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Research Article

Algorithmic aspects of quasi-total Roman domination in graphs

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Abstract: For a simple, undirected, connected graph $G(V, E)$, a function $f : V(G) \rightarrow \{0, 1, 2\}$ which satisfies the following conditions is called a quasi-total Roman dominating function (QTRDF) of G with weight $f(V(G)) = \sum_{v \in V(G)} f(v)$.
C1). Every vertex $u \in V(G)$ for which $f(u) = 0$ must be adjacent to at least one vertex v with $f(v) = 2$, and
C2). Every vertex $u \in V(G)$ for which $f(u) = 2$ must be adjacent to at least one vertex v with $f(v) \geq 1$.
For a graph G, the smallest possible weight of a QTRDF of G denoted $\gamma_{qtR}(G)$ is known as the *quasi-total Roman domination number* of G. The problem of determining $\gamma_{qtR}(G)$ of a graph G is called minimum quasi-total Roman domination problem (MQTRDP). In this paper, we show that the problem of determining whether G has a QTRDF of weight at most l is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs and planar graphs. On the positive side, we show that MQTRDP for threshold graphs, chain graphs and bounded treewidth graphs is linear time solvable. Finally, an integer linear programming formulation for MQTRDP is presented.

Keywords: Domination number, quasi-total Roman domination, complexity classes, graph classes, linear programming

AMS Subject classification: 05C69, 68Q25

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1. Introduction

Let $G(V, E)$ be a simple, undirected and connected graph. For a vertex u of G, the (*open*) *neighborhood* which is denoted by $N_G(u)$, is the set $\{v : (u, v) \in E(G)\}$ and its *degree* is $|N_G(u)|$. The *closed neighborhood* of u is $N_G[u] = \{u\} \cup N_G(u)$. *Maximum degree* of G, denoted by Δ, is $max_{u \in V(G)}|N_G(u)|$. A vertex v is called an *isolated vertex* if $|N_G(v)| = 0$. A vertex of degree $n-1$ is called a *universal vertex*, where $n = |V(G)|$. A graph formed with the vertex set $S \subseteq V(G)$ of graph G and the edge set $\{(u, v) : u, v \in S\}$ is called an *induced subgraph* of G and is denoted by $G[S]$. A subset S of $V(G)$ in a graph G is said to be *independent* if no two vertices in S are adjacent. A graph $G(V_1, V_2, E)$ is called a *split graph* if V_1 is an independent set, V_2 is a clique, $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. A graph is *chordal* if each of its cycles of length four or more has a *chord* i.e., an edge that connects two nonadjacent vertices of the cycle. For an integer $n \geq 1$, a *star graph* is a complete bipartite graph $K_{1,n}$. The maximum degree vertex of a star graph is called the *center vertex* of it. A bipartite graph $G = (X, Y, E)$ is called *tree convex* if there exists a tree $T = (X, F)$ such that, for each y in Y, the neighbors of y induce a subtree in T. When T is a star (comb), G is called *star* (*comb*) *convex bipartite* graph [14]. For undefined terminology and notations we refer to [19].
A *dominating set* (DS) of a graph G is a set $D \subseteq V(G)$ such that $\cup_{v \in D} N_G[v] = V(G)$. The *domination number* of G, which is denoted by $\gamma(G)$, is $min\{|D| : D$ is a dominating set of $G\}$. Given a graph G and a positive integer l, the domination decision problem is to check whether G has a dominating set of cardinality at most l. Literature on the concept of domination has been surveyed in [7, 8].
In 2004, the concept of Roman domination was introduced by Cockayne et al. in [2]. A function $f : V(G) \rightarrow \{0, 1, 2\}$ such that every vertex with label zero is adjacent to at least one vertex with label two is called a *Roman dominating function* (RDF) on G. A vertex is said to be *Roman-dominated* if it‘s label is one or two; or zero and adjacent to a vertex with label two. We will identify a function f with the subsets V_0, V_1, V_2 of $V(G)$ associated with it, and so we will also use the notation $f(V_0, V_1, V_2)$ for the function and these associated subsets. We use these notations interchangeably in this article. We refer to [2, 4, 5, 9–13, 16, 17] for the literature on the concept of Roman domination in graphs.

