_5, e_1,$ e_6, e_3, e_2	16 pebbles $P_1: e_5, e_2,$ e_1, e_4, e_3 & $P_2: e_5, e_4,$ e_6, e_2, e_3	16 pebbles $P_1: e_5, e_1,$ e_2, e_3, e_4 & $P_2: e_5, e_3,$ e_6, e_1, e_4	1 pebble	16 pebbles $P_1: e_5, e_1,$ e_2, e_3, e_6 & $P_2: e_5, e_4,$ e_3, e_2, e_6	32 pebbles $P_1: e_5, e_2,$ $e_1, e_4,$ e_3, e_7 & $P_2: e_5, e_3,$ $e_4, e_1,$ e_2, e_7
$s_e = e_6$	16 pebbles $P_1: e_6, e_4,$ e_5, e_2, e_1 & $P_2: e_6, e_2,$ e_3, e_4, e_1	16 pebbles $P_1: e_6, e_3,$ e_5, e_1, e_2 & $P_2: e_6, e_1,$ e_4, e_3, e_2	16 pebbles $P_1: e_6, e_4,$ e_1, e_2, e_3 & $P_2: e_6, e_2,$ e_5, e_4, e_3	16 pebbles $P_1: e_6, e_3,$ e_2, e_1, e_4 & $P_2: e_6, e_1,$ e_5, e_3, e_4	16 pebbles $P_1: e_6, e_3,$ e_2, e_1, e_5 & $P_2: e_6, e_2,$ e_3, e_4, e_5	1 pebble	32 pebbles $P_1: e_6, e_2,$ $e_5, e_4,$ e_3, e_7 & $P_2: e_6, e_3,$ $e_5, e_1,$ e_2, e_7
$s_e = e_7$	32 pebbles $P_1: e_7, e_3,$ $e_4, e_5,$ e_2, e_1 & $P_2: e_7, e_2,$ $e_6, e_3,$ e_5, e_1	32 pebbles $P_1: e_7, e_3,$ $e_6, e_1,$ e_5, e_2 & $P_2: e_7, e_5,$ $e_1, e_6,$ e_3, e_2	32 pebbles $P_1: e_7, e_2,$ $e_6, e_4,$ e_5, e_3 & $P_2: e_7, e_5,$ $e_4, e_6,$ e_2, e_3	32 pebbles $P_1: e_7, e_2,$ $e_1, e_5,$ e_3, e_4 & $P_2: e_7, e_3,$ $e_6, e_2,$ e_5, e_4	32 pebbles $P_1: e_7, e_3,$ $e_4, e_1,$ e_2, e_5 & $P_2: e_7, e_2,$ $e_1, e_4,$ e_3, e_5	32 pebbles $P_1: e_7, e_3,$ $e_4, e_5,$ e_2, e_6 & $P_2: e_7, e_2,$ $e_1, e_5,$ e_3, e_6	1 pebble

3. Detour edge cover pebbling number on some standard graphs

Theorem 1. *The detour edge cover pebbling number for the path P_n graph is $f_{ec}^*(P_n) = 2^{n-1} - 1$, for $n > 1$.*

Proof. Let P_n denote the path graph whose edge set is $\{e_1, e_2, \dots, e_{n-1}\}$. Take $2^{n-1} - 2$ pebbles for the initial distribution. Without loss of generality, let $\dot{s}_e = e_{n-1}$. Place $2^{n-1} - 2$ pebbles on the edge e_{n-1} . The edge detour distance from e_{n-1} to any other edge of P_n is at most $n - 2$. Then covering e_1 will use 2^{n-2} pebbles, covering e_2 will use 2^{n-3} pebbles, $\dots$, and covering e_{n-2} will use 2 pebbles. Then, no pebbles will be left to cover the edge e_{n-1} . Thus, $f_{ec}^*(P_n) \geq 2^{n-1} - 1$. Now we prove the sufficient part by using $2^{n-1} - 1$ pebbles on the edges of P_n .

Case 1. Let $\dot{s}_e = e_1$ or e_{n-1} .

Without loss of generality, let e_{n-1} be the source edge.

Table 3. Edge detour distance from e_{n-1} to $E(P_n)$

	e_{n-1}	e_{n-2}	e_{n-3}	e_{n-4}	$\dots$	$\dots$	$\dots$	e_3	e_2	e_1
e_{n-1}	0	1	2	3	$\dots$	$\dots$	$\dots$	$n-4$	$n-3$	$n-2$

From Table 3, we require 2^{n-2} , 2^{n-3} , and 2^{n-4} pebbles from e_{n-1} to cover e_1 , e_2 , and e_3 , respectively. Proceeding in the same way, to cover the edges e_{n-4} , e_{n-3} , and e_{n-2} , we use 2^3 , 2^2 , and 2^1 pebbles. Moreover, we require 1 pebble to cover e_{n-1} . Thus, the following series of pebbles assures covering all the edges of P_n .

$$\begin{aligned}
 & f_{ec}^*(P_n, e_1) + f_{ec}^*(P_n, e_2) + f_{ec}^*(P_n, e_3) + \dots + f_{ec}^*(P_n, e_{n-4}) \\
 & + f_{ec}^*(P_n, e_{n-3}) + f_{ec}^*(P_n, e_{n-2}) + f_{ec}^*(P_n, e_{n-1}) \\
 & = 2^{n-2} + 2^{n-3} + 2^{n-4} + \dots + 2^3 + 2^2 + 2^1 + 2^0,
 \end{aligned}$$

a geometric series with first term 2^{n-2} and common ratio $\frac{1}{2}$. Therefore, the sum of the geometric series is $2^{n-1} - 1$. Thus, the number of pebbles needed to cover $E(P_n)$ is $2^{n-1} - 1$. By symmetry, the proof follows for e_1 .

Case 2. When the middle edge is the source edge. Then, there are two subcases to consider.

Subcase 2.1. When n is even. Let $\dot{s}_e = e_i$, where $i = \frac{n}{2}$.

Table 4. Edge detour distance from $e_{\frac{n}{2}}$ to $E(P_n)$

	e_1	e_2	e_3	$\dots$	$\dots$	$e_{\frac{n}{2}-2}$	$e_{\frac{n}{2}-1}$	$e_{\frac{n}{2}}$	$e_{\frac{n}{2}+1}$	$e_{\frac{n}{2}+2}$	$\dots$	$\dots$	e_{n-3}	e_{n-2}	e_{n-1}
$e_{\frac{n}{2}}$	$\frac{n}{2} - 1$	$\frac{n}{2} - 2$	$\frac{n}{2} - 3$	$\dots$	$\dots$	2	1	0	1	2	$\dots$	$\dots$	$\frac{n}{2} - 3$	$\frac{n}{2} - 2$	$\frac{n}{2} - 1$

From Table 4, we require $2^{\frac{n}{2}-1}$, $2^{\frac{n}{2}-2}$, and $2^{\frac{n}{2}-3}$ pebbles from $e_{\frac{n}{2}}$ to cover e_1 , e_2 , and e_3 , respectively. Similarly, to cover $E(P_n)$ from $e_{\frac{n}{2}}$, we require the following series of pebble distributions.

