Communications in Combinatorics and Optimization
Vol. 4 No. 2, 2019 pp.109-122
DOI: 10.22049/CCO.2019.26446.1112

CCO
Commun. Comb. Optim.

k-Efficient partitions of graphs

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Received: 6 August 2018; Accepted: 27 February 2019
Published Online: 1 March 2019

We dedicate this paper to our colleague and friend, Professor Lutz Volkmann, on the occassion of his 75th birthday. Our best wishes and congratulations to you, Lutz!

Abstract: A set $S = \{u_1, u_2, \dots, u_t\}$ of vertices of G is an efficient dominating set if every vertex of G is dominated exactly once by the vertices of S. Letting U_i denote the set of vertices dominated by u_i, we note that $\{U_1, U_2, \dots U_t\}$ is a partition of the vertex set of G and that each U_i contains the vertex u_i and all the vertices at distance 1 from it in G. In this paper, we generalize the concept of efficient domination by considering k-efficient domination partitions of the vertex set of G, where each element of the partition is a set consisting of a vertex u_i and all the vertices at distance d_i from it, where $d_i \in \{0, 1, \dots, k\}$. For any integer $k \geq 0$, the k-efficient domination number of G equals the minimum order of a k-efficient partition of G. We determine bounds on the k-efficient domination number for general graphs, and for $k \in \{1, 2\}$, we give exact values for some graph families. Complexity results are also obtained.

Keywords: Domination; Efficient domination; Distance-k domination

AMS Subject classification: 05C69

1. Introduction

A vertex in a graph G is said to dominate itself and all the vertices adjacent to it in G. Bange et al. [1] introduced the concept of efficiency for domination and defined a set

S of vertices of G to be an efficient dominating set if every vertex of G is dominated exactly once by the vertices of S. Thus, an efficient dominating set dominates every vertex with no redundancy (no vertex dominated twice). In this paper, we study a generalization of efficiency. First we present some notation and terminology.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Two vertices v and w are *neighbors* in G if they are adjacent; that is, if $vw \in E$. The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted $N(v)$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. If S is a subset of V, then the *open neighborhood* of S is $N(S) = \cup_{x \in S} N(x)$ and the *closed neighborhood* is $N[S] = \cup_{x \in S} N[x]$. The *degree* of v is the cardinality of its open neighborhood. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set $S \subseteq V$ is called a *dominating set* if $N[S] = V$, and the *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent, and the *vertex independence number* $\alpha(G)$ equals the maximum cardinality of an independent set in G. The subgraph induced by a set S of vertices in G is denoted $G[S]$. We use the standard notation $[k] = \{1, \ldots, k\}$.

For any two vertices u and v in G, the *distance* $d(u, v)$ equals the length (number of edges) of a shortest path between u and v in G. Such a path is called a *u-v geodesic*. The eccentricity $ecc(v)$ of a vertex $v \in V$ equals the maximum length of a v-w geodesic in G. The diameter of G equals $\text{diam}(G) = \max\{ecc(v) : v \in V\}$ and the *radius* $\text{rad}(G) = \min\{ecc(v) : v \in V\}$. For any integer k, the *distance-k closed neighborhood* of a vertex $v \in V$ is $N_k[v] = \{w \in V : d(v, w) \leq k\}$, and its *distance-$k$ open neighborhood* is $N_k(v) = N_k[v] - \{v\}$. The *k-degree*, denoted $\deg_k(v)$, of a vertex v in G is $|N_k(v)|$. The *minimum k-degree* of G is $\delta_k(G) = \min\{\deg_k(v) \ : v \in V\}$ and the *maximum k-degree* of G is $\Delta_k(G) = \max\{\deg_k(v) \ : v \in V\}$. Hence, $N_1(v) = N(v)$, $\deg_1(v) = \deg(v)$, $\delta_1(G) = \delta(G)$ and $\Delta_1(G) = \Delta(G)$.

In 1975, Meir and Moon [6] introduced the concepts of distance k-domination and distance-k packing. For a positive integer k, a set $D_k \subseteq V$ is a *distance k-dominating* set in G if every vertex $v \in V$ belongs to $N_k[w]$ for some vertex $w \in D_k$. The set D_k is a *k-packing* in G if $d(x, y) > k$ for all pairs of distinct vertices x and y in D_k. The *k-domination number* $\gamma_{\leq k}(G)$ is the minimum cardinality of a distance k-dominating set in G, and the *k-packing number* $\rho_k(G)$ is the maximum cardinality of a k-packing in G. Thus for $k = 1$, the distance 1-domination number $\gamma_{\leq 1}(G)$ equals the domination number $\gamma(G)$, and the 1-packing number $\rho_1(G)$ equals the vertex independence number $\alpha(G)$.

As defined in Bange et al. [1], a dominating set S for which $|N[v] \cap S| = 1$ for all $v \in V$ is an *efficient dominating set*. Equivalently, a set S is an efficient dominating set if S is both a dominating set and a 2-packing in G. Not every graph has an efficient dominating set, for example, the cycle C_5 does not.