Quasi-total Roman domination was introduced in 2019 by S. Cabrera García et al. [6]. A function $f : V(G) \rightarrow \{0, 1, 2\}$ is said to be a *quasi-total Roman dominating function* (QTRDF) of G if the following properties hold.
C1). Every vertex $u \in V(G)$ for which $f(u) = 0$ must be adjacent to at least one vertex v with $f(v) = 2$, and
C2). Every vertex $u \in V(G)$ for which $f(u) = 2$ must be adjacent to at least one vertex v with $f(v) \geq 1$.
The weight of a QTRDF f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *quasi-total Roman domination number* is the minimum weight of a QTRDF on G and is denoted by $\gamma_{qtR}(G)$. The minimum quasi-total Roman domination problem (MQTRDP) is

to find a QTRDF of minimum weight in the input graph. The decision version of quasi-total Roman domination problem is defined as follows.

Quasi-Total-Roman-Domination-Problem (QTRDP)
Instance : A simple, undirected graph G and a positive integer k.
Question : Is $\gamma_{qtR}(G) \leq k$?
In particular, when the input graph of QTRDP is split, star convex bipartite, comb convex bipartite or planar the corresponding decision versions are denoted QTRDPS, QTRDPSC, QTRDPCC and QTRDPP respectively.

2. Complexity results

In this section, we show that QTRDPS, QTRDPSC and QTRDPCC are NP-complete by proposing a polynomial time reduction from a well-known NP-complete problem, Exact Three Set Cover ($X3SC$)[8], which is defined as follows.
Exact Three Set Cover ($X3SC$)
Instance : A set $X = \{x_1, x_2, \ldots, x_{3q}\}$, where $q \geq 1$ and another set $C = \{C_1, C_2, \ldots, C_t\}$, where C_i is a subset of X with $|C_i| = 3$.
Question : Does C have a subset C' such that $\cup_{C_i \in C'} C_i = X$ and $C_i \cap C_j = \emptyset$ for all $C_i, C_j \in C'$ and $i \neq j$.

Theorem 1. *QTRDPS is NP-complete.*

Proof. Given a split graph G and a function f, whether f is a QTRDF of weight at most k can be checked in polynomial time. Hence QTRDPS is a member of NP. Now we show that QTRDPS is NP-hard by transforming an instance (X, C) of *X3SC*, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$, to an instance (G, k) of QTRDPS as follows. Create vertices x_i for each $x_i \in X$ and c_i for each $C_i \in C$. Add edges (c_j, x_i) if $x_i \in C_j$ and (c_i, c_j) if $1 \leq i, j\ (\neq i) \leq t$. Let $k = 2q$. Since X is an independent set and C is a clique, it follows that G is a split graph as shown in Figure 1 and can be constructed from the given instance (X, C) of *X3SC* in polynomial time and (G, k) is an instance of QTRDPS.
Next we show that, *X3SC* has a solution if and only if G has a QTRDF with weight at most $2q$. Suppose C' is a solution for the given instance (X, C) of *X3SC* with $|C'| = q$. If $q = 1$ then $G = K_{1,3}$ and from [6] it follows that G has a QTRDF with weight 3. Otherwise, let $D = \{c_i : C_i \in C'\}$. We define a function $f : V(G) \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in D \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Clearly, f is a QTRDF and $f(V(G)) = 2q$.
Conversely, suppose that G has a QTRDF g with weight $2q$.
Claim 1: For each $x_i \in V(G)$, $g(x_i) = 0$.
Proof of Claim 1: By contradiction, assume that there exist some x_i's such that