$$\begin{aligned}
& f_{ec}^*(P_n, e_1) + f_{ec}^*(P_n, e_2) + f_{ec}^*(P_n, e_3) + \cdots + f_{ec}^*(P_n, e_{\frac{n}{2}-2}) \\
& + f_{ec}^*(P_n, e_{\frac{n}{2}-1}) + f_{ec}^*(P_n, e_{\frac{n}{2}}) + f_{ec}^*(P_n, e_{\frac{n}{2}+1}) \\
& + f_{ec}^*(P_n, e_{\frac{n}{2}+2}) + \cdots + f_{ec}^*(P_n, e_{n-3}) + f_{ec}^*(P_n, e_{n-2}) \\
& + f_{ec}^*(P_n, e_{n-1}) \\
& = 2^{\frac{n}{2}-1} + 2^{\frac{n}{2}-2} + 2^{\frac{n}{2}-3} + \cdots + 2^2 + 2^1 + 2^0 \\
& + 2^1 + 2^2 + \cdots + 2^{\frac{n}{2}-3} + 2^{\frac{n}{2}-2} + 2^{\frac{n}{2}-1}.
\end{aligned}$$

Thus, the number of pebbles needed to cover $E(P_n)$ is $2^{\frac{n}{2}+1} - 3$. Therefore, we used $2^{\frac{n}{2}+1} - 3$ pebbles, which is less than $2^{n-1} - 1$.

Subcase 2.2. When n is odd. Let $\dot{s}_e = e_i$, where $i = \frac{n-1}{2}$ or $\frac{n+1}{2}$.

Without loss of generality, let $e_{\frac{n-1}{2}}$ be the source edge.

Table 5. Edge detour distance from $e_{\frac{n-1}{2}}$ to $E(P_n)$

	e_1	e_2	e_3	$\cdots$	$e_{\frac{n-5}{2}}$	$e_{\frac{n-3}{2}}$	$e_{\frac{n-1}{2}}$	$e_{\frac{n+1}{2}}$	$e_{\frac{n+3}{2}}$	$\cdots$	e_{n-3}	e_{n-2}	e_{n-1}
$e_{\frac{n-1}{2}}$	$\frac{n-3}{2}$	$\frac{n-5}{2}$	$\frac{n-7}{2}$	$\cdots$	2	1	0	1	2	$\cdots$	$\frac{n-5}{2}$	$\frac{n-3}{2}$	$\frac{n-1}{2}$

From Table 5, we require $2^{\frac{n-1}{2}}$, $2^{\frac{n-3}{2}}$, and $2^{\frac{n-5}{2}}$ pebbles from $e_{\frac{n-1}{2}}$ to cover e_{n-1} , e_{n-2} , and e_{n-3} , respectively. Similarly, to cover $E(P_n)$ from $e_{\frac{n-1}{2}}$, we require the following series of pebble distributions.

$$\begin{aligned}
& f_{ec}^*(P_n, e_1) + f_{ec}^*(P_n, e_2) + f_{ec}^*(P_n, e_3) + \cdots + f_{ec}^*(P_n, e_{\frac{n-2}{2}}) \\
& + f_{ec}^*(P_n, e_{\frac{n-3}{2}}) + f_{ec}^*(P_n, e_{\frac{n-1}{2}}) + f_{ec}^*(P_n, e_{\frac{n+1}{2}}) \\
& + f_{ec}^*(P_n, e_{\frac{n+3}{2}}) + \cdots + f_{ec}^*(P_n, e_{n-3}) + f_{ec}^*(P_n, e_{n-2}) \\
& + f_{ec}^*(P_n, e_{n-1}) \\
& = 2^{\frac{n-3}{2}} + 2^{\frac{n-5}{2}} + 2^{\frac{n-7}{2}} + \cdots + 2^2 + 2^1 + 2^0 \\
& + 2^1 + 2^2 + \cdots + 2^{\frac{n-5}{2}} + 2^{\frac{n-3}{2}} + 2^{\frac{n-1}{2}}.
\end{aligned}$$

Thus, the number of pebbles needed to cover $E(P_n)$ is $3(2^{\frac{n-1}{2}} - 1)$. Therefore, we used $3(2^{\frac{n-1}{2}} - 1)$ pebbles, which is less than $2^{n-1} - 1$. By symmetry, the proof follows for $e_{\frac{n+1}{2}}$.

Case 3. When the source edge is an internal edge. Let $\dot{s}_e = e_i$, where $2 \leq i \leq n-2$, $i \neq \frac{n}{2}, \frac{n-1}{2}, \frac{n+1}{2}$.

The edge detour distance from e_i to $E(P_n)$ is at most $n-3$. When $i=2$, we require

$2^1, 2^0, 2^1, 2^2, 2^3, 2^4, \dots, 2^{n-4}$, and 2^{n-3} pebbles to cover $e_1, e_2, e_3, e_4, e_5, e_6, \dots, e_{n-2}$, and e_{n-1} , respectively. When $i = 3$, we require $2^2, 2^1, 2^0, 2^1, 2^2, 2^3, \dots, 2^{n-5}$, and 2^{n-4} pebbles to cover $e_1, e_2, e_3, e_4, e_5, e_6, \dots, e_{n-2}$, and e_{n-1} , respectively. Similarly, to cover $E(P_n)$ from e_i , we require at most $2^{n-2} + 1$ pebbles. Therefore, we used $2^{n-2} + 1$ pebbles, which is less than $2^{n-1} - 1$. Therefore, $f_{ec}^*(P_n) = 2^{n-1} - 1$. The proof is done. $\square$

Theorem 2. *The detour edge cover pebbling number for the wheel graph is $f_{ec}^*(W_n) = (2n - 1)(2^{\frac{3n-1}{2}}) + 1$ for $n \geq 3$ when n is odd, and $f_{ec}^*(W_n) = (2n - 1)(2^{\frac{3n}{2}}) + 1$ for $n \geq 4$ when n is even.*

Proof. Let $W_n = C_n + v_0$ be a wheel graph of order $n + 1$ and size $2n$ with $E(W_n) = \{e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$. Let $\{e_1, e_2, \dots, e_n\}$ be the edges incident with v_0 and $\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ be the edges of the cycle C_n .

Case 1. When n is odd.

Without loss of generality, let $s_e = e_1$. Let the initial configuration be $(2n - 1)(2^{\frac{3n-1}{2}})$ pebbles assigned to cover the edges of W_n using the edge detour paths. For $n = 3$, the graph contains 6 edges. Here, the edge detour path between any two edges lacks one edge from the total number of edges in W_3 , since the edge detour path has only 5 edges and the number of lacking edges in the edge detour path increases by one edge for every $n \geq 3$. Thus, the edge detour distance from e_1 to any other edge in W_n is $\frac{3n-1}{2}$. In order to cover the spokes $\{e_2, e_3, \dots, e_n\}$ we require $(n - 1)(2^{\frac{3n-1}{2}})$ pebbles. To cover the cycle edges in W_n we require $n(2^{\frac{3n-1}{2}})$ pebbles. Therefore, we have used $(2n - 1)(2^{\frac{3n-1}{2}})$ pebbles to cover $\{e_2, e_3, \dots, e_{2n}\}$ using the edge detour paths. Now, the edge e_1 is left uncovered. Therefore, $f_{ec}^*(W_n) \geq (2n - 1)(2^{\frac{3n-1}{2}}) + 1$. Now we prove the sufficient part by using $(2n - 1)(2^{\frac{3n-1}{2}}) + 1$ pebbles on the edges of W_n .

