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Short Note

On word-representability of Sudoku graphs

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Abstract: For a positive integer n, the Sudoku graph $\mathrm{SG}(n)$ encodes the row, column, and block constraints of an $n^2 \times n^2$ Sudoku board. A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that two distinct letters x and y alternate in w if and only if $xy \in E$. Answering a question posed by Konstantinova, we prove that $\mathrm{SG}(n)$ is not word-representable for every $n \geq 2$.

Keywords: Sudoku graph, word-representable graph, semi-transitive orientation.

AMS Subject classification: 05C62, 05C20, 68R15

1. Introduction

The *Sudoku graph* $\mathrm{SG}(n)$ is defined as the constraints graph of a generalized Sudoku puzzle of size $n^2 \times n^2$ (the classical Sudoku puzzle corresponds to $n = 3$). Formally, the vertex set of $\mathrm{SG}(n)$ is $V = \{x_{ij} \mid i, j = 0, \ldots, n^2 - 1\}$ and vertices x_{ij} and $x_{k\ell}$ are adjacent if and only if the following condition holds: $i = k$ OR $j = \ell$ OR ($\lfloor i/n \rfloor = \lfloor k/n \rfloor$ AND $\lfloor j/n \rfloor = \lfloor \ell/n \rfloor$). The graph has n^4 vertices; it is $(3n^2 - 2n - 1)$-regular and thus has $n^4(3n^2 - 2n - 1)/2$ edges.

A proper n^2-coloring of $\mathrm{SG}(n)$ is precisely a completed Sudoku board with the colors interpreted as symbols. A puzzle with clues is a precoloring-extension problem, and

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its solutions are the extensions of that partial coloring. Consequently, chromatic and partial chromatic polynomials can be used to count Sudoku squares and puzzle completions [1, 3]. The graph model also exposes symmetry and spectral structure: Sudoku graphs are integral [8], and they can be described as Cayley graphs whose automorphism groups encode many of the usual board symmetries [7]. Variants of Sudoku can similarly be studied by changing the corresponding constraint graph.

In this note, we investigate the word-representability of the Sudoku graph. Recall that a graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that, for any distinct letters x and y, the letters x and y alternate in w if and only if $xy \in E$. Here, two distinct letters x and y are said to *alternate* in w if deleting all other letters leaves a word of the form $xyxy\cdots$ or $yxyx\cdots$, of either parity. The notion of word-representability was introduced in [5], motivated by the study of Perkins semigroups; see the monograph [4] for a comprehensive account of the area.

The question of whether the Sudoku graph is word-representable was posed by Professor E. V. Konstantinova at the international conference MOTOR-2026. In this note, we answer this question in the negative by proving that the Sudoku graph is not word-representable for $n \geq 2$.

2. The non-representability result

First we need the following known criterion of word-representable graphs [2]. An orientation is *semi-transitive* if it is acyclic and, whenever it contains a directed path

$$v_0 \to v_1 \to \cdots \to v_k$$

together with the edge v_0v_k, the vertices of the path induce the transitively oriented clique, that is, $v_i \to v_j$ is present for every $i < j$. A path together with $v_0 \to v_k$ that violates this condition is called a *shortcut*. So, the orientation is semi-transitive if it contains neither shortcuts nor directed cycles. The fundamental orientation characterization is the following.

Theorem 1 ([2]). *A graph is word-representable if and only if it admits a semi-transitive orientation.*

It is also known [6] that if a graph is word-representable, then for any prescribed vertex it has a semi-transitive orientation in which that vertex is a source. The following lemma easily follows from Theorem 1.

Lemma 1 ([6]). *Let $C = x_1x_2\cdots x_mx_1$, where $m \geq 4$, be a cycle whose vertices do not induce a clique. In a semi-transitive orientation, if $m-2$ edges of C point consistently around the cycle, then the remaining two point in the opposite direction.*

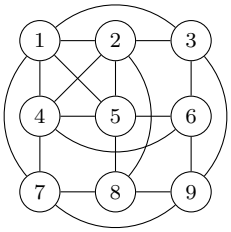

Figure 1. **The graph** G_0.

The main result of this note is the following.

Theorem 2. *The Sudoku graph* SG(n) *is non-word-representable for every* $n \geq 2$.

Proof. Consider the graph G_0 obtained from SG(2) by removing the last row and column. For brevity, we denote its vertices as shown in Fig. 1. First, we prove that G_0 is not word-representable.

Assume that G_0 has a semi-transitive orientation with the node 1 as a source. Due to the symmetry (reflection about the main diagonal), we may assume the arc $4 \to 2$. Then we have arcs $3 \to 2$ and $4 \to 7$ since otherwise 1423 or 1742 are shortcuts. Similarly, there are arcs $5 \to 2$ and $4 \to 5$ (corresponding shortcuts are 1325 and 1547). Consider two cases.

Case 1. There is an arc $7 \to 8$. Hence, $5 \to 8$ and $2 \to 8$ must exist (otherwise, we get shortcuts 1785 or 1782). By Lemma 2 applied to the cycle induced by $\{2, 3, 8, 9\}$ we have $3 \to 9 \to 8$. In order to avoid a shortcut 1397 we need an arc $7 \to 9$. Applying Lemma 2 to the cycle 47964 we get $4 \to 6 \to 9$. But now any direction of the edge 56 produces a shortcut: either 5698 or 1465, a contradiction.

Case 2. There is an arc $8 \to 7$. Then we have arcs $8 \to 2$ (otherwise there is a shortcut 1287) and hence $8 \to 5$ (shortcut 1582). Two subcases are possible.

Subcase 2.1. There is an arc $7 \to 9$. Then $8 \to 9$ appears to avoid the shortcut 17982. By Lemma 2 (for the cycle 47964) we have $4 \to 6 \to 9$. But then we cannot direct the edge 56: either 1465 or 8569 is a shortcut.

Subcase 2.2. There is an arc $9 \to 7$. It forces $9 \to 3$ (avoiding shortcut 1397) and then $9 \to 8$ (shortcut 8932). By Lemma 2 (for the cycle 56985) we have $9 \to 6 \to 5$. Hence $4 \to 6$ (otherwise, a shortcut 9647 arises), and now any orientation of the edge 36 induces a shortcut: either 3652 or 1463.

Thus, G_0 is not word-representable. For every $n \geq 2$, the graph SG(n) is an induced subgraph of SG($n+1$): given SG($n+1$), delete all vertices x_{ij} satisfying $i \geq n(n+1)$ or $j \geq n(n+1)$ or $i \equiv 0 \pmod{n+1}$ or $j \equiv 0 \pmod{n+1}$. Clearly, the remaining garph is isomorphic to $SG(n)$. Since G_0 is an induced subgraph of SG(2) and word-representability is a hereditary property, the theorem follows. □

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