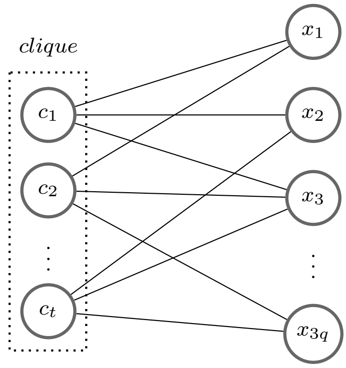

Figure 1. **Construction of a split graph from an instance of** $X3SC$

$g(x_i) \neq 0$. Let $p = |\{x_i : g(x_i) \neq 0\}|$. The number of x_i's with $g(x_i) = 0$ is $3q - p$. Since g is a QTRDF of G, each x_i with $g(x_i) = 0$ should have a neighbor c_j with $g(c_j) = 2$. Clearly $\lceil \frac{3q-p}{3} \rceil$ number of c_j's with $g(c_j) = 2$ are required. Hence $g(V(G)) \geq p + 2\lceil \frac{3q-p}{3} \rceil$, which is greater than $2q$, a contradiction. Therefore for each $x_i \in V(G)$, $g(x_i) = 0$.
Since each c_i is adjacent to three vertices in $\{x_1, x_2, \ldots, x_{3q}\}$, clearly, $|\{c_1, c_2, \ldots, c_t\} \cap V_2| = q$. Now $\{C_i : g(c_i) = 2\}$ is an exact cover for $X3SC$. □

Since any split graph is a chordal graph, we have the following.

Corollary 1. *QTRDP is NP-complete for chordal graphs.*

Theorem 2. *QTRDPSC is NP-complete.*

Proof. Given a star convex bipartite graph G and a function f, whether f is a QTRDF of weight at most k can be checked in polynomial time. Hence QTRDPSC is a member of NP. Now we show that QTRDPSC is NP-hard by transforming an instance (X, C) of *X3SC*, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$, to an instance (G, k) of QTRDPSC as follows. Create vertices x_i for each $x_i \in X$ and c_i for each $C_i \in C$ and also create vertices d, d_1, d_2, d_3. Add edges (d_i, d) for each d_i and (c_i, d) for each c_i. Also add edges (c_j, x_i) if $x_i \in C_j$. The graph constructed is shown in Figure 3. Let $A = \{d\} \cup \{x_i : 1 \leq i \leq 3q\}$ and $B = V(G) \setminus A$. The set A induces a star with vertex d as central vertex, as shown in Figure 2, and the neighbors of each element in B induce a subtree of star. Therefore G is a star convex bipartite graph and can be constructed from the given instance (X, C) of *X3SC* in polynomial time. Next we show that *X3SC* has a solution if and only if G has a QTRDF with weight at most $2q + 2$.
Suppose C' is a solution for $X3SC$ with $|C'| = q$. Let $D = \{c_i : C_i \in C'\}$. We define

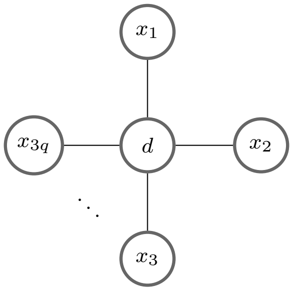

Figure 2. **Star graph associated with the star convex bipartite graph**

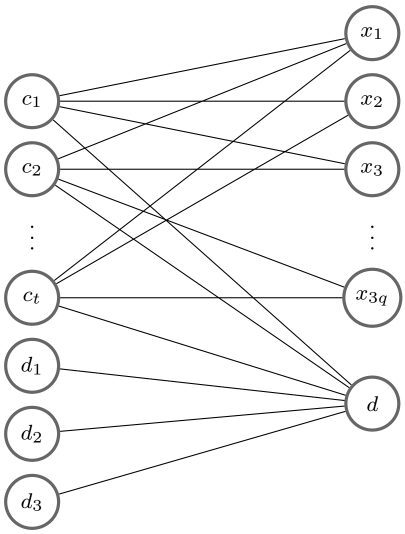

Figure 3. **Star convex bipartite graph construction from $X3SC$ instance**

a function $f : V(G) \to \{0, 1, 2\}$ as follows.