Subcase 1.1. Let any spoke be the source edge.

Without loss of generality, let $s_e = e_n$.

The edge detour distance from e_n to $\{e_1, e_2, \dots, e_{n-1}\}$ or $\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ is $\frac{3n-1}{2}$. Therefore, to cover any edge in W_n from e_n using the edge detour path we require $2^{\frac{3n-1}{2}}$ pebbles. Thus, to cover e_n we require 1 pebble, to cover $\{e_1, e_2, \dots, e_{n-1}\}$ we require $(n - 1)(2^{\frac{3n-1}{2}})$ pebbles, and to cover $\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ we require $n(2^{\frac{3n-1}{2}})$ pebbles. Thus, the number of pebbles needed to cover $E(W_n)$ using the edge detour path is $(2n - 1)(2^{\frac{3n-1}{2}}) + 1$. By symmetry, the proof follows for any source edge e_i where $1 \leq i \leq n - 1$.

Subcase 1.2. Let any cycle edge be the source edge.

Without loss of generality, let $s_e = e_{n+1}$. The edge detour distance from e_{n+1} to $\{e_1, e_2, \dots, e_n\}$ or $\{e_{n+2}, e_{n+3}, \dots, e_{2n}\}$ is $\frac{3n-1}{2}$. Therefore, to cover any edge in W_n from e_{n+1} using the edge detour path we require $2^{\frac{3n-1}{2}}$ pebbles. Thus, to cover e_{n+1} we require 1 pebble, to cover $\{e_1, e_2, \dots, e_n\}$ we require $n(2^{\frac{3n-1}{2}})$ pebbles, and to cover $\{e_{n+2}, e_{n+3}, \dots, e_{2n}\}$ we require $(n - 1)(2^{\frac{3n-1}{2}})$ pebbles. Thus, the number of pebbles needed to cover $E(W_n)$ using the edge detour path is $(2n - 1)(2^{\frac{3n-1}{2}}) + 1$. By

symmetry, the proof follows for any source edge e_i where $n + 2 \leq i \leq 2n$.

Case 2. When n is even.

Without loss of generality, let $s_e = e_1$. Let the initial configuration be $(2n - 1)(2^{\frac{3n}{2}})$ pebbles assigned to cover the edges of W_n using the edge detour paths. For $n = 4$, the graph contains 8 edges. Here, the edge detour path between any two edges lacks one edge from the total number of edges in W_4 , since the edge detour path has only 7 edges and the number of lacking edges in the edge detour path increases by one edge for every $n \geq 4$. Thus, the edge detour distance from e_1 to any other edge in W_n is $\frac{3n}{2}$. In order to cover the spokes $\{e_2, e_3, \dots, e_n\}$ we require $(n - 1)(2^{\frac{3n}{2}})$ pebbles. To cover the cycle edges in W_n we require $n(2^{\frac{3n}{2}})$ pebbles. Therefore, we have used $(2n - 1)(2^{\frac{3n}{2}})$ pebbles to cover $\{e_2, e_3, \dots, e_{2n}\}$ using the edge detour paths. Now, the edge e_1 is left uncovered. Therefore, $f_{ec}^*(W_n) \geq (2n - 1)(2^{\frac{3n}{2}}) + 1$. Now we prove the sufficient part by using $(2n - 1)(2^{\frac{3n}{2}}) + 1$ pebbles on the edges of W_n .

Subcase 2.1. Let any spoke be the source edge.

Without loss of generality, let $s_e = e_n$. The edge detour distance from e_n to $\{e_1, e_2, \dots, e_{n-1}\}$ or $\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ is $\frac{3n}{2}$. Therefore, to cover any edge in W_n from e_n using the edge detour path we require $2^{\frac{3n}{2}}$ pebbles. Thus, to cover e_n we require 1 pebble, to cover $\{e_1, e_2, \dots, e_{n-1}\}$ we require $(n - 1)(2^{\frac{3n}{2}})$ pebbles, and to cover $\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ we require $n(2^{\frac{3n}{2}})$ pebbles. Thus, the number of pebbles needed to cover $E(W_n)$ using the edge detour path is $(2n - 1)(2^{\frac{3n}{2}}) + 1$. By symmetry, the proof follows for any source edge e_i where $1 \leq i \leq n - 1$.

Subcase 2.2. Let any cycle edge be the source edge.

Without loss of generality, let $s_e = e_{n+1}$.

The edge detour distance from e_{n+1} to $\{e_1, e_2, \dots, e_n\}$ or $\{e_{n+2}, e_{n+3}, \dots, e_{2n}\}$ is $\frac{3n}{2}$. Therefore, to cover any edge in W_n from e_{n+1} using the edge detour path we require $2^{\frac{3n}{2}}$ pebbles. Thus, to cover e_{n+1} we require 1 pebble, to cover $\{e_1, e_2, \dots, e_n\}$ we require $n(2^{\frac{3n}{2}})$ pebbles, and to cover $\{e_{n+2}, e_{n+3}, \dots, e_{2n}\}$ we require $(n - 1)(2^{\frac{3n}{2}})$ pebbles. Thus, the number of pebbles needed to cover $E(W_n)$ using the edge detour path is $(2n - 1)(2^{\frac{3n}{2}}) + 1$. By symmetry, the proof follows for any source edge e_i where $n + 2 \leq i \leq 2n$. Therefore, $f_{ec}^*(W_n) = (2n - 1)(2^{\frac{3n-1}{2}}) + 1$ for $n \geq 3$ when n is odd, and $f_{ec}^*(W_n) = (2n - 1)(2^{\frac{3n}{2}}) + 1$ for $n \geq 4$ when n is even. $\square$

Theorem 3. *The detour edge cover pebbling number for the n -star graph is $f_{ec}^*(S_n) = 2n - 1$ for $n \geq 2$.*

Proof. The n -star graph is a complete bipartite graph $K_{1,n}$. Let the edge set of S_n be $\{e_1, e_2, \dots, e_n\}$. Without loss of generality, let $s_e = e_1$. Let the initial configuration be $2n - 2$ pebbles assigned to cover the edges of S_n using the edge detour paths. The edge detour distance from e_1 to any other edge in S_n is 1. In order to cover $\{e_2, e_3, \dots, e_n\}$ we require $2(n - 1)$ pebbles since there would only be one move to reach any edge, and that path would undoubtedly be the edge detour path in the n -star graph, i.e., e_i to e_1 for some i , is an edge detour path. Therefore, we have used $2n - 2$ pebbles. Now, the edge e_1 is left uncovered. Therefore, $f_{ec}^*(S_n) \geq 2n - 1$. Now

we prove the sufficient part by using $2n - 1$ pebbles on the edges of S_n .