It is important to note that if we define efficiency in terms of a partition of the vertices of G, we can say that G is 1-*efficient* if it has a set $S = \{v_1, v_2, \ldots, v_t\}$ for which $\pi = \{N[v_1], N[v_2], \ldots, N[v_t]\}$ is a (neighborhood) partition of V. In other words, G is 1-efficient if it has an efficient dominating set. More generally, a graph

G is *k-efficient*, for some positive integer k, if it has a set $S = \{v_1, v_2, \ldots, v_t\}$ for which $\pi = \{N_k[v_1], N_k[v_2], \ldots, N_k[v_t]\}$ is a (distance-k neighborhood) partition of V. Clearly, every graph G is rad-efficient, since for any vertex v for which $ecc(v)$ =rad(G), we have $N_{\text{rad}}[v] = V$.
We introduce the following generalization of efficiency, where the neighborhood distances are allowed to vary.

Definition 1. A partition $\pi = \{N_{i_1}[v_1], \ldots, N_{i_t}[v_t]\}$ is called a *k-efficient partition* of V if for every $j \in [t]$, we have $i_j \in \{0, 1, \ldots, k\}$. The vertices $v_1, v_2, \ldots, v_t$ will be called the *essential vertices* of the partition π.

Definition 2. For any integer $k \geq 0$, the *k-efficient domination number* of G, denoted $\varepsilon_k(G)$, equals the minimum order of a k-efficient partition of G. A k-efficient partition of G having cardinality $\varepsilon_k(G)$ is called an *ε_k-partition* of G.

Since the partition of the vertices of G into singleton sets is a 0-efficient partition of V, every graph G has a 0-efficient partition and $\varepsilon_0(G) = |V|$. It follows that every graph has a k-efficient partition for all $k \geq 0$ since a 0-efficient partition is a k-efficient partition. Recall that the cycle C_5 does not have an efficient dominating set, that is, C_5 is not 1-efficient. However, with the generalization of efficiency, C_5 has a 1-efficient partition. For example, $\{N_1[v_1], N_0[v_3], N_0[v_4]\}$ is an ε_1-partition of the 5-cycle $v_1v_2v_3v_4v_5v_1$, and so $\gamma(C_5) = 2 < \varepsilon_1(C_5) = 3$.
In this paper, we study the k-efficient domination number with a focus on $\varepsilon_1(G)$ and $\varepsilon_2(G)$, that is, we are interested primarily in minimum order partitions of the vertex set V into distance-0, distance-1 and distance-2 neighborhoods. Obviously, for any graph G, $\varepsilon_i(G) \geq \varepsilon_j(G)$ for $i \leq j$. We note that for every integer $t \geq 1$, there exists a graph G_t such that $\varepsilon_1(G_t) = \varepsilon_2(G_t) = t$. Indeed, if $t = 1$, then let G_1 be any nontrivial star, and if $t \geq 2$, then let G_t be the connected graph obtained from $t-1$ disjoint cycles C_6 sharing a same vertex.
In Section 2, we discuss the relationship between the k-efficient partitions and broadcasts in graphs. In Section 3, we give bounds on the k-efficient domination number, and in Section 4, we give exact values of the 1-efficient domination number for some graph families. In Section 5, we show that the decision problems associated with $\varepsilon_1(G)$ and $\varepsilon_2(G)$ are NP-complete, even when restricted to bipartite graphs. Finally, in Section 6 we mention several problems for further study.

2. Broadcast Domination

An equivalent formulation of k-efficient partitions can be given in terms of the concept of broadcasting in graphs. Broadcasts were introduced by Erwin [3] in 2004 and extended by Dunbar et al. [2] in 2006.
A function $f : V \to \{0, 1, 2, \ldots, \text{diam(G)}\}$ on the vertex set V of a connected graph $G = (V, E)$ is called a *broadcast* if for every vertex $v \in V$, $f(v) \leq ecc(v)$. The *cost*

$f(V)$ of a broadcast f is $f(V) = \Sigma_{v \in V} f(v)$. For a broadcast f, let $V_f^0 = \{v : f(v) = 0\}$ and $V_f^+ = V - V_f^0 = \{u : f(u) > 0\}$. The vertices in V_f^+ are called *broadcast* vertices. For a broadcast f and a broadcast vertex v, the *broadcast neighborhood* of v is the set $N_f[v] = \{u : d(u, v) \leq f(v)\}$. We say that every vertex in the broadcast neighborhood $N_f[v]$ can *hear* a broadcast from v, or is *broadcast dominated by* v.
A vertex u with $f(u) = 0$ *hears* a broadcast if there exists a vertex $v \in V_f^+$ where $d(u, v) \leq f(v)$. The set of vertices that a vertex u hears is the set $H(u) = \{v \in V_f^+ : d(u, v) \leq f(v)\}$. Thus, every broadcast vertex automatically hears itself. Define $H(f) \subseteq V$ to equal the set of vertices that hear a broadcast from at least one vertex $v \in V_f^+$. Finally, we say that a broadcast g satisfies $g \leq f$, if for every vertex $v \in V$, $g(v) \leq f(v)$. Notice that when we say $g \leq f$ we are saying that for at least one vertex v, $g(v) < f(v)$.
A broadcast f is called a *dominating broadcast* if $H(f) = V$, that is, for every vertex $u \in V$ with $f(u) = 0$, there exists a broadcast vertex v with $d(u, v) \leq f(v)$, or equivalently, if $H(u) \neq \emptyset$. The *broadcast domination number* $\gamma_b(G)$ of a graph G equals the minimum weight $f(V)$ of a dominating broadcast f in G. We say that a dominating broadcast f is *minimal* if there does not exist a dominating broadcast g for which (i) $g \leq f$.
The following characterization of minimal dominating broadcasts is due to Erwin [3].