$$f(u) = \begin{cases} 2, & \text{if } u \in D \text{ or } u = d \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Clearly, f is a QTRDF and $f(V(G)) = 2q + 2$.
Conversely, suppose that G has a QTRDF g with weight $k = 2q + 2$. We state the following claim without proof.
Claim 2: $g(\{x_1, x_2, \ldots, x_{3q}\}) = 0$ and $g(d) + g(d_1) + g(d_2) + g(d_3) \geq 2$.
Clearly, $|\{c_1, c_2, \ldots, c_t\} \cap V_2| = q$. Now $\{C_i : g(c_i) = 2\}$ is an exact cover for $X3SC$.
□

Theorem 3. *QTRDPCC is NP-complete.*

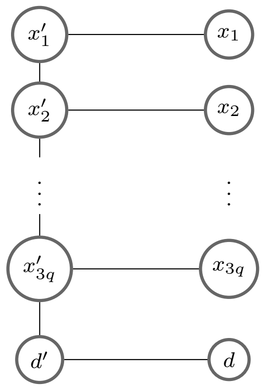

Figure 4. **Comb graph associated with the comb convex bipartite graph**

Proof. Clearly, QTRDPCC is a member of NP. Now we show that QTRDPCC is NP-hard by transforming an instance (X, C) of *X3SC*, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$, to an instance (G, k) of QTRDPCC as follows. Create vertices x_i, x'_i for each $x_i \in X$ and c_i for each $C_i \in C$ and also create vertices d, d', d_1, d_2, d_3. Add edges (c_j, x_i) if $x_i \in C_j$, (d_i, d) for each d_i and (c_i, d), (c_i, d') for each c_i. Also add edges by joining each c_j to every x'_i. The graph constructed is shown in Figure 5. Let $A = \{d, d'\} \cup \{x_i, x'_i : 1 \leq i \leq 3q\}$ and $B = V(G) \setminus A$. The set A induces a comb with elements $\{x'_i : 1 \leq i \leq 3q\ \} \cup \{d'\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\ \} \cup \{d\}$ as teeth, as shown in Figure 4, and the neighbors of each element in B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance (X, C) of *X3SC* in polynomial time. Next we show that $X3SC$ has a solution if and only if G has a QTRDF with weight at most $2q + 2$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We construct a QTRDF f on G same as in Theorem 2. Clearly $f(V(G)) = 2q + 2$.

The proof of converse is similar to the proof given in Theorem 2. □

Since star convex bipartite graphs and comb convex bipartite graphs are subclasses of tree convex bipartite graphs, the following result is immediate.

Corollary 2. *QTRDP is NP-complete for tree convex bipartite graphs.*

Next, we show that QTRDP is NP-complete for planar graphs by providing a polynomial-time transformation from a known NP-complete problem, called domination decision problem for planar graphs, which is defined as follows.

Domination-Decision-Problem-Planar-Graphs (DDPG)

INSTANCE : An undirected, planar graph G and an integer k.
QUESTION : Is $\gamma(G) \leq k$?