Assume the edge e_n has zero pebbles. Then, there is k such that the pebbles on the edge e_k , $p(e_k) \geq 2$, where $1 \leq k \leq n - 1$. Using the pigeonhole principle, it is possible to shift a pebble from e_k to e_n . Then, to cover the remaining $n - 1$ edges is achievable by using $2n - 3$ pebbles. Therefore, $f_{ec}^*(S_n) = 2n - 1$ for $n \geq 2$. The proof is done. $\square$

Theorem 4. *The detour edge cover pebbling number for the cycle graph is*

$$f_{ec}^*(C_n) = \sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i} + 1 \quad \text{for } n \geq 3 \text{ when } n \text{ is odd,}$$

and

$$f_{ec}^*(C_n) = \sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}} + 1 \quad \text{for } n \geq 4 \text{ when } n \text{ is even.}$$

Proof. Consider the edge set of C_n to be $\{e_1, e_2, \dots, e_n\}$.

Case 1. When n is odd.

Without loss of generality, let $\dot{s}_e = e_1$. Let the initial configuration be $\sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i}$ pebbles assigned to cover the edges of C_n using the edge detour paths. The edge detour distance from e_1 to any other edge in C_n is at most $n - 1$. In order to cover the edges e_2 and e_n we require $2(2^{n-1})$ pebbles. To cover the edges e_3 and e_{n-1} we require $2(2^{n-2})$ pebbles. Proceeding in the same way, to cover the edges $e_{\frac{n-1}{2}}$ and $e_{\frac{n+1}{2}}$ we require $2(2^{\frac{n-1}{2}})$ pebbles. Therefore, we have used $\sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i}$ pebbles to cover $\{e_2, e_3, \dots, e_n\}$ using the edge detour paths. Now, the edge e_1 is left uncovered. Therefore, $f_{ec}^*(C_n) \geq \sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i} + 1$. Now we prove the sufficient part by using $\sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i} + 1$ pebbles on the edges of C_n . Without loss of generality, let $\dot{s}_e = e_n$.

Table 6. Edge detour distance from e_n to $E(C_n)$

	e_1	e_2	e_3	$\dots$	$\dots$	$e_{\frac{n-1}{2}}$	$e_{\frac{n+1}{2}}$	$\dots$	$\dots$	e_{n-3}	e_{n-2}	e_{n-1}	e_n
e_n	$n - 1$	$n - 2$	$n - 3$	$\dots$	$\dots$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\dots$	$\dots$	$n - 3$	$n - 2$	$n - 1$	0

From Table 6, we require $2(2^{n-1})$ pebbles from e_n to cover the edges e_1 and e_{n-1} , we require $2(2^{n-2})$ pebbles from e_n to cover the edges e_2 and e_{n-2} . Proceeding in the same way, to cover the edges $e_{\frac{n-1}{2}}$ and $e_{\frac{n+1}{2}}$ we require $2(2^{\frac{n-1}{2}})$ pebbles. Moreover, we require 1 pebble to cover e_n . Thus, the number of pebbles needed to cover $E(C_n)$ using the edge detour path is $\sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i} + 1$. By symmetry, the proof follows for any source edge e_i where $1 \leq i \leq n - 1$.

Case 2. When n is even.

Without loss of generality, let $\dot{s}_e = e_1$. Let the initial configuration be $\sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}}$ pebbles assigned to cover the edges of C_n using the edge detour paths.

The edge detour distance from e_1 to any other edge in C_n is at most $n - 1$. In order to cover the edges e_2 and e_n we require $2(2^{n-1})$ pebbles. To cover the edges e_3 and e_{n-1} we require $2(2^{n-2})$ pebbles. Proceeding in the same way, to cover the edge $e_{\frac{n}{2}+1}$ we require $2^{\frac{n}{2}}$ pebbles. Therefore, we have used $\sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}}$ pebbles to cover $\{e_2, e_3, \dots, e_n\}$ using the edge detour paths. Now, the edge e_1 is left uncovered. Therefore, $f_{ec}^*(C_n) \geq \sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}} + 1$. Now we prove the sufficient part by using $\sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}} + 1$ pebbles on the edges of C_n . Without loss of generality, let $s_e = e_n$.

Table 7. Edge detour distance from e_n to $E(C_n)$

	e_1	e_2	e_3	$\dots$	$\dots$	$e_{\frac{n}{2}+1}$	$\dots$	$\dots$	e_{n-3}	e_{n-2}	e_{n-1}	e_n
e_n	$n - 1$	$n - 2$	$n - 3$	$\dots$	$\dots$	$\frac{n}{2}$	$\dots$	$\dots$	$n - 3$	$n - 2$	$n - 1$	0

From Table 7, we require $2(2^{n-1})$ pebbles from e_n to cover the edges e_1 and e_{n-1} , we require $2(2^{n-2})$ pebbles from e_n to cover the edges e_2 and e_{n-2} . Proceeding in the same way, to cover the edge $e_{\frac{n}{2}+1}$ we require $2^{\frac{n}{2}}$ pebbles. Moreover, we require 1 pebble to cover e_n . Thus, the number of pebbles needed to cover $E(C_n)$ using the edge detour path is $\sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}} + 1$. By symmetry, the proof follows for any source edge e_i where $1 \leq i \leq n - 1$. Therefore, $f_{ec}^*(C_n) = \sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i} + 1$ for $n \geq 3$ when n is odd, and $f_{ec}^*(C_n) = \sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}} + 1$ for $n \geq 4$ when n is even. $\square$

4. Detour edge cover pebbling number on staircase graphs

Staircase graphs, particularly generalized forms such as arithmetic and geometric staircase graphs [17], have been extensively studied for their labeling properties, including total edge irregularity strength and cordial labeling.

The structure of staircase graphs can be leveraged in clustering algorithms and for understanding connectivity patterns within networks. Each step in these graphs represents a constant value over a specific interval, making it straightforward to identify where changes occur. Furthermore, staircase compatibility can be applied to optimization problems involving m -cliques in m -partite graphs, leading to totally uni-modular linear programming formulations [1].

Overall, staircase graphs are valuable tools in Graph theory, offering a structured way to model complex systems, provide insights into their properties through labeling and coloring, and exhibiting potential applications in computer science and related fields. Let $n^*(C_4)$ be a graph, consisting of n copies of C_4 with a vertex common in between any two consecutive duplicates of C_4 except the first and last copies of C_4 . A staircase graph, $S_{1,n}$, of order n ($n \geq 2$ and $n \in \mathbb{Z}$) is obtained by merging $(2n - 2)$ edges of $n^*(C_4)$ with $(2n - 2)$ edges in the shortest path $V_{c1} - V_{c3}$ of the $(n - 1)$ -th staircase along the steps of the $(n - 1)$ -th staircase.

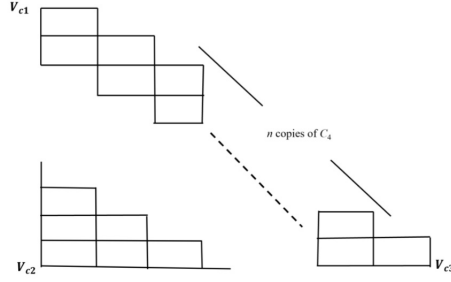


Figure 2. A staircase graph $S_{1,n}$ with corner vertices.