Theorem 1. (Erwin [3]) *A dominating broadcast f on a graph G is minimal if and only if the following two conditions are satisfied:*

1. *for every broadcast vertex v with $f(v) \geq 2$, there exists a vertex $u \in V_f^0$ such that $H(u) = \{v\}$, and $d(u, v) = f(v)$, and*
2. *for every broadcast vertex v with $f(v) = 1$, there exists a vertex $u \in N[v]$ such that $H(u) = \{v\}$.*

Let $S \subseteq V$ be a minimal dominating set in a graph G. The *characteristic function* f_S of S is the broadcast function defined as follows: $f_S(v) = 0$ if $v \notin S$, and $f_S(v) = 1$ if $v \in S$.

Proposition 1. (Erwin [3]) *If $S \subseteq V$ is a minimal dominating set in a graph G, then the characteristic function f_S is a minimal dominating broadcast.*

Corollary 1. *For any graph G, $\gamma_b(G) \leq \gamma(G)$.*

It is worth pointing out that the broadcast domination number of a graph G can be considerably smaller than its domination number. For example, let $S(K_{1,n})$ denote a *subdivided star*, that is, a graph having one central vertex of degree n, to which are attached n paths of length 2. It is easy to see that $\gamma_b(S(K_{1,n})) = 2 < \gamma(S(K_{1,n})) = n$.
A broadcast f is called *efficient* if every vertex hears exactly one broadcast, that is, for every vertex $v \in V$, $|H(v)| = 1$. The minimum weight of an efficient broadcast is denoted $\gamma_{eb}(G)$. In [2], Dunbar et al. prove the following.

Theorem 2. (Dunbar et al. [2]) *Every graph G has an efficient γ_b-broadcast, that is, for any graph G, $\gamma_b(G) = \gamma_{eb}(G)$.*

This theorem means that every connected graph G can achieve $\gamma_b(G)$ with an efficient broadcast f. An efficient broadcast is equivalent to a k-efficient partition $\pi = \{N_{i_1}[v_1], \ldots, N_{i_t}[v_t]\}$, where (i) $1 \leq k = max\{f(v) : v \in V\}$, and (ii) for every $j \in [t]$, we have $i_j \in [k]$, that is, the value $i_j > 0$ always holds.
Notice that in broadcast domination, one is interested in the minimum value of the sum of the neighborhood distances, $f(V) = \Sigma_{i=1}^{t} i_j$, of a dominating broadcast, while in determining the k-efficient domination number, one is only interested in the minimum number of neighborhoods required, that is, the minimum t for which a k-efficient partition of order t exists. Further, for k-efficient domination, the distance-i neighborhoods of vertices for $0 \leq i \leq k$ are considered; while in broadcast domination, there is no upper limit on the value assigned to a broadcast vertex, other than $f(v) \leq \mathrm{diam(G)}$.

3. Bounds on $\varepsilon_k(G)$

Our first three observations follow directly from the definitions and previous comments.

Observation 3. For any graph G of order n and integers j and k, $1 \leq j \leq k$, $\varepsilon_k(G) \leq \varepsilon_j(G) \leq \varepsilon_0(G) = n$.

Observation 4. For any graph G, $\varepsilon_k(G) = 1$ for all $k \geq \mathrm{rad(G)}$.

Since the essential vertices of any k-efficient partition of G form a distance k-dominating set, the k-efficient domination number of every graph is bounded below by the distance k-domination number.

Observation 5. For every graph G and positive integer k, $\varepsilon_k(G) \geq \gamma_{\leq k}(G)$.

If we consider efficient distance k-dominating sets as a generalization of efficient dominating sets, then it is trivial that graphs G having such sets satisfy $\varepsilon_k(G) = \gamma_{\leq k}(G)$. In particular, if G has an efficient dominating set, then $\varepsilon_1(G) = \gamma(G)$. However, the converse is not true as can be seen by the connected graph G obtained from two cycles C_{2k+2} having one vertex in common. Then $\varepsilon_k(G) = \gamma_{\leq k}(G) = 3$, but G does not have an efficient distance k-dominating set.
Next we give a necessary condition for graphs G with $\varepsilon_k(G) = \gamma_{\leq k}(G)$.