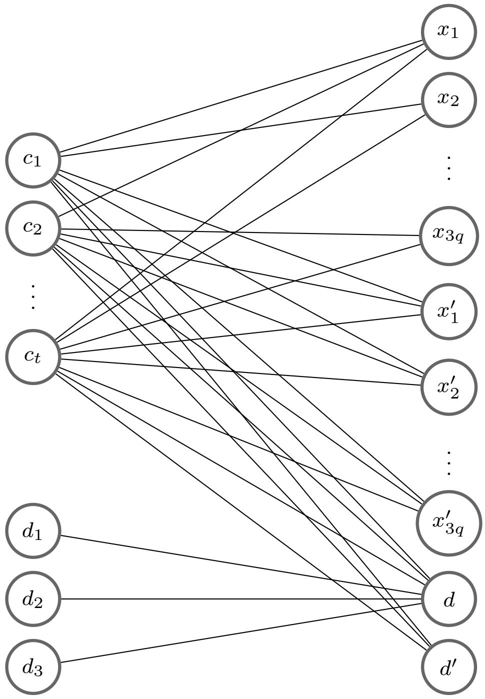

Figure 5. **Comb convex bipartite graph construction from $X3SC$ instance**

Theorem 4. *QTRDPP is NP-complete.*

Proof. Clearly, QTRDPP is in NP. We transform an instance (G, r) of DDPG with vertex set $\{v_1, v_2, \ldots, v_n\}$ to an instance (H, s) of QTRDPP as follows.
$V(H) = V(G) \cup \{a_i, b_i, c_i : i \in \{1, 2, \ldots, n\}\}$ and
$E(H) = E(G) \cup \{(v_i, a_i), (a_i, b_i), (b_i, c_i), (v_i, c_i) : i \in \{1, 2, \ldots, n\}\}$.
An illustration to the construction of graph H from G is shown in Figure 6. It is easy to verify that H is a planar graph and can be constructed from the given instance (G, r) of DDPG in polynomial time. Next we show that G has a dominating set of size at most r if and only if H has a QTRDF with weight at most $s = r + 3n$.

Suppose D be a dominating set of size at most r in G. We define a function $f : V(H) \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{v_i, a_i : v_i \in D\} \text{ or } v \in \{b_i : v_i \notin D\} \\ 1, & \text{if } v \in \{a_i : v_i \notin D\} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Clearly, f is a QTRDF of H and $\gamma_{qtR}(H) \leq s$.
Conversely, suppose that g be a QTRDF of H with weight at most s. Clearly $g(v_i) +$

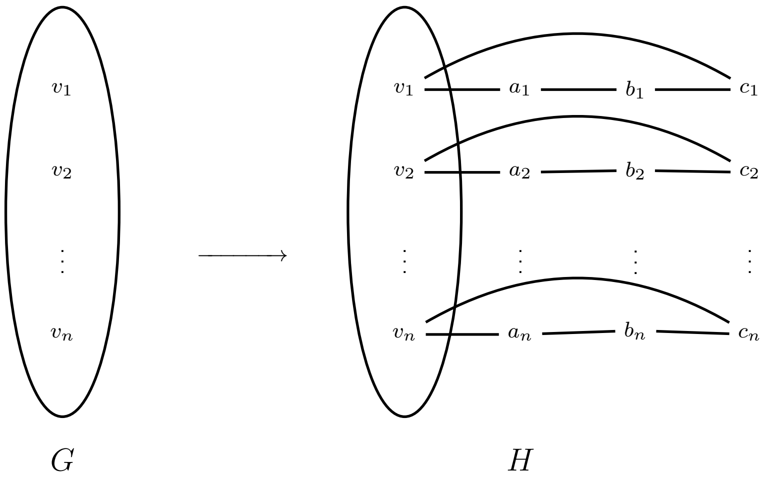

Figure 6. **An illustration to the construction of graph H from the graph G**

$g(a_i) + g(b_i) + g(c_i) \geq 3$ for all i, $1 \leq i \leq n$. From g we construct a QTRDF g' of H such that $g'(v_i) = 0$ or 2, for all i, $1 \leq i \leq n$ as follows. If $g(v_i) + g(a_i) + g(b_i) + g(c_i) \geq 4$ for some i, then assign weights as $g'(v_i) = 2$, $g'(a_i) = 2$, $g'(b_i) = 0$ and $g'(c_i) = 0$. Otherwise, assign weights as $g'(v_i) = 0$, $g'(a_i) = 1$, $g'(b_i) = 2$ and $g'(c_i) = 0$. It is easy to verify that g' is a QTRDF of H with weight at most s. Let $D = \{v_i : g'(v_i) = 2\}$. By contradiction, it can be shown that D is a dominating set of G with $|D| \leq r$. □

3. Threshold graphs

Here, we determine the quasi-total Roman domination number of threshold graphs. A graph G is *threshold* if and only if the following conditions hold [15]:
i). $V(G)$ is partitioned into two disjoint sets, a clique Q and an independent set R
ii). There exists a permutation $(q_1, q_2, \ldots, q_p)$ of vertices of Q such that $N_G[q_1] \subseteq N_G[q_2] \subseteq \ldots \subseteq N_G[q_p]$, and
iii). There exists a permutation $(r_1, r_2, \ldots, r_i)$ of vertices of R such that $N_G(r_1) \supseteq N_G(r_2) \supseteq \ldots \supseteq N_G(r_i)$.