The 1-step staircase graph is a cycle graph of order 4. The corner-vertex of $S_{1,n}$ is defined as a vertex of degree 2. Therefore, for any $S_{1,n}$, the number of corner vertices is exactly 3, namely V_{c1} , V_{c2} , and V_{c3} . The number of vertices and edges in $S_{1,n}$ are $p = \frac{n^2+5n+2}{2}$ and $q = n^2+3n$. Let the edge set of vertical edges be $\{e_{1,v}, e_{2,v}, \dots, e_{\frac{n^2+3n}{2}, v}\}$ and the edge set of horizontal edges be $\{e_{1,h}, e_{2,h}, \dots, e_{\frac{n^2+3n}{2}, h}\}$.

Theorem 5. Let $S_{1,n}$ be a staircase graph of order n (n is even and $n \geq 4$). Then

$$f_{ec}^*(S_{1,n}) = 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) + 1,$$

when $n = 4$, and

$$\begin{aligned} f_{ec}^*(S_{1,n}) &= 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ &\quad + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots \\ &\quad + 2^{n^2+2n-3}(29) + 2 \cdot 2^{n^2+\frac{3n}{2}+1} + 1, \end{aligned}$$

when $i = 4$ and $n \geq 6$.

Proof. Without loss of generality, let $s_e = e^*$, where e^* is the outer horizontal edge incident with V_{c3} . Let the initial configuration be

$$\begin{aligned} &6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ &\quad + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) \\ &\quad + 2 \cdot 2^{n^2+\frac{3n}{2}+1} \end{aligned}$$

pebbles assigned to cover the edges of $S_{1,n}$ using the edge detour paths where $i = 4$. For $n = 4$, the graph contains 28 edges. Here, the edge detour path lacks two edges from the total number of edges in $S_{1,4}$, since the edge detour path has only 26 edges

and the number of lacking edges in the edge detour path increases by two edges for every $n \geq 4$. Thus, the edge detour distance from e^* to any other edge in $S_{1,n}$ is at most $n^2 + 2n + 1$. In order to cover the edges $e_{1,v}$, $e_{1,h}$, and $e_{2,v}$ we require $2^{n^2+2n+1} + 2 \cdot 2^{n^2+2n}$ pebbles. To cover the edges $e_{2,h}$ and $e_{3,h}$ we require $2^{n^2+2n+1} + 2^{n^2+2n}$ pebbles, to cover the edges $e_{3,v}$, $e_{4,v}$, and $e_{5,v}$ we require $2^{n^2+2n+1} + 2^{n^2+2n} + 2^{n^2+2n-1}$ pebbles, to cover the edges $e_{4,h}$, $e_{5,h}$, and $e_{6,h}$ we require $2^{n^2+2n} + 2^{n^2+2n-1} + 2^{n^2+2n-2}$ pebbles and to cover the edges $e_{6,v}$, $e_{7,v}$, $e_{8,v}$, and $e_{9,v}$ we require $2^{n^2+2n+1} + 2^{n^2+2n} + 2^{n^2+2n-1} + 2^{n^2+2n-2}$ pebbles. Proceeding in the same way, to cover the edges from $e_{\frac{n^2+n+2}{2},h}$ to $e_{\frac{n^2+3n-2}{2},h}$ we require $(n-1)2^{n^2+2n}$ pebbles, to cover the edges from $e_{\frac{n^2+n}{2},v}$ to $e_{\frac{n^2+3n}{2},v}$ we require $2^{n^2+2n+1} + (n-1)2^{n^2+2n} + 2^{n^2+2n-1}$ pebbles and to cover the edges from $e_{\frac{n^2-n+2}{2},h}$ to $e_{\frac{n^2+n}{2},h}$ we require $2^{n^2+2n+1} + 2^{n^2+2n} + (n-4)2^{n^2+2n-1} + 2 \cdot 2^{n^2+2n-2}$ pebbles. Next, to cover the remaining edges for every $n \geq 6$, we require $i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) + 2 \cdot 2^{n^2+\frac{3n}{2}+1}$ pebbles where $i = 4$. Therefore, we have used

$$\begin{aligned} & 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ & + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) \\ & + 2 \cdot 2^{n^2+\frac{3n}{2}+1} \end{aligned}$$

pebbles to cover $\{e_1, e_2, \dots, e_{n^2+3n-1}\}$ using the edge detour paths. Now, the edge $e_{\frac{n^2+3n}{2},h}$ is left uncovered. Therefore,

$$\begin{aligned} f_{ec}^*(S_{1,n}) & \geq 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ & + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) \\ & + 2 \cdot 2^{n^2+\frac{3n}{2}+1} + 1. \end{aligned}$$

Now we prove the sufficient part by using

$$\begin{aligned} & 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ & + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) \\ & + 2 \cdot 2^{n^2+\frac{3n}{2}+1} + 1 \end{aligned}$$

pebbles on the edges of $S_{1,n}$.

Case 1. When the horizontal edge is the source edge.

Without loss of generality, let $s_e = e_{\frac{n^2+3n}{2},h}$. From Table 8, we require $2^{n^2+2n+1} + 2 \cdot 2^{n^2+2n}$ pebbles from $e_{\frac{n^2+3n}{2},h}$ to cover the edges $e_{1,v}$, $e_{1,h}$, and $e_{2,v}$, we require $2^{n^2+2n+1} + 2^{n^2+2n}$ pebbles to cover the edges $e_{2,h}$ and $e_{3,h}$. Proceeding in the same way, to cover the edges from $e_{\frac{n^2+n}{2},v}$ to $e_{\frac{n^2+3n}{2},v}$ we require $2^{n^2+2n+1} + (n-1)2^{n^2+2n} +$

Table 8. Edge detour distance from $e_{\frac{n^2+3n}{2},h}$ to $E(S_1, n)$

	$e_{1,v}$	$e_{2,v}$	$e_{3,v}$	$e_{4,v}$	$e_{5,v}$	..
$e_{\frac{n^2+3n}{2},h}$	$n^2 + 2n + 1$	$n^2 + 2n$	$n^2 + 2n + 1$	$n^2 + 2n$	$n^2 + 2n - 1$	..

$e_{1,h}$	$e_{2,h}$	$e_{3,h}$	..	$e_{\frac{n^2+3n-2}{2},v}$	$e_{\frac{n^2+3n}{2},v}$	..	$e_{\frac{n^2+3n-4}{2},h}$	$e_{\frac{n^2+3n-2}{2},h}$
$n^2 + 2n$	$n^2 + 2n + 1$	$n^2 + 2n$	..	$n^2 + 2n - 1$	$n^2 + 2n$	..	$n^2 + 2n$	$n^2 + 2n$

2^{n^2+2n-1} pebbles. Moreover, to cover the edges from $e_{\frac{n^2-n+2}{2},h}$ to $e_{\frac{n^2+n}{2},h}$ we require $2^{n^2+2n+1} + 2^{n^2+2n} + (n-4)2^{n^2+2n-1} + 2 \cdot 2^{n^2+2n-2}$ pebbles. Thus, the number of pebbles needed to cover $E(S_{1,n})$ using the edge detour path is

$$\begin{aligned}
& 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\
& + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) \\
& + 2 \cdot 2^{n^2+\frac{3n}{2}+1} + 1,
\end{aligned}$$

where $i = 4$. By symmetry, the proof follows for any source edge e_k where $e_{1,h} \leq k \leq e_{\frac{n^2+3n-2}{2},h}$.