Proposition 2. *For any graph G with $\varepsilon_k(G) = \gamma_{\leq k}(G)$ for $k \geq 1$, the set of essential vertices from any ε_k-partition of G is independent.*

Proof. Let G be a graph with $\varepsilon_k(G) = \gamma_{\leq k}(G) = t$, and let $\{N_{i_1}[v_1], \ldots, N_{i_t}[v_t]\}$ be an ε_k-partition of V, where $k \geq 1$ and $i_j \in \{0, 1, \ldots, k\}$ for every $j \in [t]$. Suppose, to the contrary, that $S = \{v_1, v_2, \ldots, v_t\}$ is not independent, and let v_p and v_q two adjacent vertices in S. Since $N_{i_p}[v_p] \cap N_{i_q}[v_q] = \emptyset$, we deduce that $i_p = i_q = 0$. But since $k \geq 1$, the set $S - \{v_q\}$ is a distance-k dominating set of G. Hence, $\gamma_{\leq k}(G) \leq |S - \{v_q\}| = |S| - 1 < \varepsilon_k(G)$, contradicting the fact that $\varepsilon_k(G) = \gamma_{\leq k}(G)$. □

Let $i(G)$ denote the minimum cardinality of an independent dominating set of G.

Corollary 2. *If G is a graph with $\varepsilon_1(G) = \gamma(G)$, then $\gamma(G) = i(G) = \varepsilon_1(G)$.*

Theorem 6. *Let G be a graph of order n, k be a positive integer, and $S \subseteq V$ be any $2k$-packing set of a graph G. Then*

$$\varepsilon_k(G) \leq n - |N_k(S)|.$$

Proof. Let $k \geq 1$ and let $S = \{v_1, v_2, \ldots, v_s\}$ be a $2k$-packing set of G (not necessarily of maximum order), and let $V - N_k[S] = \{u_1, u_2, \ldots, u_t\}$. Then

$$\{N_k[v_1], \ldots, N_k[v_s], N_0[u_1], \ldots, N_0[u_t]\}$$

is a k-efficient partition of G. Therefore,

$$\begin{aligned} \varepsilon_k(G) &\leq s + t \\ &= |S| + |V - N_k[S]| \\ &= n - |N_k(S)|. \end{aligned}$$

□

If we choose $S = \{v\}$, where v is a vertex of maximum k-degree, then we have the following corollary.

Corollary 3. *If G is a graph of order n and k is a positive integer, then*

$$\varepsilon_k(G) \leq n - \Delta_k(G).$$

Moreover, if S is a maximum $2k$-packing set in G, then $|N_k(S)| \geq \delta_k|S|$, and so Theorem 6 leads to the following.

Corollary 4. *If G is a graph of order n and $\delta(G) = \delta$, then*

$$\varepsilon_1(G) \leq n - \delta\rho_2(G).$$

Corollary 5. *If G is a graph of order n and $\delta_2(G) = \delta_2$, then*

$$\varepsilon_2(G) \leq n - \delta_2\rho_4(G).$$

Before showing that the upper bound of Corollary 4 is reached for all regular graphs, we shall show that this upper bound can also be strict and the difference $(n - \delta\rho_2(G)) - \varepsilon_1(G)$ may be arbitrarily large.

Proposition 3. *For every positive integer p, there exists a graph G_p of order n_p such that*

$$(n_p - \delta\rho_2(G_p)) - \varepsilon_1(G_p) \geq p.$$

Proof. Let $H_i = K_{2,3}$ and denote by x_i and y_i the vertices of $V(H_i)$ of degree three. For a positive integer p, let G_p be a graph obtained from $H_1, H_2, \ldots, H_p$ by adding $p-1$ edges between vertices of degree two so that G_p is connected. Clearly, $n_p = 5p$ and $\delta(G_p) = \delta = 2$. Since no two vertices in a maximum 2-packing can be in the same subgraph H_i, it follows that $\rho_2(G_p) = p$.
On the other hand, $\{N_1[x_1], \ldots, N_1[x_p], N_0[y_1], \ldots, N_0[y_p]\}$ is a 1-efficient partition of G_p, and so, $\varepsilon_1(G_p) \leq 2p$. Therefore, $(n_p - \delta\rho_2(G_p)) - \varepsilon_1(G_p) \geq p$. □

Theorem 7. *If G is an r-regular graph of order n, then $\varepsilon_1(G) = n - r\rho_2(G)$.*

Proof. Let G be an r-regular graph of order n. By Corollary 4, we have that $\varepsilon_1(G) \leq n - r\rho_2(G)$. Suppose, to the contrary, that $\varepsilon_1(G) < n - r\rho_2(G)$.
Let $\varepsilon_1(G) = p$ and $\{N_1[v_1], \ldots, N_1[v_q], N_0[v_{q+1}], \ldots, N_0[v_p]\}$ be an ε_1-partition of V. Note that $q \geq 1$, and since $S_1 = \{v_1, v_2, \ldots, v_q\}$ is a 2-packing set of G, we have $q \in [\rho_2(G)]$. Now using the fact that G is r-regular, we deduce that $\varepsilon_1(G) = n - r|S_1|$. Thus, $n - r|S_1| = \varepsilon_1(G) < n - r\rho_2(G)$ implies that $|S_1| > \rho_2(G)$, which is impossible. Hence, $\varepsilon_1(G) = n - r\rho_2(G)$. □

We close this section by giving a Nordhaus-Gaddum bound for $\varepsilon_k(G) + \varepsilon_k(\overline{G})$ in terms of the order of the graph G.