Theorem 5. *If G is a connected threshold graph then*

$$\gamma_{qtR}(G) = \begin{cases} 1, & \text{if } G \cong K_1 \\ 2, & \text{if } G \cong K_2 \\ 3, & \text{otherwise} \end{cases} \tag{4}$$

Proof. Clearly $\gamma_{qtR}(K_i) = i$ for $i = 1$ or 2. Let G be a connected threshold graph with at least three vertices. Also assume that G has p (≥ 1) clique vertices and i (≥ 1)

independent vertices as described above. We define a function $f : V(G) \to \{0,1,2\}$ as follows.

$$f(v) = \begin{cases} 2, & \textit{if } v = q_p \\ 1, & \textit{if } v = r_i \\ 0, & \textit{otherwise} \end{cases} \tag{5}$$

Clearly, f is a QTRDF and $\gamma_{qtR}(G) \leq 3$. Since G has at least three vertices by the definition of QTRDF it follows that $\gamma_{qtR}(G) \geq 3$. Hence the theorem. □

If G is a disconnected threshold graph with $k(\geq 2)$ connected components G_1,G_2,…,G_k then $\gamma_{qtR}(G) = \sum_{i=1}^{k} \gamma_{qtR}(G_i)$.

Now, the result below follows from Theorem 5 and the fact that the ordering of dominating vertices of threshold graph can be found in linear time [15].

Theorem 6. *MQTRDP is linear time solvable for threshold graphs.*

4. Chain graphs

In this section, we determine the quasi-total Roman domination number of chain graphs. A bipartite graph $G = (X,Y,E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, that is, the vertices of X can be linearly ordered, say $(x_1, x_2, ..., x_s)$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_s)$. If $G = (X,Y,E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain. An ordering $\alpha = (x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_t)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_s)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \ldots \supseteq N_G(y_t)$. Every chain graph admits a chain ordering [20].

Theorem 7. *Let $G = (X,Y,E)$ be a connected chain graph. Then*

$$\gamma_{qtR}(G) = \begin{cases} 2, & if \;\; G \cong K_2 \\ 3, & if \;\; |X| = 1 \;\; and \;\; |Y| = t \;(\geq 2) \\ 4, & if \;\; |X| = s \;(\geq 2) \;\; and \;\; |Y| = t \;(\geq 2) \end{cases} \tag{6}$$

Proof. Let G be a chain graph with $|X| = s$ and $|Y| = t$, where $s, t \geq 1$. Clearly $\gamma_{qtR}(G) = 2$ when $G \cong K_2$ and $\gamma_{qtR}(G) = 3$ when $G \cong K_{1,r}$, where $r \geq 2$. Next, we assume that $s \geq 2$ and $t \geq 2$. Now we define a function $f : V(G) \to \{0,1,2\}$ as follows.

$$f(v) = \begin{cases} 2, & \textit{if } v = x_s \;\; or \;\; y_t \\ 0, & \textit{otherwise} \end{cases} \tag{7}$$

Clearly f is a QTRDF and $\gamma_{qtR}(G) \leq 4$. By contradiction it can be easily shown that $\gamma_{qtR}(G) \geq 4$. Therefore in this case, $\gamma_{qtR}(G) = 4$. Now from Theorem 7 and the fact that the chain ordering of a chain graph can be computed in linear time [18] the below theorem follows. □