Case 2. When the vertical edge is the source edge.

Without loss of generality, let $s_e = e_{\frac{n^2+3n}{2},v}$. From Table 9, we require $2^{n^2+2n} +$

Table 9. Edge detour distance from $e_{\frac{n^2+3n}{2},v}$ to $E(S_1, n)$

	$e_{1,v}$	$e_{2,v}$	$e_{3,v}$	$e_{4,v}$	$e_{5,v}$	..
$e_{\frac{n^2+3n}{2},v}$	$n^2 + 2n$	$n^2 + 2n - 1$	$n^2 + 2n$	$n^2 + 2n - 1$	$n^2 + 2n - 2$	..

$e_{1,h}$	$e_{2,h}$	$e_{3,h}$	..	$e_{\frac{n^2+3n-4}{2},v}$	$e_{\frac{n^2+3n-2}{2},v}$	..	$e_{\frac{n^2+3n-2}{2},h}$	$e_{\frac{n^2+3n}{2},h}$
$n^2 + 2n - 1$	$n^2 + 2n$	$n^2 + 2n - 1$	..	$n^2 + 2n - 1$	$n^2 + 2n - 2$	..	$n^2 + 2n - 1$	$n^2 + 2n - 1$

$2 \cdot 2^{n^2+2n-1}$ pebbles from $e_{\frac{n^2+3n}{2},v}$ to cover the edges $e_{1,v}$, $e_{1,h}$, and $e_{2,v}$, we require $2^{n^2+2n} + 2^{n^2+2n-1}$ pebbles to cover the edges $e_{2,h}$ and $e_{3,h}$ and we require $2^{n^2+2n} + 2^{n^2+2n-1} + 2^{n^2+2n-2}$ pebbles to cover the edges $e_{3,v}$, $e_{4,v}$, and $e_{5,v}$. Proceeding in the same way, covering $e_{\frac{n^2+3n-4}{2},v}$ and $e_{\frac{n^2+3n-2}{2},v}$ will use $2^{n^2+2n-1} + 2^{n^2+2n-2}$ pebbles. Moreover, covering $e_{\frac{n^2+3n-2}{2},h}$ and $e_{\frac{n^2+3n}{2},h}$ will use $2 \cdot 2^{n^2+2n-1}$ pebbles. Thus, the number of pebbles needed to cover $E(S_{1,n})$ using the edge detour path is less than

$$\begin{aligned}
& 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\
& + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) \\
& + 2 \cdot 2^{n^2+\frac{3n}{2}+1} + 1,
\end{aligned}$$

where $i = 4$. By symmetry, the proof follows for any source edge e_k where $e_{1,v} \leq k \leq e_{\frac{n^2+3n-2}{2},v}$. Therefore,

$$f_{ec}^*(S_{1,n}) = 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) + 1,$$

when $n = 4$, and

$$\begin{aligned} f_{ec}^*(S_{1,n}) &= 6(2^{n^2+2n+1}) + (2n+5)(2^{n^2+2n}) + n(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ &\quad + i(2^{n^2+2n-6})57 + (i+1)(2^{n^2+2n-6})41 + \dots + 2^{n^2+2n-3}(29) \\ &\quad + 2 \cdot 2^{n^2+\frac{3n}{2}+1} + 1, \end{aligned}$$

where $i = 4$ and $n \geq 6$ when n is even. □

Example 1. We can check the result on $S_{1,6}$ (see Figure 3). Here, the edge detour path from $e_{27,h}$ to $e_{1,v}$ is mentioned and the edges that are not in the edge detour path are also highlighted.

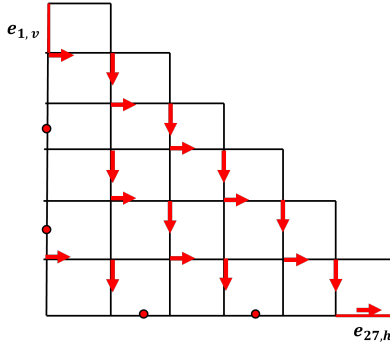


Figure 3. A staircase graph $S_{1,6}$.

Theorem 6. Let $S_{1,n}$ be a staircase graph of order n (n is odd and $n \geq 5$). Then $f_{ec}^*(S_{1,n}) = 7(2^{n^2+2n+1}) + \frac{3n+15}{2}(2^{n^2+2n}) + \frac{3n+5}{2}(2^{n^2+2n-1}) + 6(2^{n^2+2n-2}) + 2(2^{n^2+2n-3}) + 1$; when $n = 5$ and

$$\begin{aligned} f_{ec}^*(S_{1,n}) &= 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) + 2^{n^2+2n-3} \\ &\quad + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} \\ &\quad - 2^{\frac{n^2+3n+6}{2}} + 1; \end{aligned}$$

when $i = 5$ and $n \geq 7$.

Proof. Without loss of generality, let $\dot{s}_e = e_{6,v}$. Let the initial configuration be

$$\begin{aligned} & 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ & + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\ & + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} \end{aligned}$$

pebbles assigned to cover the edges of $S_{1,n}$ using the edge detour paths where $i = 5$. For $n = 5$, the graph contains 40 edges. Here, the edge detour path lacks three edges from the total number of edges in $S_{1,5}$, since the edge detour path has only 37 edges and the number of lacking edges in the edge detour path increases by two edges for every $n \geq 5$. Thus, the edge detour distance from $e_{6,v}$ to any other edge in $S_{1,n}$ is at most $n^2 + 2n + 1$. In order to cover the edges $e_{1,v}$, $e_{1,h}$, and $e_{2,v}$ we require $2^{n^2+2n+1} + 2 \cdot 2^{n^2+2n}$ pebbles. To cover the edges $e_{2,h}$ and $e_{3,h}$ we require $2^{n^2+2n+1} + 2^{n^2+2n}$ pebbles, to cover the edges $e_{3,v}$, $e_{4,v}$, and $e_{5,v}$ we require $2^{n^2+2n+1} + 2^{n^2+2n} + 2^{n^2+2n-1}$ pebbles, to cover the edges $e_{4,h}$, $e_{5,h}$, and $e_{6,h}$ we require $2^{n^2+2n} + 2^{n^2+2n-1} + 2^{n^2+2n-2}$ pebbles and to cover the edges $e_{7,v}$, $e_{8,v}$, and $e_{9,v}$ we require $2^{n^2+2n-1} + 2^{n^2+2n-2} + 2^{n^2+2n-3}$ pebbles. Proceeding in the same way, to cover the edges from $e_{\frac{n^2+n+2}{2},h}$ to $e_{\frac{n^2+3n}{2},h}$ we require $n(2^{n^2+2n})$ pebbles, to cover the edges from $e_{\frac{n^2+n}{2},v}$ to $e_{\frac{n^2+3n}{2},v}$ we require $2^{n^2+2n+1} + \frac{n+1}{2}(2^{n^2+2n}) + \frac{n-1}{2}(2^{n^2+2n-1})$ pebbles and to cover the edges from $e_{\frac{n^2-n+2}{2},h}$ to $e_{\frac{n^2+n}{2},h}$ we require $2^{n^2+2n+1} + 2^{n^2+2n} + (n-2)2^{n^2+2n-1}$ pebbles. Next, to cover the remaining edges for every $n \geq 7$, we require $i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}}$ pebbles where $i = 5$. Therefore, we have used

$$\begin{aligned} & 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ & + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\ & + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} \end{aligned}$$

pebbles to cover $\{e_1, e_2, \dots, e_{n^2+3n-1}\}$ using the edge detour paths. Now, the edge $e_{6,v}$ is left uncovered. Therefore,

$$\begin{aligned} f_{ec}^*(S_{1,n}) & \geq 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\ & + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\ & + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} + 1. \end{aligned}$$