Theorem 8. *For every graph G of order n and positive integer k,*

$$\varepsilon_k(G) + \varepsilon_k(\overline{G}) \leq n + 1.$$

Proof. By Corollary 3, $\varepsilon_k(G) \leq n - \Delta_k(G)$. Let v be a vertex having k-degree $\Delta_k(G)$ in G. Let $\{u_1, u_2, \ldots, u_t\}$
be the neighbors of v in G. Note that $\deg(v) = t \leq \deg_k(v) = \Delta_k(G)$ in G. Then in $\overline{G}$, the partition

$$\{N_1[v]], N_0[u_1], \ldots, N_0[u_t]\}$$

is a k-efficient partition of $V(\overline{G})$. Hence, $\varepsilon_k(G) + \varepsilon_k(\overline{G}) \leq n - \Delta_k(G) + 1 + t \leq n - \Delta_k(G) + 1 + \Delta_k(G) = n + 1$. □

4. Exact Values of $\varepsilon_1(G)$ for Some Graphs G

We begin with characterizations of graphs G with $\varepsilon_1(G) \in \{1, 2\}$. The first result applies to $\varepsilon_k(G)$ all $k \geq 0$.

Proposition 4. *For every graph G of order n, $\varepsilon_k(G) = 1$ if and only if $\Delta_k(G) = n - 1$.*

Proof. Let v be a vertex of degree $\Delta_k(G) = n-1$ in G. Then $\{N_k[v]\}$ is a k-efficient partition of V, so $\varepsilon_k(G) = 1$. Clearly, if $\varepsilon_k(G) = 1$, then $\Delta_k(G) = n - 1$. □

Proposition 5. *For every graph G, $\varepsilon_1(G) = 2$ if and only if either*

i) G has two components G_1 and G_2 such that $\varepsilon_1(G_1) = \varepsilon_1(G_2) = 1$, or

ii) G is connected such that either $\Delta(G) = n-2$ or G has a minimum efficient dominating set of cardinality 2.

Proof. Let G be a graph such that $\varepsilon_1(G) = 2$, and let $\{N_i[x], N_j[y]\}$ be an $\varepsilon_1(G)$-partition of V, where $i, j \in \{0, 1\}$. Clearly, G can have at most two components. Now, if G is not connected, then G has exactly two components, G_1 and G_2, such that $\varepsilon_1(G_1) = \varepsilon_1(G_2) = 1$. Hence, (i) holds. Thus, we may assume that G is connected. It follows that $\max\{i, j\} \neq 0$. Without loss of generality, we assume that $i = 1$. Moreover, by Proposition 4, we have $\Delta(G) \leq n - 2$. Now if $j = 0$, then $V - N[x] = \{y\}$, and so, $\Delta(G) = n-2$ and (ii) holds. Therefore, we may assume that $j = 1$. Thus, $\{x, y\}$ is a minimum efficient dominating set of G, and so, (ii) holds. The converse is straightfoward. □

Next provide exact values of $\varepsilon_k(G)$ for some particular graphs G.

Theorem 9. *If G is a graph of order n and diameter 2, then $\varepsilon_1(G) = n - \Delta(G)$.*

Proof. By Corollary 3, $\varepsilon_1(G) \leq n - \Delta(G)$. Since diam(G) = 2, $\Delta(G) \geq 2$. Suppose that $\varepsilon_1(G) < n - \Delta(G)$, and let $\pi = \{N_{i_1}[v_1], \ldots, N_{i_t}[v_t]\}$ be an ε_1-partition of V, where $i_j \in \{0, 1\}$ for all $j \in [t]$. Clearly, if $i_j = 0$ for every j, then $|\pi| = n$, a contradiction. Suppose now there are two indices p and q such that $i_p = i_q = 1$. Then vertices v_p and v_q are at distance at least three from each other, contradicting the fact that G has diameter two. Hence, there is exactly one index j such that $i_j = 1$. But then $|\pi| = |V| - |N_1(v_j)| = n - \deg(v_j) < n - \Delta(G)$, implies that $|N_1(v_j)| > \Delta(G)$, which is impossible. Therefore, $\varepsilon_1(G) = n - \Delta(G)$. □

As an immediate consequence of Theorem 9, we have the following.

Corollary 6. *If G is a complete bipartite graph $K_{p,q}$ with $p \geq q \geq 1$, then $\varepsilon_1(G) = p$.*

Next we give the 1-efficient domination number of cycles and paths. It is well-known (see [4]) that if G is a path P_n or a cycle C_n, then $\gamma(G) = \lceil n/3 \rceil$, $\rho_2(P_n) = \lceil n/3 \rceil$ and $\rho_2(C_n) = \lfloor n/3 \rfloor$. Hence, Theorem 7 leads to the following result.