Theorem 8. *MQTRDP can be solved in linear time for chain graphs.*

5. Bounded treewidth graphs

A *tree decomposition* of a graph H is a tree T_1 with the vertex set $V(T_1) = \{Z_1, Z_2, \ldots, \}$, a subset of the power set of $V(H)$ with the following requirements.
i). $V(H) = \bigcup_{Z_v \in V(T_1)} Z_v$
ii). $\forall (u, v) \in E(H)$, there exists a vertex $Z_t \in V(T_1)$ such that $u, v \in Z_t$, and
iii). $\forall v \in V(H)$, the induced subgraph $\{Z_t : v \in Z_t \text{ and } Z_t \in V(T_1)\}$ is a subtree of T.
Then the tree decomposition T_1 of H is said to have *width* equals to $max\{|Z_t| - 1 : Z_t \in V(T_1)\}$. The *treewidth* is the smallest width of a tree decomposition of a graph. A graph problem for bounded treewidth graphs is linear time solvable if there exists a counting monadic second-order logic (CMSOL) formula for it [1]. We show that QTRDP can be expressed in CMSOL.

Theorem 9. *Given a graph G and a positive integer k, QTRDP can be expressed in CMSOL.*

Proof. Let $f : V(G) \to \{0, 1, 2\}$ be a function. Also, for $j = 0, 1$ or 2, let $V_j = \{v \mid g(v) = j\}$. A CMSOL formula for the QTRDP is expressed as follows.

$$Quasi\_Total\_Rom\_Dom(V(G)) = (f(V(G)) \leq k) \wedge \exists V_0, V_1, V_2\ \forall p(p \in V_1 \vee (p \in V_2 \wedge \exists q \in V_1 \cup V_2 \wedge edge(p, q)) \vee (p \in V_0 \wedge \exists q \in V_2 \wedge edge(p, q))),$$

where $edge(p, q)$ holds true iff $(p, q) \in E(G)$. □

Now, the theorem below follows from Courcelle's result [3] and Theorem 9.

Theorem 10. *MQTRDP for graphs with treewidth at most a constant is solvable in linear time.*

6. Integer linear programming formulation

Let $G = (V, E)$ be an undirected graph, with $|V(G)| = n$, $|E(G)| = m$ and $f : V(G) \to \{0, 1, 2\}$ be a QTRDF of G. Here we present an integer linear program (ILP) model for MQTRDP. This model uses two sets of binary variables. Specifically, for each vertex $v \in V(G)$, we define

$$a_v = \begin{cases} 1, & f(v) = 1 \\ 0, & \text{otherwise} \end{cases} \qquad b_v = \begin{cases} 1, & f(v) = 2 \\ 0, & \text{otherwise} \end{cases}$$

The ILP model of the MQTRDP can now be formulated as
Determine :

$$min\{\sum_{v\in V(G)}(a_v+2b_v)\} \tag{8}$$

subject to constraints:

$$a_v+b_v+\sum_{u\in N(v)}b_u\geq 1,\ v\in V(G) \tag{9}$$

$$b_v\leq \sum_{u\in N(v)}(a_u+b_u),\ v\in V(G) \tag{10}$$

$$a_v+b_v\leq 1,\ v\in V(G) \tag{11}$$

$$a_v,b_v\in\{0,1\} \tag{12}$$

The objective function given in Equation 8 minimizes the weight of QTRDF. Constraint 9 ensures Roman domination condition i.e., either a vertex is assigned label 1 or 2, or if the label assigned is zero then it is adjacent to a vertex with label 2. Constraint 10, guarantees that every vertex with label two has at least one neighbor with a non-zero label. Condition 11, guarantees that exactly one label is assigned to every vertex and the condition 12 ensures that the variables are binary in nature.
In the proposed ILP model, the number of variables is $2n$ and the constraints is $3n$.

7. Conclusion

In this paper, we have shown that the problem of determining if a graph has a quasi-total Roman domination number of at most k is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs and planar graphs. Investigating the algorithmic complexity of these problems for other subclasses of bipartite graphs and chordal graphs remains open. Next, it is shown that MQTRDP is linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs. Finally, an integer linear programming formulation for the quasi-total Roman domination problem is proposed. Designing better ILP formulations for the quasi-total Roman domination problem is interesting.

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