Now we prove the sufficient part by using

$$\begin{aligned}
& 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\
& + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\
& + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} + 1
\end{aligned}$$

pebbles on the edges of $S_{1,n}$.

Case 1. When the vertical edge is the source edge.

Without loss of generality, let $\dot{s}_e = e_{6,v}$. From Table 10, we require $2^{n^2+2n+1} +$

Table 10. Edge detour distance from $e_{6,v}$ to $E(S_1, n)$

	$e_{1,v}$	$e_{2,v}$	$e_{3,v}$	$e_{4,v}$	$e_{5,v}$	..
$e_{6,v}$	$n^2 + 2n + 1$	$n^2 + 2n$	$n^2 + 2n + 1$	$n^2 + 2n$	$n^2 + 2n - 1$	..

$e_{1,h}$	$e_{2,h}$	$e_{3,h}$	..	$e_{\frac{n^2+3n-2}{2},v}$	$e_{\frac{n^2+3n}{2},v}$	..	$e_{\frac{n^2+3n-2}{2},h}$	$e_{\frac{n^2+3n}{2},h}$
$n^2 + 2n$	$n^2 + 2n + 1$	$n^2 + 2n$	..	$n^2 + 2n - 1$	$n^2 + 2n - 1$	..	$n^2 + 2n$	$n^2 + 2n$

$2 \cdot 2^{n^2+2n}$ pebbles from $e_{6,v}$ to cover the edges $e_{1,v}$, $e_{1,h}$, and $e_{2,v}$, we require $2^{n^2+2n+1} + 2^{n^2+2n}$ pebbles to cover the edges $e_{2,h}$ and $e_{3,h}$. Proceeding in the same way, to cover the edges from $e_{\frac{n^2+n}{2},v}$ to $e_{\frac{n^2+3n}{2},v}$, we require $2^{n^2+2n+1} + \left(\frac{n+1}{2}\right)2^{n^2+2n} + \left(\frac{n-1}{2}\right)2^{n^2+2n-1}$ pebbles. Moreover, to cover the edges from $e_{\frac{n^2+n+2}{2},h}$ to $e_{\frac{n^2+3n}{2},h}$ we require $n(2^{n^2+2n})$ pebbles. Thus, the number of pebbles needed to cover $E(S_{1,n})$ using the edge detour path is

$$\begin{aligned}
& 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\
& + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\
& + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} + 1,
\end{aligned}$$

where $i = 5$. By symmetry, the proof follows for any source edge e_k where $e_{1,v} \leq k \leq e_{\frac{n^2+3n}{2},v}$ and $e_k \neq e_{6,v}$.

Case 2. When the horizontal edge is the source edge.

Without loss of generality, let $\dot{s}_e = e_{7,h}$. From Table 11, we require $2^{n^2+2n-1} + 2 \cdot 2^{n^2+2n-2}$ pebbles from $e_{7,h}$ to cover the edges $e_{1,v}$, $e_{1,h}$, and $e_{2,v}$ and we require $2^{n^2+2n-1} + 2^{n^2+2n-2}$ pebbles to cover the edges $e_{2,h}$ and $e_{3,h}$. Proceeding in the same way, covering $e_{\frac{n^2+3n-2}{2},v}$ and $e_{\frac{n^2+3n}{2},v}$ will use $2^{n^2+2n-3} + 2^{n^2+2n-4}$ pebbles. Moreover, covering $e_{\frac{n^2+3n-2}{2},h}$ and $e_{\frac{n^2+3n}{2},h}$ will use $2 \cdot 2^{n^2+2n-3}$ pebbles. Thus, the

Table 11. Edge detour distance from $e_{7,h}$ to $E(S_1, n)$

	$e_{1,v}$	$e_{2,v}$	$e_{3,v}$	..	$e_{1,h}$	$e_{2,h}$
$e_{7,h}$	$n^2 + 2n - 1$	$n^2 + 2n - 2$	$n^2 + 2n - 1$	..	$n^2 + 2n - 2$	$n^2 + 2n - 1$

$e_{3,h}$	..	$e_{\frac{n^2+3n-2}{2},v}$	$e_{\frac{n^2+3n}{2},v}$	..	$e_{\frac{n^2+3n-2}{2},h}$	$e_{\frac{n^2+3n}{2},h}$
$n^2 + 2n - 2$	..	$n^2 + 2n - 3$	$n^2 + 2n - 4$	..	$n^2 + 2n - 3$	$n^2 + 2n - 3$

number of pebbles needed to cover $E(S_{1,n})$ using the edge detour path is less than

$$\begin{aligned}
& 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\
& + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\
& + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} + 1,
\end{aligned}$$

where $i = 4$. By symmetry, the proof follows for any source edge e_k where $e_{1,h} \leq k \leq e_{\frac{n^2+3n}{2},h}$ and $e_k \neq e_{7,h}$. Therefore,

$$\begin{aligned}
f_{ec}^*(S_{1,n}) &= 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\
& + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\
& + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} + 1,
\end{aligned}$$

when $n = 4$, and

$$\begin{aligned}
f_{ec}^*(S_{1,n}) &= 7(2^{n^2+2n+1}) + \frac{3n+17}{2}(2^{n^2+2n}) + \frac{3n+3}{2}(2^{n^2+2n-1}) + 4(2^{n^2+2n-2}) \\
& + 2^{n^2+2n-3} + i(2^{n^2+2n-3})3 + i(2^{n^2+2n-4})11 + \dots \\
& + \left(\frac{n-5}{2}\right)2^{n^2+2n-2}(27) + \left(\frac{n+33}{4}\right)2^{n^2+2n-1} - 2^{\frac{n^2+3n+6}{2}} + 1,
\end{aligned}$$

where $i = 5$ and $n \geq 7$ when n is odd. $\square$

Example 2. We can check the result on $S_{1,7}$ (see Figure 4). Here, the edge detour path from $e_{6,v}$ to $e_{1,v}$ is mentioned and the edges that are not in the edge detour path are also highlighted.