Corollary 7. *For every cycle C_n with $n \geq 3$,*

$$\varepsilon_1(C_n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod 3 \\ (n+2)/3 & \text{if } n \equiv 1 \pmod 3 \\ (n+4)/3 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proposition 6. *For every path P_n on n vertices,*

$$\gamma(P_n) = i(P_n) = \varepsilon_1(P_n) = \lceil n/3 \rceil .$$

Proof. Since the result holds for $n \leq 2$, we may assume that $n \geq 3$. By Observation 5, $\varepsilon_1(P_n) \geq \gamma(P_n) = \lceil n/3 \rceil$. Let $v_1, v_2, \ldots, v_n$ be the path P_n. Let $\pi = \{N_1[v_{3i+2}] : 0 \leq i \leq \lfloor \frac{n-2}{3} \rfloor\}$. If $n \equiv 0, 2 \bmod 3$, then π is a 1-efficient partition of $V(P_n)$. If $n \equiv 1 \bmod 3$, then $\pi \cup \{N_0[v_n]\}$ is a 1-efficient partition of $V(P_n)$. In any case, $\varepsilon_1(P_n) \leq \lceil n/3 \rceil$, and the equality follows. $\square$

We next extend Proposition 6 to 2-by-n and 3-by-n grid graphs. An *m-by-n grid graph* $G_{m,n}$ is the Cartesian product $G_{m,n} = P_m \square P_n = (V(P_m) \times V(P_n), E_{m,n})$, where two vertices (v_i, v_j) and (v_k, v_l) are adjacent in $E_{m,n}$ if and only if either $v_i = v_k$ and v_j is adjacent to v_l in P_n or v_i is adjacent to v_k in P_m and $v_j = v_l$. For ease of discussion, we will consider the m-by-n grid as a m-by-n array and denote by $v_{i,j}$ the vertex in the *ith* row and *jth* column for $i \in [m]$ and $j \in [n]$.

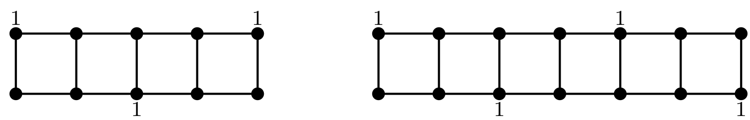


Figure 1. **Examples 2-by-n Grids, n odd**

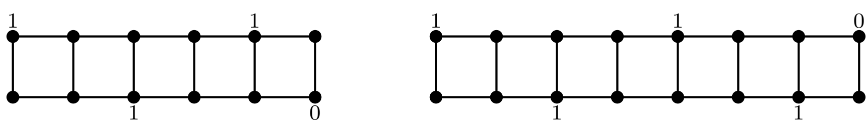
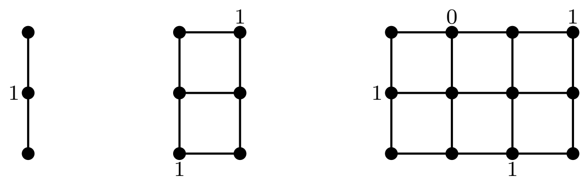


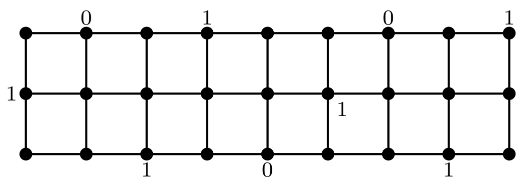
Figure 2. **Examples 2-by-n Grids, n even**

Theorem 10. *For any 2-by-n grid graph G, $\varepsilon_1(G) = \gamma(G) = \lfloor (n+2)/2 \rfloor$.*

Proof. The value of the domination number of 2-by-n grid graphs in the statement of the theorem was originally determined by Jacobson and Kinch in 1983 [5].
By Observation 5, $\varepsilon_1(G) \geq \gamma(G) = \lfloor (n+2)/2 \rfloor$. To complete the proof, it suffices to give a 1-efficient partition of order $\lfloor (n+2)/2 \rfloor$. We build such a partition by selecting the essential vertices as follows: beginning with vertex $v_{1,1}$, select a vertex from each odd labeled column by alternating rows. If n is odd, we are finished. If n is even, then we add the vertex from the nth column that is in a different row from the vertex selected from column $n-1$. Thus, if n is odd, the set of selected essential vertices is $\{v_{1,1}, v_{2,3}, v_{1,5}, v_{2,7}, \ldots, v_{i,n-2}, v_{3-i,n}\}$, where i is the appropriate row label in the sequence. If n is even, then the set is $\{v_{1,1}, v_{2,3}, v_{1,5}, v_{2,7}, \ldots, v_{j,n-3}, v_{3-j,n-1}, v_{j,n}\}$, where j is the appropriate row label in the sequence. For examples, see Figures 1 and 2, where the selected essential vertices are labeled either 0 or 1 depending on the k-neighborhood, $k \in \{0,1\}$, used in the 1-efficient partition.
Then if n is odd, $\{N_1[v_{1,1}], N_1[v_{2,3}], \ldots, N_1[v_{i,n}]\}$, where i is the appropriate row label, is a 1-efficient partition of V having cardinality $\lfloor (n+2)/2 \rfloor$. For even n, $\{N_1[v_{1,1}], N_1[v_{2,3}], \ldots, N_1[v_{i,n-3}], N_1[v_{3-i,n-1}], N_0[v_{i,n}]\}$, where i is the appropriate row label, is a 1-efficient partition of V having cardinality $\lfloor (n+2)/2 \rfloor$. □