5. Applications

The mathematical game of pebbling, played on a graph G . Many variations on this pebbling concept have been presented over the years. Here, some applications of

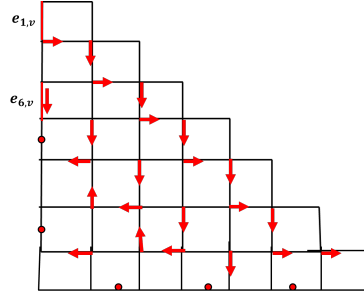


Figure 4. A staircase graph $S_{1,7}$.

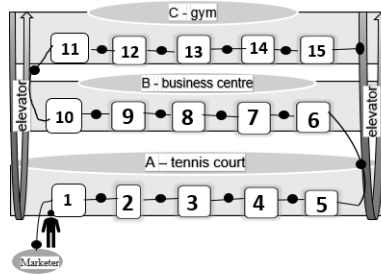


Figure 5. A first-class hotel.

the detour edge cover pebbling concept has been discovered. To start with, let us construct a situation of a marketer of the sales and marketing department visiting all the rooms in the first-class hotel (see Figure 5) to promote his business. The figure shows that the hotel has 15 rooms. Here, the ground floor has 5 rooms and using the elevator the marketer goes to the next floor and visits the next 5 rooms on the first floor and similarly for the second floor. To enhance the guest comfort and experience, this hotel provides amenities A, B, and C facing the rooms on each floor. Consider the rooms as edges and the corners of each room as vertices.

Now, the marketer must visit all the 15 rooms and he starts from the 1st room and reaches the 15th room using the edge detour path. Thus, all the edges are in the edge detour path connecting the edges e_1 and e_{15} . The marketer is carrying small gift baskets along with bouquets of flowers and the marketer needs to give a commission to the hotel manager of 50% of what he has when he visits each room. The number of gift baskets will be considered as pebbles, and the discarded pebbles as commission. Therefore, when he reaches room 13, he should have one gift basket and worth money of 6 gift baskets i.e., from the 6 gift baskets, 3 gift basket's amount will be sent to the hotel manager as commission and remaining 3 gift baskets will be left. Among which, one will be given to the room 14, and out of two, one will be considered as commission and the remaining one will be given to room 15.

Therefore, initially, the marketer requires an amount of 32,767, which includes 15 gift baskets and the total commission. Since the detour edge cover pebbling number of

path graph is $2^{n-1} - 1$, we could solve this.

Furthermore, let us now analyse another authentic application. The Golden Quadrilateral [12] is a national highway network in India. It forms a quadrilateral by connecting 84 major cities in India. Primitively, the pebbling concept is widely used in transportation. Let's look at how it works on this variation of graph pebbling.

Here, the fuel which is a consumable resource can be considered as a pebble and the network as a edge detour path. Also, consider the cities as edges, and their endpoints as vertices. Now, fuel will be decreased for each move since it will cause emissions during transportation, and the goods which must arrive in each city will be considered as the pebble that will be left in each city. Now, we can apply the concept of detour edge cover pebbling number of path graph to find out the quantity of the fuel required. Conclusively, by using this detour edge cover pebbling concept, we can also develop a Game App by designing graphics and make the business profitable.

Numerous applications remain to be explored.

6. Comparative interpretation table

We have explored the limitations mentioned below and the number of pebbles required is different for the detour pebbling number and detour monophonic vertex cover pebbling number. These parameters sparked our interest to find out the detour edge cover pebbling number by incorporating the detour pebbling using the edge detour path. The various graph pebbling numbers shown in the table 12 demonstrate the uniqueness of the detour edge cover pebbling number of a graph.

Table 12. Numerous pebbling numbers in some standard graphs

Detour pebbling number [14]	Detour monophonic vertex cover pebbling number [4]	Detour edge cover pebbling number
$f^*(P_n) = 2^{n-1}$	$\mu_{vc}(P_n) = \sum_{i=0}^{\frac{n-2}{2}} 2^{2i}; n \text{ is even}$ & $\mu_{vc}(P_n) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} 2^{2i+1}; n \text{ is odd}$	$f_{ec}^*(P_n) = 2^{n-1} - 1$
$f^*(W_n) = 2^n$	$\mu_{vc}(W_n) = \sum_{i=0}^{\frac{n-3}{2}} 2^i + 5; n \text{ is even}$ & $\mu_{vc}(W_n) = 2 \left(\sum_{j=\lfloor \frac{n}{4} \rfloor + 1}^{\lfloor \frac{n}{2} \rfloor - 1} 2^{2j} \right) + 2^{\lfloor \frac{n}{2} \rfloor} + 3;$ $n \text{ is odd}, n \equiv 1 \pmod{4}$ & $\mu_{vc}(W_n) = 2 \left(\sum_{k=\lfloor \frac{n}{4} \rfloor + 1}^{\lfloor \frac{n}{2} \rfloor - 1} 2^{2k} \right) + 3;$ $n \text{ is odd}, n \equiv 3 \pmod{4}$	$f_{ec}^*(W_n) = (2n-1)2^{\frac{3n-1}{2}} + 1;$ $n \text{ is odd}$ & $f_{ec}^*(W_n) = (2n-1)2^{\frac{3n}{2}} + 1;$ $n \text{ is even.}$
$f^*(S_n) = n + 2$	-	$f_{ec}^*(S_n) = 2n - 1$
$f^*(C_n) = 2^{n-1}$	$\mu_{vc}(C_n) = \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-2} 2^k + 3; n \text{ is odd}$ & $\mu_{vc}(C_n) = 2 \left(\sum_{j=\lfloor \frac{n}{4} \rfloor + 1}^{\frac{n}{2}-1} 2^{2j} \right) + 1;$ $n \text{ is even}, n \equiv 2 \pmod{4}$ & $\mu_{vc}(C_n) = 2 \left(\sum_{i=\frac{n}{4}+1}^{\frac{n}{2}-1} 2^{2i} \right) + 2^{\frac{n}{2}} + 1;$ $n \text{ is even}, n \equiv 0 \pmod{4}$	$f_{ec}^*(C_n) = \sum_{i=1}^{\frac{n-1}{2}} 2 \cdot 2^{n-i} + 1; n \text{ is odd}$ & $f_{ec}^*(C_n) = \sum_{i=1}^{\frac{n}{2}-1} 2 \cdot 2^{n-i} + 2^{n-\frac{n}{2}} + 1;$ $n \text{ is even.}$

7. Conclusion

In Section 2, we provided a detailed, step-by-step guide to illustrate the process of finding the detour edge cover pebbling number, using a concrete example. In Section

3, we found the detour edge cover pebbling number for some standard graphs. In Section 4, we analysed the staircase graph and found the detour edge cover pebbling for staircase graphs. In Section 5, we discussed some applications of the detour edge cover pebbling concept. In the previous section, pre-existing limitations are addressed, and the detour edge cover pebbling concept is compared to some existing theories using a comparative interpretation table, demonstrating its uniqueness. The following problems are open.

Problem 1. Determine the detour edge cover pebbling number of other graph classes such as double staircase graphs, generalised Petersen graphs etc. Some applications are addressed. Finding the applications in the fields of Machine learning and Deep learning is an open problem.

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Conflict of Interest: According to the author(s), there is no conflict of interest.

Data Availability: Data sharing does not apply to this article because there were no datasets generated or analysed during the current study.

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