(a) $n=1$ (b) $n=2$ (c) $n=4$



(d) $n=9$

Figure 3. **Base Patterns 3-by-n Grid**

Theorem 11. *For any 3-by-n grid graph $G_{3,n}$, $\varepsilon_1(G_{3,n}) = n$, unless $n=3$, in which case $\varepsilon_1(G_{3,n}) = 4$.*

Proof. We first show that for every $n \geq 1$, $\varepsilon_1(G_{3,n}) \geq n$. Assume that for some n, $\varepsilon_1(G_{3,n}) < n$. This means that there exists at least one column, none of the three vertices in which are essential vertices in any ε_1-partition of $G_{3,n}$. Hence, each of these three vertices must appear in a 1-neighborhood of an essential vertex in either the preceding or succeeding column. This means that at least two of these three essential vertices must appear in the same column, implying that the distance between these two essential vertices is at most 2, contradicting the fact that the essential vertices in any 1-efficient partition form a 2-packing. Hence, $\varepsilon_1(G_{3,n}) \geq n$.

It is straightforward to check that $\varepsilon_1(G_{3,3}) = 4$. It remains to show that except for $n = 3$, every 3-by-n grid graph has an 1-efficient partition of order n. Again we build a 1-efficient partition by selecting essential vertices. For this selection, we assign the label of i for some $i \in \{0, 1\}$ to an essential vertex v to indicate that the $N_i[v]$ is in the 1-efficient partition.

For $n = 1$, assign the label of 1 to vertex $v_{2,1}$. For $n = 2$, assign the label of 1 to each of $v_{3,1}$ and $v_{1,2}$. For $n = 4$, assign 1 to each of $v_{2,1}$, $v_{3,3}$, and $v_{1,4}$, and assign 0 to $v_{1,2}$. For $n = 6$, use the assignment of labels for $n = 4$ on the first four columns, and assign 1 to $v_{2,6}$ and 0 to $v_{3,5}$. If $n = 9$, then use the assignment of labels for $n = 6$ for the first six columns, and assign 1 to each of $v_{3,8}$ and $v_{1,9}$ and 0 to $v_{1,7}$. See Figure 3 for illustration of these labelings.

To complete the set of essential vertices, we consider n modulo 3. We refer to the labeling shown in Figure 4 of three consecutive columns, where the third vertex in the first row and the second vertex in the third row are labeled 1 and the first vertex in the second row is labeled 0, as pattern R. If $n = 3j$ for $j \geq 4$, then use the labeling for $n = 9$ on the first nine columns and repeat the pattern R labeling $j - 3$ times to label the remaining grid columns. If $n = 3j + 1$ for $j \geq 2$, then use the labeling for $n = 4$ on the first four columns and repeat the pattern R labeling $j - 1$ times to label the remaining grid columns. If $n = 3j + 2$ for $j \geq 1$, then use the labeling for $n = 2$ on the first two columns and repeat the pattern R labeling j times to label the remaining grid columns. This completes the proof. □

It is interesting to compare $\varepsilon_1(G_{3,n}) = n$ with the domination number $\gamma(G_{3,n}) = \lfloor \frac{3n+4}{4} \rfloor$, which was originally determined by Jacobson and Kinch in 1983 [5].

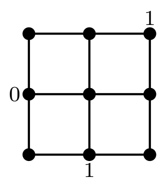


Figure 4. **Pattern** R

5. Complexity

Our aim in this section is to show that the decision problems associated with $\varepsilon_1(G)$ and $\varepsilon_2(G)$ are NP-complete, even when restricted to bipartite graphs.

r-EFFICIENCY PARTITION PROBLEM (r-EPP)
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have an r-efficient partition of order at most k?

We will show that this problem is NP-complete by reducing the well-known NP-complete problem, Exact-3-Cover (X3C), to r-EPP.

EXACT 3-COVER (X3C)
Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 12. *1-EPP is NP-Complete for bipartite graphs.*

Proof. Clearly, 1-EPP is a member of $\mathcal{NP}$, since we can check in polynomial time that a partition of order at most k is 1-efficient. Now let us show how to transform any instance of X3C into an instance G of 1-EPP so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.
For each $x_i \in X$, we create a cycle $C_4 : x_i u_i v_i w_i x_i$. For each $C_j \in C$, we create a path $P_3 : a_j b_j c_j$. Now to obtain a graph G, we add edges $c_j x_i$ if $x_i \in C_j$. Clearly, G is a bipartite graph. Set $k = t + 4q$.
Suppose that the instance X, C of X3C has a solution C'. We construct a 1-efficient partition π of order k by putting in π the sets $N_1[v_1]$, $N_1[v_2]$,..., $N_1[v_{3q}]$. Moreover, for each $C_j \in C'$, we put in π the sets $N_1[c_j]$ and $N_0[a_j]$, and for each $C_j \notin C'$, we put in π the set $N_1[b_j]$. Note that since C' exists, its cardinality is precisely q, and so the number of essential vertices c_j's in π is q, having disjoint neighborhoods in $\{x_1, x_2, \ldots, x_{3q}\}$. Hence, π is a 1-efficient partition of V of order $3q + 2q + (t - q) = k$.
Conversely, suppose that G has a 1-efficient partition π of order at most k. Clearly, at least one vertex of each path $a_j b_j c_j$ is essential in π. Likewise, at least one vertex from the set $\{u_i, v_i, w_i\}$ is essential in π; we can assume, without loss of generality, that it is vertex v_i. Using the fact that each c_j has exactly three neighbors in X, and since we can not consider more than q other essential vertices in π, the only way to spread the vertices of X in π is to consider the 1-neighborhood of q vertices of $\{c_1, c_2, \ldots, c_t\}$. Consequently, $C' = \{C_j : c_j \text{ is essential in } \pi\}$ is an exact cover for C. $\square$

Theorem 13. *2-EPP is NP-Complete for bipartite graphs.*

Proof. Clearly, 2-EPP is a member of $\mathcal{NP}$, since we can check in polynomial time that a partition of order at most k is 2-efficient. Now let us show how to transform any instance of X3C into an instance G of 2-EPP so that one of them has a solution

if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.
For each $x_i \in X$, we build a connected graph H_i obtained from a 6-cycle $y_i^1 y_i^2 y_i^3 y_i^4 y_i^5 y_i^6 y_i^1$ by adding a vertex x_i and the edge $x_i y_i^1$. For each $C_j \in C$, we create a path $P_4 : c_j u_j v_j w_j$. Now to obtain a graph G, we add edges $c_j x_i$ if $x_i \in C_j$. Clearly, G is a bipartite graph. Set $k = t + 4q$.
Suppose that the instance X, C of X3C has a solution C'. We construct a 2-efficient partition π of order k by putting in π the sets $N_2[y_1^4]$, $N_2[y_2^4]$,..., $N_2[y_{3q}^4]$. Moreover, for each $C_j \in C'$, we put in π the sets $N_2[c_j]$ and $N_0[w_j]$, and for each $C_j \notin C'$, we put in π the set $N_p[v_j]$, where $p \in \{1, 2\}$ depending on whether c_j belongs or not to $N_2[c_d]$ for some $C_d \in C'$, respectively. We note that since C' exists, its cardinality is precisely q, and so the number of essential vertices c_j's in π is q, having disjoint neighborhoods in $\{x_1, x_2, \ldots, x_{3q}\}$. Moreover, it is easy to see that π is a 2-efficient partition of V of order $3q + 2q + (t - q) = k$.
Conversely, suppose that G has a 2-efficient partition π of order at most k. Clearly, for each j at least one vertex of $\{u_j, v_j, w_j\}$ is essential in π. Likewise, for each i, at least one vertex of $\{y_i^1, y_i^2, y_i^3, y_i^4, y_i^5, y_i^6\}$ is essential in π. Using the fact that each c_j has exactly three neighbors in X, and since we can not consider more than q other essential vertices in π, the only way to spread in π the remaining vertex of each C_6 as well as those of X is to consider the 2-neighborhood of q vertices of $\{c_1, c_2, \ldots, c_t\}$. Consequently, $C' = \{C_j : c_j \text{ is essential in } \pi\}$ is an exact cover for C. □

6. Open Problems

We conclude by mentioning several open problems.

Problem 1. Design algorithms to compute the values of $\varepsilon_1(T)$ and $\varepsilon_2(T)$ for any tree T.

Problem 2. Characterize the graphs G (or at least trees) such that $\varepsilon_1(G) = \gamma(G)$.

Problem 3. Characterize the graphs G (or at least trees) such that $\varepsilon_1(G) = \varepsilon_2(G)$.

Problem 4. Determine $\varepsilon_1(G_{m,n})$ for every *m-by-n grid graph* $G_{m,n}$.

References

[1] D.W. Bange, A.E. Barkauskas, and P.J. Slater, *Efficient dominating sets in graphs*, Applications of Discrete Math. , R. D. Ringeisen and F. S. Roberts, eds., SIAM, Philadelphia (1988), 189–199.

[2] J.E. Dunbar, D.J. Erwin, T.W. Haynes, S.M. Hedetniemi, and S.T. Hedetniemi, *Broadcasts in graphs*, Discrete Appl. Math. **154** (2006), no. 1, 59–75.

[3] D.J. Erwin, *Dominating broadcasts in graphs*, Bull. Inst. Comb. Appl. **42** (2004), 89–105.

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[5] M.S. Jacobson and L.F. Kinch, *On the domination number of products of graphs: I*, Ars Combin. **18** (1983), 33–44.

[6] A. Meir and J. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975), no. 1, 225–233.